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Final Report

Volume 1

OPUS: Optimal Projection for Uncertain Systems

Dennis S. Bernstein, Harris Corporation

Wassim M. Haddad, Florida Institute of Technology

For
Air Force Office of Scientific Research (AFOSR)
Bolling Air Force Base
Washington, DC 20332

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HARRIS CORPORATION GOVERNMENT AEROSPACE SYSTEMS DIVISION
P.O. BOX 94000, MELBOURNE, FLORIDA 32902 (407) 727-5115

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Abstract

OPUS (Optimal Projection for Uncertain Systems) is a unified approach to control-system design and analysis for high-performance, multivariable applications such as large flexible space structures. OPUS yields low-order, robust controllers that meet both time- and frequency-domain objectives. This final report discusses progress achieved during the previous three years in the areas of robust control, fixed-structure control, sampled-data control, tracking control, and nonlinear control.

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Abstract

OPUS (Optimal Projection for Uncertain Systems) is a unified approach to control-system design and analysis for high-performance, multivariable applications such as large flexible space structures. OPUS yields low-order, robust controllers that meet both time- and frequency-domain objectives. This final report discusses progress achieved during the previous three years in the areas of robust control, fixed-structure control, sampled-data control, tracking control, and nonlinear control.

1 Introduction

1.1 Overview of OPUS: Scope, Philosophy and Goals

Over the past 10–15 years controls researchers have come to the realization that classical controls analysis and design techniques are inadequate in the face of modern large-scale, high-performance applications. In particular, the principal motivation for OPUS (Optimal Projection for Uncertain Systems) is the problem of vibration suppression in large lightweight flexible space structures. Such systems are typically characterized by high-dimensional, highly uncertain models with multiple inputs and outputs. In addition, stringent performance specifications in the face of high disturbance levels place severe demands on existing control-design techniques. Specifically, performance tradeoffs involving sensors, processors, actuators, and identification accuracy must be cut as tightly as possible to minimize hardware and testing costs. For feasibility and cost effectiveness, system design must also be performed efficiently with respect to human and computer resources.

The goal of OPUS is to develop a mathematically rigorous, unified control-design methodology that directly addresses these technology issues.[1] In particular, optimal projection theory addresses the need for low-order, high-performance controllers that can be implemented on-board for real-time operation. Low-order controllers are necessitated by cost, weight, and reliability constraints associated with space-qualified processors. Furthermore, OPUS incorporates a fundamental theory of robust controller synthesis to account for unavoidable modeling uncertainties arising for reasons such as material and manufacturing variations, thermal and aging effects, and limited identification accuracy in a 1-g environment. The principal contribution of OPUS is a unified design methodology that simultaneously

[1]See Appendix A for an extensive review of OPUS

accounts for both real-time processor constraints and modeling uncertainty. Under this program OPUS has been extended to a large class of problems in systems and control theory. The current scope of the theory includes

1. Reduced-order modeling, estimation and control;
2. Robust estimation and control via quadratic Lyapunov functions including robust performance;
3. A unified approach to H_2 and H_∞ control;
4. A general theory of fixed-structure control;
5. Extensions to sampled-data and decentralized control.

One goal of OPUS is to capture as many design constraints as possible within a single system of design equations. Of particular interest is recent extensions to H_∞ design. As shown in [I.29] (Appendix B), we have developed a method for directly imbedding H_∞ design constraints within OPUS theory and thus, in particular, within LQG. These results are given by a system of modified Riccati equations that directly generalize LQG theory and that have the potential for significant computational savings compared to existing H_∞ synthesis methods. An additional example is provided by the results obtained in [I.20,I.30,I.35,I.43] (Appendix D) for both robust stability and robust H_2 performance via fixed-order compensation in the presence of parametric uncertainty.

There are several justifications for this line of research. First, and most obvious, is the fact that our results show that numerous design constraints can be captured simultaneously within a design theory that directly generalizes LQG theory. Such an approach provides the capability for simultaneously performing multiple design tradeoffs for multivariable feedback systems with respect to competing constraints such as sensor noise, control authority, controller order, robustness, disturbance attenuation, mean-square error, sample rate, degree of decentralization, etc. Next we stress that rather than being ad hoc constructions, these design equations follow directly from the optimality of physically meaningful well-defined performance criteria. These results are thus useful in assessing the suboptimality of alternative methods.

Overall, OPUS can be viewed as a *theory* for characterizing solutions to constrained control-design problems. Transforming OPUS into a *practical* design methodology requires the development of effective computational algorithms. Such development has been in progress at Harris Corporation and Florida Institute of Technology for several years [I.23,I.41,II.117,II.124]. Using homotopic continuation methods, efficient numerical algorithms have been developed that account for the structure of these modified Riccati equations and their coupling terms. Homotopy algorithms offer several advantages over both gradient-based and Newton-type methods. For example, homotopy methods have a strong theoretical foundation based upon topological degree theory, while in practice these methods address the issues of startup, convergence, and global optimality. Homotopy algorithms have also reached a high degree of maturity and availability with the advent of HOMPACK described in

> L. T. Watson, S. C. Billups and A. P. Morgan, "A Suite of Codes for Globally Convergent Homotopy Algorithms," *ACM Trans. Math Software*, Vol. 13, pp. 281–310, 1987.

1.2 Impact of OPUS on Related Programs

At Harris, OPUS has been used in the NASA CSI (Control-Structure Interaction) program administered by NASA Langley Research Center. As a contractor to NASA under this program, Harris has implemented robust optimal projection controllers to demonstrate vibration suppression on the ACES testbed at Marshall Space Flight Center and the Minimast structure at NASA Langley. Details of these experiments can be found in [II.118].

Outside of Harris, OPUS has been applied to an experimental truss structure at Sandia Laboratories. As described in

> Peterson, L. D., Allen, J. J., Lauffer, J. P., and Miller, A. K., "An Experimental and Analytical Synthesis of Controller Structure Design," *Proceedings of the 30th Structures, Structural Dynamics, and Materials Conference*, paper 89–1170, Mobile, AL, April 1989.
>
> Peterson, L. D., "An Optimal Projection Controller for an Experimental Truss Structure," *Proceedings of the AIAA Guidance,*

Navigation, and Control Conference, Boston, MA, August 1989, pp. 77–88; *J. Guid. Contr. Dyn.*, Vol. 14, pp. 241–250, 1991.

experimental results demonstrated close agreement with theoretical predictions for low-order, practically implementable controllers. There are also ongoing programs at Sandia that are utilizing OPUS for active vibration suppression. In fact, Sandia National Laboratories is funding the development at Harris of advanced homotopy algorithms.

1.3 Goals of this Report

This report has two main objectives. First, we shall summarize progress achieved under the program during the past three years. Since most of the technical results are given in the extensive appendices, we shall only summarize these results and remark on their significance and interrelationship. Our second objective is to describe the status of ongoing research activities.

2 Robust Control

The problem of robust control design and analysis constitutes a significant challenge in mathematical systems theory which, at the same time, addresses a fundamental issue in the *practical* implementation of feedback control systems, namely, modeling uncertainty. Modeling uncertainty must be characterized and quantified so that it can be accounted for within the control-design process. Furthermore, a controller can only be considered robust when it is guaranteed to provide specified stability and performance over a prescribed set of uncertain plant variations.

For vibration suppression in flexible structures, modeling uncertainty is often significant and thus the need for robust control may be severe.

2.1 H_∞ Control

Strictly speaking, H_∞ control refers to the problem of disturbance attenuation with L_2 signal norms on disturbances and performance variables. The induced H_∞ transfer function norm thus corresponds to the worst-case

disturbance attenuation. By introducing suitable dynamic weightings, H_∞ design provides the means for loop shaping and hence stability robustness.

During the past several years it has become clear that suboptimal H_∞ design can be performed by utilizing algebraic Riccati equations. In [I.29] (Appendix B) we formulated a fixed-order control-design problem with a constraint on the H_∞ norm of the closed-loop transfer function. By using an algebraic Riccati equation to enforce the H_∞ constraint, optimality conditions for characterizing feedback gains that minimize an H_2 cost bound were obtained. The results derived In [I.29] thus address both H_2 and H_∞ design aspects. This result provides the means to tradeoff rms (quadratic performance) and stability robustness to unstructured uncertainty.

An additional feature of [I.29] is the treatment of both full- and reduced-order controllers via optimal projection design. Hence, these results provide a unified setting for H_2/H_∞/reduced-order control design. The extension to reduced-order control is useful for applications in which high plant order and dynamic weightings would otherwise lead to excessively high-order controllers.

The reduced-order H_∞ design results have been extended to two related problems, namely, H_2/H_∞ model reduction and H_2/H_∞ estimation (see [I.31] and [I.28] in Appendix B). For the control problem, the results of [I.29] were generalized in [I.45] (see Appendix B) to include additional aspects such as disturbance/performance feedthrough. Finally, the relationship between the H_2 norm bound (an entropy functional) and the H_2 norm itself is explored in [I.40] (Appendix B).

2.2 Robustness with Structured Uncertainty

Our approach to robust control in the presence of structured uncertainty was originally inspired by the effects of multiplicative white noise within a linear-quadratic optimization problem. Optimal controllers designed in the presence of such disturbances, it was reasoned, are automatically desensitized to *actual* plant parameter variations. This idea was confirmed empirically by numerical studies [I.5] which showed an efficient design tradeoff between performance and robustness in the presence of structured real-valued parameter variations.

What was lacking, however was a rigorous proof that such controllers

are guaranteed to be robust for all cases in which the design equations are solvable. This issue was addressed in [I.13] where it was shown that a second-moment stochastic stability condition in the presence of a time-exponential cost weighting induces a Lyapunov function that guarantees deterministic robust stability over a prescribed range of parameter variations. This result thus provides a link between stochastic optimal control and deterministic robust stability theory.

Although robust stability is extremely important in applications, it is often desirable in practice to obtain, in addition, a bound on worst-case performance degradation over a class of plant perturbations. To this end we have extended optimal projection theory to encompass the problem of robust H_2 performance. Specifically, as shown in [I.30] (Appendix D), the multiplicative white noise model yields, in addition, a bound on worst-case performance.

A more basic problem than robust synthesis is the problem of robust *analysis*. That is, given a particular control design, determine the class of perturbations under which the closed-loop system remains stable along with a bound on worst-case performance. Results for this problem were obtained in [I.33] (Appendix C) by means of the Lyapunov equation

$$0 = AQ + QA^T + \Omega + V, \tag{1}$$

where Ω is a fixed matrix that provides a bound for perturbations ΔA of A. A related idea developed in [I.44] (Appendix C) involves replacing Ω by an operator $\hat{\Omega}(Q)$ resulting in

$$0 = AQ + QA^T + \hat{\Omega}(Q) + V. \tag{2}$$

The operator $\hat{\Omega}$ is chosen to bound terms of the form $\Delta AQ + Q\Delta A^T$, where ΔA is an uncertain perturbation of the dynamics matrix A. This approach leads to an *a priori* robustness test, i.e., a "yes/no" test for a given uncertainty set in contrast to the results of [I.33] which are of an *a posteriori* nature. Using (2) also unifies several approaches studied in the literature for control design. Specifically, setting

$$\hat{\Omega}(Q) = \sum_{i=1}^{p} \delta_i \mid A_i Q + Q A_i^T \mid, \tag{3}$$

where $| \cdot |$ is the eigenvalue absolute value operator, corresponds to the approach of

S. S. L. Chang and T. K. C. Peng, "Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters," *IEEE Trans. Autom. Contr.*, Vol. AC-17, pp. 474-483, 1972.

while the quadratic (in Q) bound

$$\hat{\Omega}(Q) = D + QEQ, \tag{4}$$

which is related to H_∞ theory, was studied in

I. R. Petersen and C. V. Hollot, "A Riccati Equation Approach to the Stabilization of Uncertain Systems," *Automatica*, Vol. 22, pp. 397-411, 1986.

Furthermore, the linear (in Q) bound

$$\hat{\Omega}(Q) = \sum_{i=1}^{p} \delta_i(\alpha_i Q + \alpha_i^{-1} A_i Q A_i^T) \tag{5}$$

corresponds to the multiplicative white noise approach. The quadratic bound (4) and the linear bound (5) were developed for robust controller synthesis in [I.20] and [I.43], respectively (Appendix D). Both the linear and quadratic bounds were considered within the context of H_∞ control in [I.35] (Appendix D).

In general these bounds are sufficient for robust stability but, except in special cases, are not necessary. Furthermore, the bounds hold for complex variations as well thus leading to additional conservatism. To reduce this conservatism one can utilize positivity theory and the circle and Popov criteria, to which we now turn.

2.3 Robustness to Positive Real Uncertainty

As a first step in reducing the small-gain-type conservatism of quadratic Lyapunov functions, we have addressed the use of phase information in robust analysis and synthesis. The importance of addressing phase information in controlling flexible structures and its relationship to real parameter uncertainty are discussed in [II.123] (Appendix M). To include phase information

in the modeling of uncertainty, we applied the Riccati equation characterization of positive real transfer functions due to Anderson and Vongpanitlerd to a robust control problem involving positive real uncertainty [I.54] (Appendix M). It turns out that the Riccati equation that guarantees robust stability for positive real uncertainty is actually a quadratic Lyapunov bound in the sense of the theory developed in [I.44]. This result thus unifies small gain and positive real stability robustness analysis.

These results have also been extended to encompass the classical Popov criterion [62,64] (Appendix M). The principal feature of these results is the fact that they are based upon parameter-dependent Lyapunov functions which effectively exclude time-varying parameter perturbations. Hence these results are significantly less conservative when the plant uncertainty involves constant real parameters. The results of [64] encompass full-state, full-order, and reduced-order controller synthesis in H_2 and H_2/H_∞ settings.

3 Fixed-Structure Control

In practical control design it is usually necessary to focus on controllers that satisfy structural constraints such as compensator order, decentralization, etc. Several aspects of the fixed-structure problem are discussed in this section.

3.1 Finite-Dimensional Control of Distributed Parameter Systems

Distributed parameter systems such as large flexible structures are inherently infinite dimensional. That is, it is not generally possible to specify a "modeling bandwidth" prior to the control-design process. Implementable controllers, in contrast, are invariably constrained to be finite dimensional and preferably of as low order as possible. The need for such controllers was discussed at length in [I.4], where the optimal projection equations were generalized to characterize finite-dimensional, fixed-order controllers for infinite-dimensional systems. This result provides a path to controller design that avoids both model and controller reduction.

The usual approach to finite-dimensional controller design involves con-

structing a sequence of full-order controllers for a sequence of plant approximations of increasing order. The drawback to such an approach is that for a given controller order there is no guarantee that the corresponding controller is optimal over the class of controllers of the given order, thus yielding controllers of unnecessarily high dimension in order to meet performance requirements.

To overcome the suboptimality problem and to limit the compensator order in accordance with implementation constraints, the optimal projection approach can be used to constrain the controller order even as the discretization order increases [I.4]. This approach has been implemented using spline approximations in [I.42] (Appendix E). For a heat equation and a delay system, optimal projection designs were obtained. For the heat equation these results comprise a sequence of first-order controllers for plant discretizations up to 32nd order. These results should be contrasted with the functional gains for the same example which require considerably greater processor capacity for real-time implementation with virtually the same closed-loop performance.

3.2 Controller Complexity Constraints

In control-system design it is often desirable to implement the simplest possible controller for achieving performance objectives. By "simplest" we are referring to a reduction in control system complexity which can be measured in several ways, for example:

- sensor/actuator requirements (throughput and memory)
- degree of decentralization (sensor/actuator communication)
- controller memory (dynamic order and pole location)
- control law logic (mode switching, limit checking, and fault detection)

To address the issues of controller complexity, we have developed a theory of fixed-structure design that permits considerable flexibility in assigning the structure of the controller architecture. This approach is based upon the results obtained in [II.89] (Appendix F) which shows that a dynamic controller of arbitrary structure can be recast as a static output feedback

controller for a suitably modified plant. The static output feedback problem has a highly structured form not considered previously in the literature, namely, decentralized control allowing for the repetition of gains in different feedback paths. This gain repetition provides additional flexibility in addressing problems such as reliable stabilization of multiple plants.

The problem defined in [II.89] was addressed in [II.91] which developed a theory of H_2/H_∞ control design. By extending results obtained earlier for H_2 decentralized control in [I.34], the results of [II.91] include, in addition, a constraint on H_∞ performance. The key feature of [II.91] is the treatment of the singular feedback control gain from nonnoisy measurements to unweighted controls. These results also clarify relationships with singular control theory and, in particular, the singular LQG problem to which we now turn.

3.3 Singular Control Problems

The fixed-structure approach provides an additional tool for addressing the longstanding problem of singularities in optimal feedback design. To address the singular problem, we have reexamined the simplest possible case, namely full-state feedback with totally singular control weighting [II.122] (Appendix G). To better understand the fixed-structure approach as applied to this singular problem, the optimality conditions were derived in [II.122] by means of four distinct methods, namely, the Goh transformation, a perturbation method, the generalized Legendre-Clebsch conditions, and the fixed structure technique. Applying these different techniques side-by-side has proven to be useful in illuminating the subtleties of this problem.

The advantage of the fixed-structure approach over the other techniques studied in [II.122] is the ease with which it addresses feedback constraints. Thus using this technique we have undertaken a renewed attack on the singular LQG problem [I.67] (Appendix G). The fixed structure approach also provides the ability to limit the number of signal differentiators utilized in the feedback controller.

Finally, the fixed-structure approach extends directly to the H_2/H_∞ problem. The results given in [II.54] (Appendix G) represent the most general singular H_∞ results currently available.

3.4 Pole Placement

As an additional extension of fixed-structure design, we have addressed the problem of designing feedback compensators with constrained pole locations. The basic idea is to extend LQG controller design to include a constraint on the closed-loop poles. To this end we have considered static and dynamic compensation with a variety of constraint regions, including a circle, ellipse, parabola, and strip. The results obtained in [I.55] (Appendix H) demonstrate a tradeoff between pole location for transient response and steady-state control effort. An important extension for future research is the incorporation of eigenvector constraints as well to achieve disturbance decoupling.

4 Sampled-Data Control

The discussion in the previous sections has focused on continuous-time systems subject to continuous-time (analog) controllers. In practice, however, controller implementation will almost invariably utilize digital controllers within the context of sampled-data control systems. Rigorous consideration of such systems is critical, particularly for distributed parameter systems which possess modal frequencies beyond the Nyquist rate of *any* digital controller. Hence, a rigorous theory of sampled-data control design must be developed that accounts precisely for all effects arising from analog-to-digital and digital-to-analog operations.

Optimal projection theory for discrete-time systems was developed in [I.6] and applied to sampled-data systems in [I.7]. As a next step it is desirable to obtain robust control results. To this end, the optimal projection equations for reduced-order discrete-time control in the presence of multiplicative white noise were obtained in [I.10]. After these results were obtained, it became clear that a true sampled-data robustness theory must account for the special matrix structure that arises from the sampling process. For example, if $A+\Delta A$ denotes the continuous-time dynamics matrix, where A is the nominal matrix and ΔA denotes uncertainty, then the equivalent discrete-time dynamics matrix is given by $e^{(A+\Delta A)h}$, where h is the sample interval. Because of the exponential, however, this discrete-time dynamics matrix does not have the form considered in the discrete-time theory. Moreover, a finite approximation for the exponential will not be valid in the

presence of system time constants near or above the sample rate.

One attempt to bound this discrepancy resulted in new inequalities in [I.19] (Appendix J), while an alternative approach based upon quadratic Lyapunov bounds was developed in [I.37] (Appendix J).

In practical applications there arise additional sampled-data effects that must also be accounted for. Specifically, real systems invariably involve multiple, nonsynchronized A/D and D/A devices working at different rates within the context of a decentralized controller architecture. Preliminary analysis of several such cases reveals enormous underlying complexity that must be accounted for in designing implementable control laws.

Our next goal is the problem of multirate sampled-data control. As a first step in addressing the multirate problem, we have obtained results for multirate estimation [I.66] (Appendix J). The estimator has a discrete-time periodic structure to directly account for the multirate timing sequence of the measurements. This result is based upon a discrete-time periodic stability condition and involves a coupled system of Riccati equations corresponding to each subinterval of the period. Extensions to multirate control are in progress.

5 Tracking Control

The theory discussed in Sections 2–4 addresses the problem of feedback control for regulation in the presence of stochastic disturbances. Many control problems are, however, of a tracking or servomechanism nature. While a limited class of such problems can be recast without loss of generality as regulation problems, many important ones cannot. For example, the standard transformations given in

> H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley, New York, 1972.
>
> B. A. Francis, *A Course in* H_∞ *Control Theory*, Springer-Verlag, New York, 1987.

assume that the command signals can be represented as an augmentation of the plant dynamics. There are many important cases, such as the tracking

of steps and ramps, that must be represented by uncontrollable, unstable (i.e., unstabilizable) dynamics, where this transformation cannot be applied. Furthermore, such transformations often ignore controller effort.

As a first step in addressing these issues, we have considered the problem of regulation about a prescribed nonzero set point, which corresponds to the step command tracking problem. Our work in this area was originally motivated by results obtained in

> Z. Artstein and A. Leizarowitz, "Tracking Periodic Signals with the Overtaking Criterion," *IEEE Trans. Autom. Contr.*, Vol. AC–30, pp. 1123-1126, 1985.
>
> A. Leizarowitz, "Tracking Nonperiodic Trajectories with the Overtaking Criterion," *Appl. Math. Optim.*, Vol. 14, pp. 155–171, 1986.
>
> A. Leizarowitz, "Infinite Horizon Stochastic Regulation and Tracking with the Overtaking Criterion," *Stochastics*, 1987.

References [I.9,I.17,I.24] (Appendix I) present general solutions to the nonzero set point problem for static and dynamic controllers. Note that these controllers involve two components, namely, a closed-loop feedback component similar to a regulator and an open-loop feedforward component which has no counterpart in the standard theory and which cannot be obtained from standard transformations.

Recent activities have focused on extending the nonzero set point results to broader classes of command and disturbance signals. It turns out that the challenging case (as with steps and ramps) involves signals generated by unstable command or reference dynamics. As a critical first step in addressing this problem we have considered the problem of reduced-order steady-state observer design for *unstable* plants [I.38,I.47] (Appendix I). Previous results on reduced-order estimation obtained in [I.2] were limited to stable systems.

Our present goal is to extend the results of [I.47] to the problem of command following with unstabilizable dynamics. This result involves a nontrivial extension of the results of [I.47] to incorporate the command dynamics within the compensator [II.143] (Appendix I).

6 Nonlinear Control

6.1 Motivation

In view of the requirements of control-system practice, we are primarily interested in developing nonlinear control-system synthesis methods with the following attributes:

- yield automonous (time-invariant) controllers for simplicity of implementation
- yield feedback controllers that are independent of initial conditions for disturbance rejection and robustness
- yield continuous control signals to avoid impulsive inputs

Of course there are numerous applications in which some or all of the above requirements may not apply. For example, a precise spacecraft slewing maneuver may be best achieved by a time-varying finite-interval control law obtained by solving a two-point boundary value problem. Nevertheless, our requirements lead to a useful class of implementable feedback controllers.

There currently exist a variety of methods for synthesizing nonlinear feedback control systems, including local linearization, global linearization, variable structure control, Lyapunov function techniques, and differential-geometric methods. Our attention is focussed primarily on optimality-based methods. Specifically, we have considered synthesis methods that yield nonlinear feedback controllers as a consequence of minimizing a specified performance (cost) functional. A principal motivation is the desire to extend linear optimality-based methods such as LQG and H_∞ theory.

In practice, optimality-based design theories provide the ability to quantify and optimize the performance of the closed-loop system. Moreover, the structure and characteristics of the feedback control law arise as a consequence of variational principles thus providing a rigorous foundation for the design task. Such methods often provide an optimality context for controllers designed by other methods.

The Maximum Principle is, of course, an optimality-based result which is applicable to a wide range of time-varying, nonlinear plants. However, since

the Maximum Principle does not generally provide *feedback* controllers, robustness in the presence of uncertainties such as external disturbances and plant modeling errors is difficult to achieve. Furthermore, infinite-horizon controllers (discussed in more detail below) are difficult to achieve. Nevertheless, the Maximum Principle remains within our sphere of interest both for its fundamental nature and for its connections to Hamilton-Jacobi-Bellman theory to which we now turn.

6.2 HJB Theory

Hamilton-Jacobi-Bellman theory has its roots in the classical Hamilton-Jacobi partial differential equation as well as the dynamic programming technique of Bellman. In its most general form, the theory involves a partial differential equation whose solution yields an optimal controller. For practical purposes we assume analytic data with an infinite horizon cost criteria. As shown in

> D. L. Lukes, "Optimal Regulation of Nonlinear Dynamical Systems," *SIAM J. Contr.*, Vol. AC-16, pp. 87–88, 1971.

under these assumptions the HJB equation has an analytic solution $V(x)$ which is Lyapunov function that guarantees global stability of the closed-loop system. Furthermore, the optimal control is given in the form of a *feedback* law which is independent of the initial condition. A simplified framework for this theory is given in [I.57] (Appendix K). Particular attention is paid to the closed-form solution given in

> R. W. Bass and R. D. Webber, "Optimal Nonlinear Feedback Control Derived from Quartic and Higher-Order Performance Criteria," *IEEE Trans. Autom. Contr.*, Vol. AC-11, pp. 448–454, 1966.

Connections are also made to related results given in

> J. L Speyer, "A Nonlinear Control Law for a Stochastic Infinite Time Problem," *IEEE Trans. Autom. Contr.*, Vol. AC–21, pp. 560–564, 1976.

In spite of the appealing nature of HJB theory, its current state of development entails several limitations in addressing real-world problems. In particular, these include 1) the ability to design static and dynamic output-feedback compensators; 2) the treatment of external disturbances; and 3) the development of robust controllers for uncertain plants. Hence one of our principal goals is to extend the current state of the theory by removing these limitations.

Our first goal is to eliminate the requirement that the full state be available to implement the feedback law. For the class of control problems that we are addressing there are typically only a small number of measurements available for a high-dimensional plant. An indirect solution to this problem is to implement an estimator to reconstruct the state from the available measurements. However, besides being a difficult problem itself, there is no reason to expect that certainty equivalence (separation) will hold in the presence of nonlinearities. Consequently, we must consider the problem of designing output-feedback laws that are pre-constrained to operate solely upon the available measurements. Such controllers may be either static (proportional) or dynamic.

To achieve this goal it appears natural to develop a *fixed-structure* HJB theory in which one can prespecify the structure of the feedback law with respect to, say, the order of nonlinearities appearing in the dynamic compensator. The actual gain maps can then be determined by solving algebraic relations in much the same way full-state feedback controllers are obtained. Preliminary results in this direction have been obtained in

J. J. Beaman, "Nonlinear Quadratic Gaussian Control," *Int. J. Contr.*, Vol. 39, pp. 343-361, 1984.

Another limitation of most existing HJB theory is the lack of design results in the presence of external disturbances. In linear control theory such disturbances can be modeled as additive plant disturbances, and, in the case of output feedback, as additive sensor noise. The treatment of external disturbances is possible by means of stochastic HJB theory, although with greater complexity.

When the performance functional $\int L(x,u)$ involves terms of order x^p, where $p < 2$, then we call the cost criterion *subquadratic*. Subquadratic cost criteria pay close attention to the behavior of the state near the origin

since, for example, $x^{\frac{1}{2}} >> x^2$ for x near zero. Our interest in subquadratic cost criteria stems from the fact that optima controllers for such criteria are sublinear and thus exhibit finite settling time behavior. This phenomenon has been studied in

> S. V. Salehi and E. P Ryan, "On Optimal Nonlinear Feedback Regulation of Linear Plants," *IEEE Trans. Autom. Contr.*, Vol. AC–27, pp. 1260–1264, 1982.
>
> V. T. Haimo, "Finite Time Controllers," *SIAM J. Contr. Optim.*, Vol. 24, pp. 768–770, 1986.

and applied to spacecraft control in

> S. V. Salehi and E. P. Ryan, "Optimal Nonlinear Feedback Regulation of Spacecraft Angular Momentum," *Optim. Contr. Appl. Meth.*, Vol. 5, pp. 101–110, 1984.

These results yield finite interval controllers even though the original cost criterion was defined on the infinite horizon. Hence one advantage of this approach for certain applications is to obtain finite-interval controllers without the computational complexities c wo-point boundary value problems. We also note that if the order of the subquadratic state terms appearing in the cost functional is sufficiently small, then the controllers actually optimize a minimum-time cost criterion. Currently, such results are only obtainable using the maximum principle which generally does not yield feedback controllers. Hence subquadratic cost criteria permit a unified treatment of a broad range of design goals and provide the tools for developing connections with alternative methods.

Appendix A

Chapter 10

Optimal Projection Approach to Robust Fixed-Structure Control Design

Dennis S. Bernstein* and David C. Hyland†
Harris Corporation, Melbourne, Florida

Overview of the Optimal Projection Approach to Control Design

THE challenge of modern control theory research is to develop effective methodologies for multivariable control system design. Both the importance and difficulty of the task should not be underestimated. Although three decades of intensive effort have now been devoted to this task, its attainment continues to elude us. Meanwhile, the challenges of high-performance control system design for complex systems continue to mount. As sensor/processor/actuator technologies continue to improve, there arises a corresponding burden on the control system designer to effectively utilize the available hardware; to minimize design, implementation, testing, and maintenance costs; and to guarantee the highest possible levels of performance and reliability.

Optimal projection theory is one approach to this challenge. The optimal projection approach was originally developed to address the problem of fixed-order linear-quadratic dynamic compensator design to obtain low-order, high-performance controllers.[1-14] The motivation for restricting the controller order arises from limitations on computer processing capability typical of applications such as vibration suppression in large space structures. Subsequently, a decade of effort has extended the optimal projection approach to address a broad range of issues in multivariable control system design, including the following:

1) distributed-parameter systems[15-17];
2) discrete-time and sampled-data systems[18,19];

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*Staff Engineer, Structural Controls Group.
†Senior Principal Engineer and Group Leader, Structural Controls Group.

3) decentralized control[20,21];
4) model reduction[22–24];
5) H_2/H_∞ control[25–30];
6) parameter-robust control[31–56];
7) fixed-structure design[57,58]; and
8) pole placement.[59]

The key feature of optimal projection theory is that all of the preceding issues are addressed within a *common* mathematical framework that effectively permits simultaneous treatment of multiple design goals. For example, results on reduced-order design and parameter-robust design can be merged to produce parameter-robust, reduced-order designs.

As its name implies, optimal projection theory is an optimal control methodology. However, it is a *constrained* optimal control methodology. That is, optimal projection theory does not seek to optimize a performance measure per se, but rather seeks to optimize performance within a class of controllers satisfying specified side constraints such as robust stability or fixed controller order. Unconstrained optimal controllers, although globally optimal in a narrow sense, may be quite unacceptable in meeting additional specifications. For example, for linear plants with Gaussian white noise disturbances, linear-quadratic Gaussian (LQG) controllers provide the best nominal quadratic performance compared to all (even nonlinear) feedback controllers, yet the order of an LQG controller is equal to the order of the plant and hence may be impossible or impractical to implement if the plant is of high order. Furthermore, LQG controllers may be arbitrarily sensitive to plant parameter variations and unmodeled dynamics. Optimal projection theory overcomes these defects by generalizing LQG theory to include these and other design aspects. Precisely how these design aspects are incorporated as side constraints is the subject of subsequent sections of this chapter.

How the Optimal Projection Approach Differs from Other Fixed-Structure Design Methods

The most basic constraint incorporated within optimal projection theory involves fixing the order of the feedback compensator to be less than that of the plant. This is the fixed-order dynamic compensation problem. The fundamental nature of this problem is reflected by the extensive literature devoted to it.[60–83] In the standard approach to this problem, a quadratic criterion is minimized over the class of compensator gain matrices of given dimension. By setting the cost gradients to zero, the first-order necessary conditions for the problem consist of a collection of complicated nonlinear matrix equations without apparent structure. Because the equations are complicated, they are difficult to analyze, insight into their solution structure is lacking, and search methods are often the only viable numerical algorithms.

Optimal projection theory, however, makes no attempt to solve these complicated nonlinear matrix equations. Rather, its point of departure

involves transforming these complicated nonlinear matrix equations into a highly structured form that is consistent with LQG theory. The key to this transformation is the recognition that the optimality conditions give rise to an idempotent matrix, that is, a nonorthogonal projection. The transformation yields a highly structured system of four coupled algebraic matrix equations consisting of two modified Riccati equations and two modified Liapunov equations. The coupling terms, which depend on the projection, clearly show that there is no estimator/regulator separation in the reduced-order case. However, if the controller order is chosen to be equal to the plant order, then the projection becomes the identity, the coupling terms are zero, the four equations reduce to two standard Riccati equations, and the LQG controller is recovered.

These four optimal projection equations present new challenges and opportunities. Mathematical challenges arise in analyzing these equations. Since the Riccati and Liapunov equations are nonstandard, their analysis requires extensions of existing techniques as well as the development of new methods. In addition, the coupling between the equations presents additional difficulties. Hence, the equations present nontrivial mathematical challenges relevant to a problem of practical interest.

On the other hand, practical benefits lie in the structured form of the equations. We already have a crucial insight, namely, the mechanism responsible for the breakdown of estimator/regulator separation, i.e., the coupling terms. As will be seen in the section on relationships between optimal projection theory and model/controller reduction techniques, it is precisely these coupling terms that are responsible for the suboptimality of controller reduction techniques. The main practical benefit of these equations is the opportunity to develop novel computational methods. By exploiting the Riccati and Liapunov structure, a new class of algorithms can be constructed as described in the section on numerical solution of the optimal projection equations. Furthermore, the optimal projection equations provide insight into the multiplicity of solutions and the guaranteed stability of the closed-loop system. To illustrate these points, we now turn to a brief review of basic optimal projection theory.

Brief Review of Basic Optimal Projection Theory

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{1}$$

$$y(t) = Cx(t) + D_2 w(t) \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^\ell$, and $w \in \mathbb{R}^p$ is white noise with unit intensity I_p. The intensities of $D_1 w(t)$ and $D_2 w(t)$ are thus given, respectively, by $V_1 \triangleq D_1 D_1^T \geq 0$ and $V_2 \triangleq D_2 D_2^T > 0$. For convenience we assume that $V_{12} \triangleq D_1 D_2^T = 0$; i.e., the plant disturbance and measurement noise are uncorrelated. The goal of the fixed-order dynamic compensation problem is

to determine an n_cth-order optimal dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3}$$

$$u(t) = C_c x_c(t) \tag{4}$$

that minimizes the steady-state performance criterion

$$J(A_c,B_c,C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[x^T(t)R_1 x(t) + u^T(t)R_2 u(t)] \tag{5}$$

where $x_c \in \mathbb{R}^{n_c}$, $n_c \leqslant n$, $R_1 = R_1^T \geq 0$, and $R_2 = R_2^T > 0$.

The $\tilde{n}$-dimensional closed-loop system corresponding to Eqs. (1) and (2) ($\tilde{n} \triangleq n + n_c$) can be expressed as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t) \tag{6}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix} \tag{7}$$

In addition, the performance (5) can be expressed as

$$J(A_c,B_c,C_c) = \lim_{t\to\infty} \mathbb{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)] \tag{8}$$

where

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix} \tag{9}$$

To guarantee that the cost J is finite and independent of initial conditions, we restrict our attention to the set of n_cth-order stabilizing compensators

$$\mathscr{S}_{n_c} \triangleq \{(A_c,B_c,C_c) \colon \tilde{A} \text{ is asymptotically stable}\} \tag{10}$$

We require no explicit characterization of $\mathscr{S}_{n_c}$; its role is theoretical only. However, we shall *assume* that $\mathscr{S}_{n_c}$ is not empty.

Now let $(A_c,B_c,C_c) \in \mathscr{S}_{n_c}$ so that $\tilde{A}$ is asymptotically stable, and let $\tilde{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ and $\tilde{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ be the closed-loop steady-state covariance and its dual, i.e.,

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} \tag{11}$$

$$0 = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{R} \tag{12}$$

where

$$\tilde{V} \triangleq \tilde{D}\tilde{D}^T = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix} \tag{13}$$

Then the performance (5) can be expressed as

$$J(A_c,B_c,C_c) = \operatorname{tr} \tilde{Q}\tilde{R} = \operatorname{tr} \tilde{P}\tilde{V} \tag{14}$$

Also, if $\tilde{Q}$ and $\tilde{P}$ are partitioned according to

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \qquad Q_1 \in \mathbb{R}^{n\times n}, \qquad Q_2 \in \mathbb{R}^{n_c\times n_c} \tag{15}$$

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \qquad P_1 \in \mathbb{R}^{n\times n}, \qquad P_2 \in \mathbb{R}^{n_c\times n_c} \tag{16}$$

then Q_1 is the covariance of the plant states, Q_2 is the covariance of the compensator states, and Q_{12} is the cross covariance of the plant and controller states.

Since the value of J is independent of the internal realization of the compensator, *in the following* we will further restrict our attention to minimal compensators. Thus, we define the admissible set as

$$\mathscr{S}_{n_c}^+ = \{(A_c,B_c,C_c) \in \mathscr{S}_{n_c} \colon (A_c,B_c) \text{ is controllable, } (A_c,C_c) \text{ is observable}\} \tag{17}$$

Note that $\mathscr{S}_{n_c}^+$ is an open set.

Before we present the main results we review a lemma that is relevant to the solution of the optimal fixed-order compensation problem (see Refs. 7 and 56). This result follows from *Theorem* 6.2.5 of Ref. 84.

Lemma 1. Suppose $\hat{Q} \in \mathbb{R}^{n\times n}$ and $\hat{P} \in \mathbb{R}^{n\times n}$ are symmetric and nonnegative-definite and assume rank $\hat{Q}\hat{P} = n_c$. Then the following statements hold:

(*i*) There exists invertible $W \in \mathbb{R}^{n\times n}$ such that $W^{-1}\hat{Q}W^{-T}$ and $W^T\hat{P}W$ are both diagonal.

(*ii*) $\hat{Q}\hat{P}$ is diagonalizable and has nonnegative eigenvalues.

(*iii*) The $n \times n$ matrix

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{18}$$

where $(\ )^{\#}$ denotes the group or Drazin generalized inverse,[85] is idempotent; i.e., τ is an oblique projection, and

$$\operatorname{rank} \tau = n_c \tag{19}$$

Furthermore, $\tau_\perp \triangleq I_n - \tau$ satisfies

$$\operatorname{rank} \tau_\perp = n - n_c \tag{20}$$

(*iv*) Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n_c} > 0$ denote the positive eigenvalues of $\hat{Q}\hat{P}$, and let invertible $W \in \mathbb{R}^{n \times n}$ be such that

$$\hat{Q}\hat{P} = W \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{r_c} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} W^{-1} \tag{21}$$

and define the ith eigenprojection $\Pi_i[\hat{Q}\hat{P}]$ of $\hat{Q}\hat{P}$ by

$$\Pi_i[\hat{Q}\hat{P}] \triangleq W e_i e_i^T W^{-1} \tag{22}$$

where e_i is the ith column of I_n. Then

$$\hat{Q}\hat{P} = \sum_{i=1}^{n_c} \sigma_i \Pi_i[\hat{Q}\hat{P}] \tag{23}$$

and

$$\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}\hat{P}] = W \begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \tag{24}$$

(*v*) There exist $G, \Gamma \in \mathbb{R}^{n_c \times n}$ and nonsingular $M \in \mathbb{R}^{n_c \times n_c}$ such that

$$\hat{Q}\hat{P} = G^T M \Gamma \tag{25}$$

$$\Gamma G^T = I_{n_c} \tag{26}$$

(*vi*) If G, Γ, and M are as in (*v*), then

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} M = n_c \tag{27}$$

$$(\hat{Q}\hat{P})^{\#} = G^T M^{-1} \Gamma \tag{28}$$

$$\tau = G^T \Gamma \tag{29}$$

$$\tau G^T = G^T, \qquad \Gamma\tau = \Gamma \tag{30}$$

(*vii*) The matrices G, Γ, and M given in (*v*) are unique except for a change of basis in $\mathbb{R}^{n_c}$; i.e., if G', Γ', and M' also satisfy property (*v*), then there exists nonsingular $S \in \mathbb{R}^{n_c \times n_c}$ such that $G' = S^T G$, $\Gamma' = S^{-1}\Gamma$, and $M' = S^{-1}MS$. Furthermore, all such M are diagonalizable with positive eigenvalues.

(*viii*) If G, Γ, and M are as in (*v*), then M is given by

$$M = \Gamma \hat{Q} \Gamma^T G \hat{P} G^T \tag{31}$$

and the $n_c \times n_c$ matrices $\Gamma\hat{Q}\Gamma^T$ and $G\hat{P}G^T$ are positive-definite.

(*ix*) If $\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c$, then there exists a (nonunique) invertible $W \in \mathbb{R}^{n \times n}$ such that

$$\hat{Q} = W \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} W^T \tag{32}$$

$$\hat{P} = W^{-T} \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \tag{33}$$

where $\Lambda \in \mathbb{R}^{n_c \times n_c}$ and $\Omega \in \mathbb{R}^{n_c \times n_c}$ are diagonal and nonsingular. In addition,

$$\hat{Q} = \tau\hat{Q} = \hat{Q}\tau^T = \tau\hat{Q}\tau^T \tag{34}$$

$$\hat{P} = \tau^T\hat{P} = \hat{P}\tau = \tau^T\hat{P}\tau \tag{35}$$

Remark 1. The transformation W in statement (*ix*) meets the requirements of statement (*iv*).

Definition 1. A triple (G, M, Γ) satisfying statement (*v*) of *Lemma* 1 is called a "projective factorization" of $\hat{Q}\hat{P}$.

To optimize Eq. (14) subject to the constraint (11), form the Lagrangian

$$\mathscr{L}(A_c, B_c, C_c, \tilde{Q}, \tilde{P}, \lambda_0) \triangleq \operatorname{tr}[\lambda_0 \tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}] \tag{36}$$

where the Lagrange multipliers, scalar $\lambda_0 \geq 0$ and $\tilde{P} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, are not both zero. It can be shown[7] that without loss of generality we can set $\lambda_0 = 1$. The stationary conditions are then given by

$$\frac{\partial \mathscr{L}}{\partial \tilde{P}} = 0, \qquad \frac{\partial \mathscr{L}}{\partial \tilde{Q}} = 0 \tag{37}$$

$$\frac{\partial \mathscr{L}}{\partial A_c} = 0, \qquad \frac{\partial \mathscr{L}}{\partial B_c} = 0, \qquad \frac{\partial \mathscr{L}}{\partial C_c} = 0 \tag{38}$$

Definition 2. A compensator (A_c, B_c, C_c) is an *extremal* of the optimal fixed-order dynamic compensation problem if it satisfies the stationary conditions [Eqs. (37) and (38)].

Definition 3. A compensator (A_c, B_c, C_c) is an *admissible extremal* of the optimal fixed-order dynamic compensation problem if it is an extremal and is also in $\mathscr{S}_{n_c}^+$.

Remark 2. The optimal (admissible) fixed-order dynamic compensator (if it exists) can be found by computing all admissible extremals.

We can now state the basic result of optimal projection theory, namely, the necessary conditions for characterizing admissible extremals of the optimal fixed-order dynamic compensation problem. For convenience define

$$\Sigma \triangleq BR_2^{-1}B^T, \qquad \bar{\Sigma} \triangleq C^T V_2^{-1} C \tag{39}$$

Theorem 1. Suppose (A_c,B_c,C_c) is an admissible extremal of the fixed-order dynamic compensation problem. Then there exist nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ such that A_c, B_c, and C_c are given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^T \tag{40}$$

$$B_c = \Gamma Q C^T V_2^{-1} \tag{41}$$

$$C_c = -R_2^{-1} B^T P G^T \tag{42}$$

for some projective factorization (G,M,Γ) of $\hat{Q}\hat{P}$ and such that the following conditions are satisfied:

$$0 = AQ + QA^T + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{43}$$

$$0 = A^T P + PA + R_1 - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp \tag{44}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{45}$$

$$0 = (A - Q\bar{\Sigma})^T\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp \tag{46}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{47}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{48}$$

Furthermore, the extremal cost is given by

$$J(A_c,B_c,C_c) = \operatorname{tr}[QR_1 + P(Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)] \tag{49}$$

or, equivalently,

$$J(A_c,B_c,C_c) = \operatorname{tr}[PV_1 + Q(P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp)] \tag{50}$$

Remark 3. Notice that in the full-order case $n_c = n$ it follows that $\tau = I_n$; hence, without loss of generality one can choose $G = \Gamma = I_n$. In this case Eqs. (43) and (44) reduce to the standard observer and regulator Riccati equations, and Eqs. (40–42) yield the usual LQG expressions. Finally, in the full-order case $\tau_\perp = 0$, and Eqs. (49) and (50) reduce to Eq. (5.77) of Ref. 62.

Remark 4. The alternative projective factorization $(S^T G, S^{-1}MS, S^{-1}\Gamma)$ of $\hat{Q}\hat{P}$ corresponds to a change of basis of the compensator, i.e., replacing (A_c,B_c,C_c) by $(S^{-1}A_cS, S^{-1}B_c, C_cS)$. Hence, all compensator realizations equivalent to Eqs. (40–42) are effectively characterized by the optimal projection equations.

Remark 5. The plant state covariance Q_1, the controller state covariance Q_2, and the cross covariance of the plant and controller states Q_{12} are given by

$$Q_1 = Q + \hat{Q}, \qquad Q_{12} = \hat{Q}\Gamma^T, \qquad Q_2 = \Gamma\hat{Q}\Gamma^T \tag{51}$$

Similarly, the corresponding partitions of $\tilde{P}$ are given by

$$P_1 = P + \hat{P}, \qquad P_{12} = -\hat{P}G^T, \qquad P_2 = G\hat{P}G^T \tag{52}$$

The optimal projection equations [Eqs. (43–46)] are coupled by the terms $\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T$ and $\tau_\perp^T P\Sigma P\tau_\perp$. If these coupling terms were not present, then Eqs. (43) and (44) could be solved independently of Eqs. (45) and (46). The "strength" of these coupling terms is dependent on R_2 and V_2, which essentially govern regulator and estimator authority relative to the state weighting R_1 and plant disturbance intensity V_1. Hence, if $\|R_1\| \gg \|R_2\|$ and $\|V_1\| \gg \|V_2\|$, then one would expect the terms $\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T$ and $\tau_\perp^T P\Sigma P\tau_\perp$ to lead to strong coupling and hence play a significant role in determining the optimal compensator gains. The strength of these coupling terms is also affected by the choice of n_c. If, for example, $n_c \approx n$, then rank $\tau_\perp = n - n_c \approx 0$ so that $\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T$ is reduced relative to $Q\bar{\Sigma}Q$. However, if $n_c \ll n$, then rank $\tau_\perp = n - n_c \approx n$ so that the term $\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T$ is comparable to $Q\bar{\Sigma}Q$. Note also that the plus sign associated with this term shows that it counteracts the usual quadratic term $Q\bar{\Sigma}Q$ in accordance with the excluded portion of the compensator.

The next result is a restatement of *Theorem* 1 in the form of sufficient conditions. Furthermore, conditions are given to guarantee closed-loop stability.

Theorem 2. Suppose there exist nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ satisfying Eqs. (43–48) and let A_c, B_c, and C_c be given by Eqs. (40–42). Then the compensator (A_c,B_c,C_c) is an extremal of the optimal fixed-order dynamic compensation problem. Furthermore, the following are equivalent:

$$\tilde{A} \text{ is stable} \tag{53}$$

$$(\tilde{A},\tilde{D}) \text{ is stabilizable} \tag{54}$$

$$(\tilde{A},\tilde{R}^{1/2}) \text{ is detectable} \tag{55}$$

In addition,

$$(A_c,B_c) \text{ is controllable} \Leftrightarrow A_c + B_cCG^T \text{ is stable} \tag{56}$$

$$(A_c,C_c) \text{ is observable} \Leftrightarrow A_c - \Gamma BC_c \text{ is stable} \tag{57}$$

Proof. That the compensator (A_c,B_c,C_c) is an extremal follows immediately from the proof of *Theorem* 1 presented in Refs. 7 and 56. It is also shown in Refs. 7 and 56 that, if nonnegative-definite Q, P, $\hat{Q}$, and $\hat{P}$ satisfy Eqs. (43–48) and the compensator (A_c,B_c,C_c) is given by Eqs. (40–42), then (independent of the stability of $\tilde{A}$), there exist $(n+n_c)\times(n+n_c)$ real matrices $\tilde{P}$ and $\tilde{Q}$ satisfying Eqs. (11) and (12) and having partitioned forms [Eqs. (15) and (16)] with the partitions satisfying Eqs. (51) and (52). It then follows that $\tilde{Q}$ and $\tilde{P}$ can be expressed as

$$\tilde{Q}=\begin{bmatrix} Q & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix} I_n\\ \Gamma\end{bmatrix}\hat{Q}[I_n \quad \Gamma^T] \tag{58}$$

$$\tilde{P}=\begin{bmatrix} P & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix} -I_n\\ G\end{bmatrix}\hat{P}[-I_n \quad G^T] \tag{59}$$

Thus,

$$\tilde{Q}\geqslant 0, \qquad \tilde{P}\geqslant 0 \tag{60}$$

Obviously Eq. (53) implies Eqs. (54) and (55). To show that Eqs. (55) and (56) imply Eq. (53), use Eqs. (11) and (12) and *Lemma* 12.2 of Ref. 86.

Next it follows from (*vii*) of *Lemma* 1 that the (2,2) block Q_2 of $\tilde{Q}$ satisfies

$$Q_2=\Gamma\hat{Q}\Gamma^T>0 \tag{61}$$

and it is shown in Ref. 7 that Q_2 satisfies

$$0=(A_c+B_cCG^T)Q_2+Q_2(A_c+B_cCG^T)^T+B_cV_2B_c^T \tag{62}$$

The equivalence [Eq. (56)] then follows from Eqs. (61), (62), and *Lemmas* 2.1 and 12.2 of Ref. 86. The equivalence [Eq. (57)] follows in similar fashion by noting that the (2,2) block P_2 of $\tilde{P}$ satisfies

$$P_2>0 \tag{63}$$

and

$$0=(A_c-\Gamma BC_c)^TP_2+P_2(A_c-\Gamma BC_c)+C_c^TR_2C_c \tag{64}$$

Theorem 1 states conditions that all *admissible* extremals must satisfy. *Theorem* 2 includes a converse result which shows that each compensator (A_c,B_c,C_c) that satisfies the conditions of *Theorem* 1 is an extremal. Since it is possible that multiple extrema exist, the optimal projection equations may have multiple solutions. However, it is now conjectured that in many practical situations only one solution exists to the optimal projection equations.

Numerical Solution of the Optimal Projection Equations

Despite the analytical insights gained from the optimal projection equations, this approach to controller design would be little more than a theoretical curiosity without concrete, practical benefits. That such benefits are forthcoming will be demonstrated in the next section, where comparisons are made to suboptimal controller reduction methods. However, before doing so we will provide a brief overview of numerical algorithms for solving the optimal projection equations.

Essentially, two distinct approaches have been developed for solving the optimal projection equations: iterative refinement of the projection and homotopic continuation. We shall make no attempt to provide an in-depth description of these algorithms since to do so would require a lengthy development. Rather, a high-level summary will be given to convey the nature of the approaches.

The iterative refinement method, as its name implies, attempts to satisfy the four coupled equations by progressively refining the projection matrix until a solution is achieved.[4,6,11] The basic idea is to initialize $\tau=I_n$ so that the starting solution is the full-order LQG solution. Once $\hat{Q}$ and $\hat{P}$ are computed, the key step is to select τ to be a sum of eigenprojections of $\hat{Q}\hat{P}$ [see Eq. (22)]. Note that at this stage $\hat{Q}$ and $\hat{P}$ will generally have full rank so that the rank conditions [Eq. (47)] are not satisfied. There are essentially two methods for selecting eigenprojections: balancing or component cost analysis. Balancing selects the eigenprojections of $\hat{Q}\hat{P}$ in accordance with the largest eigenvalues of $\hat{Q}\hat{P}$. Roughly speaking, this choice truncates from the compensator those states that are least controllable and observable. The second approach to eigenprojection selection is via component cost analysis. This approach is somewhat more relevant to the optimization problem since it deletes those eigenprojections that contribute most greatly to the cost. Pertinent references to both balancing and component cost analysis will be given in the next section in connection with controller reduction methods.

The second numerical approach to solving the optimal projection equations is based on homotopic continuation methods. To illustrate such methods, suppose we wish to solve $f(x)=0$ where $f:\mathbb{R}^n\to\mathbb{R}^n$. The idea is to replace this problem with a simpler problem whose solution is known. For example, let $x_0\in\mathbb{R}^n$ be arbitrary, let $\alpha\in[0,1]$, and define the homotopy function

$$H(x(\alpha),\alpha)\triangleq\alpha f(x(\alpha))+(1-\alpha)[f(x(\alpha))-f(x_0)] \tag{66}$$

For $\alpha=0$ the problem $H(x(0),0)=0$ has a known solution, namely, $x(0)=x_0$. However, for $\alpha=1$, solutions to the problem $H(x(1),1)=0$ are precisely solutions to the original problem $f(x)=0$ with $x=x(1)$. Once a homotopy function has been chosen the objective is to follow a path from the initial solution of $H(x(0),0)=0$ to the desired solution of $H(x(1),1)=0$. This can be done by integrating Davidenko's equation

$$\frac{\mathrm{d}x(\alpha)}{\mathrm{d}\alpha}=[H_x(x(\alpha),\alpha)]^{-1}H_\alpha(x(\alpha),\alpha), \qquad \alpha\in[0,1] \tag{67}$$

with the initial condition given by the known solution $x(0)$ of $H(x(0),0)=0$. An alternative but related approach called a "discrete homotopy" involves defining a sequence of points

$$0=\alpha_0<\alpha_1<\ldots<\alpha_N=1$$

and solving $H(x(\alpha_{k+1}),\alpha_{k+1})=0$ given $x(\alpha_k)$ using a local iteration scheme. Although this approach avoids direct integration of Davidenko's equation, it does, nevertheless, follow the homotopy path from $x(0)$ to $x(1)$. References 88–101 describe the theoretical basis for homotopic continuation as well as a variety of applications.

Below we briefly describe a homotopy algorithm for solving the optimal projection equations developed in Refs. 10 and 13. This algorithm uses both discrete homotopy and continuous homotopy techniques. The overall algorithm is a discrete homotopy algorithm. However, the local iteration scheme that advances the discrete homotopy uses continuous homotopies to solve certain sets of nonlinear algebraic equations that are subsets of the optimal projection equations. We begin by introducing some additional notation and definitions.

For the homotopy parameter $\alpha\in[0,1]$ define

$$A(\alpha)\triangleq A_0+f_1(\alpha)[A-A_0] \tag{68}$$

$$B(\alpha)\triangleq B_0+f_2(\alpha)[B-B_0] \tag{69}$$

$$C(\alpha)\triangleq C_0+f_3(\alpha)[C-C_0] \tag{70}$$

$$R_1(\alpha)\triangleq (R_1)_0+f_4(\alpha)[R_1-(R_1)_0] \tag{71}$$

$$R_2^{-1}(\alpha)\triangleq (R_2^{-1})_0+f_5(\alpha)[R_2^{-1}-(R_2^{-1})_0] \tag{72}$$

$$V_1(\alpha)\triangleq (V_1)_0+f_6(\alpha)[V_1-(V_1)_0] \tag{73}$$

$$V_2^{-1}(\alpha)\triangleq (V_2^{-1})_0+f_7(\alpha)[V_2^{-1}-(V_2^{-1})_0] \tag{74}$$

$$\Sigma(\alpha)\triangleq B(\alpha)R_2^{-1}(\alpha)B^T(\alpha) \tag{75}$$

$$\boldsymbol{\Sigma}(\alpha)\triangleq C^T(\alpha)V_2^{-1}(\alpha)C(\alpha) \tag{76}$$

where the functions $f_j(\ )$ are monotonically increasing functions on $[0,1]$ satisfying

$$f_j(0)=0 \quad \text{and} \quad f_j(1)=1, \quad j=1,2,\ldots,7 \tag{77}$$

and the initial matrices [i.e., the matrices expressed by the notation $(\ )_0$] are chosen according to the following general guidelines:

1) A_0, B_0, C_0, $(R_1)_0$, $(R_2^{-1})_0$, $(V_1)_0$, and $(V_2^{-1})_0$ should be chosen to yield a low authority controller;

2) V_0, C_0, $(R_2^{-1})_0$, and $(V_2^{-1})_0$ should also be chosen such that, for each $\alpha\in[0,1]$,

$$\mathcal{R}(\Sigma(\alpha))=\mathcal{R}(\Sigma), \qquad \mathcal{R}(\boldsymbol{\Sigma}(\alpha))=\mathcal{R}(\boldsymbol{\Sigma}) \tag{78}$$

where $\mathcal{R}(\ )$ denotes range.

Now, for some positive integer N let $\{\alpha_k\}_{k=0}^N$ be a sequence with the property

$$0=\alpha_0<\alpha_1<\ldots<\alpha_{N-1}<\alpha_N=1 \tag{79}$$

Equations (68–77) and (79) will be used to define a discrete homotopy that is the basis of the homotopy algorithm for solving the optimal projection equations.

Remark 6. The choice $f_j(\alpha)=\alpha$ for $j\in\{1,2,\ldots,7\}$ gives monotonically increasing functions satisfying the conditions (77).

For $i\in\{0,1,2,\ldots\}$ we will subsequently consider the equations

$$\begin{aligned}0&=A(\alpha)Q^{(i+1)}+Q^{(i+1)}A^T(\alpha)+V_1(\alpha)-Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)Q^{(i+1)}\\&\quad+\tau_\perp^{(i)}Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)Q^{(i+1)}(\tau_\perp^{(i)})^T\end{aligned} \tag{80}$$

$$\begin{aligned}0&=A^T(\alpha)P^{(i+1)}+P^{(i+1)}A(\alpha)+R_1(\alpha)-P^{(i+1)}\Sigma(\alpha)P^{(i+1)}\\&\quad+(\tau_\perp^{(i)})^TP^{(i+1)}\Sigma(\alpha)P^{(i+1)}\tau_\perp^{(i)}\end{aligned} \tag{81}$$

$$\begin{aligned}0&=[A(\alpha)-\Sigma(\alpha)P^{(i+1)}]\hat{Q}^{(i+1)}+\hat{Q}^{(i+1)}[A(\alpha)-\Sigma(\alpha)P^{(i+1)}]^T\\&\quad+Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)Q^{(i+1)}-\tau_\perp^{(i+1)}Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)Q^{(i+1)}(\tau_\perp^{(i+1)})^T\end{aligned} \tag{82}$$

$$\begin{aligned}0&=[A(\alpha)-Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)]^T\hat{P}^{(i+1)}+\hat{P}^{(i+1)}[A(\alpha)-Q^{(i+1)}\boldsymbol{\Sigma}(\alpha)]\\&\quad+P^{(i+1)}\Sigma(\alpha)P^{(i+1)}-(\tau_\perp^{(i+1)})^TP^{(i+1)}\Sigma(\alpha)P^{(i+1)}\tau_\perp^{(i+1)}\end{aligned} \tag{83}$$

$$\operatorname{rank}\hat{Q}^{(i+1)}=\operatorname{rank}\hat{P}^{(i+1)}=\operatorname{rank}\hat{Q}^{(i+1)}\hat{P}^{(i+1)}=n_c \tag{84}$$

$$\tau^{(i+1)}=\hat{Q}^{(i+1)}\hat{P}^{(i+1)}[\hat{Q}^{(i+1)}\hat{P}^{(i+1)}]^{\#} \tag{85}$$

Also, define

$$\begin{aligned}\Delta_Q^{(i)}(\alpha)&\triangleq A(\alpha)Q^{(i)}+Q^{(i)}A^T(\alpha)+V_1(\alpha)-Q^{(i)}\boldsymbol{\Sigma}(\alpha)Q^{(i)}\\&\quad+\tau_\perp^{(i)}Q^{(i)}\boldsymbol{\Sigma}(\alpha)Q^{(i)}(\tau_\perp^{(i)})^T\end{aligned} \tag{86}$$

$$\Delta_P^{(i)}(\alpha) \triangleq A^T(\alpha)P^{(i)} + P^{(i)}A(\alpha) + R_1(\alpha) - P^{(i)}\Sigma(\alpha)P^{(i)} + (\tau_\perp^{(i)})^T P^{(i)}\Sigma(\alpha)P^{(i)}\tau_\perp^{(i)} \quad (87)$$

$$\Delta_{\hat{Q}}^{(i)}(\alpha) \triangleq [A(\alpha) - \Sigma(\alpha)P^{(i)}]\hat{Q}^{(i)} + \hat{Q}^{(i)}[A(\alpha) - \Sigma(\alpha)P^{(i)}]^T + Q^{(i)}\Sigma(\alpha)Q^{(i)} - \tau_\perp^{(i)}Q^{(i)}\Sigma(\alpha)Q^{(i)}(\tau_\perp^{(i)})^T \quad (88)$$

$$\Delta_{\hat{P}}^{(i)}(\alpha) \triangleq [A(\alpha) - Q^{(i)}\Sigma(\alpha)]^T\hat{P}^{(i)} + \hat{P}^{(i)}[A(\alpha) - Q^{(i)}\Sigma(\alpha)] + P^{(i)}\Sigma(\alpha)P^{(i)} - (\tau_\perp^{(i)})^T P^{(i)}\Sigma(\alpha)P^{(i)}\tau_\perp^{(i)} \quad (89)$$

Equations (80) and (81) are identical in form to Eqs. (43) and (44), and Eqs. (82–85) are identical in form to Eqs. (45–48).

In the homotopy algorithm the matrix functions $\Delta_Q^{(i)}(\alpha)$, $\Delta_P^{(i)}(\alpha)$, $\Delta_{\hat{Q}}^{(i)}(\alpha)$, and $\Delta_{\hat{P}}^{(i)}(\alpha)$, given by Eqs. (86), (87), (88), and (89), respectively, are considered equation errors. The normalized equation errors norms are defined by

$$e_Q^{(i)}(\alpha) \triangleq \|\Delta_Q^{(i)}(\alpha)\|/\|V_1(\alpha)\| \quad (90)$$

$$e_P^{(i)}(\alpha) \triangleq \|\Delta_P^{(i)}(\alpha)\|/\|R_1(\alpha)\| \quad (91)$$

$$e_{\hat{Q}}^{(i)}(\alpha) \triangleq \|\Delta_{\hat{Q}}^{(i)}(\alpha)\|/\|Q^{(i)}\Sigma(\alpha)Q^{(i)}\| \quad (92)$$

$$e_{\hat{P}}^{(i)}(\alpha) \triangleq \|\Delta_{\hat{P}}^{(i)}(\alpha)\|/\|P^{(i)}\Sigma(\alpha)P^{(i)}\| \quad (93)$$

where $\| \ \|$ denotes the maximum absolute value of the matrix elements. The maximum normalized error norm $e_{\max}^{(i)}(\alpha)$ is defined by

$$e_{\max}^{(i)}(\alpha) \triangleq \max\{e_Q^{(i)}(\alpha), e_P^{(i)}(\alpha), e_{\hat{Q}}^{(i)}(\alpha), e_{\hat{P}}^{(i)}(\alpha)\} \quad (94)$$

Figure 1 presents a general flowchart of the homotopy algorithm. The outer (k) loop corresponds to a discrete homotopy that is advanced by the iteration scheme described by the inner (i) loop. The inner loop requires the solution of Eqs. (80) and (81) and then the simultaneous solution of Eqs. (82–85). These solutions can be obtained by using continuous homotopies.[13] The sequence $\{\epsilon_k\}_{k=0}^N$ of "small" positive numbers determines how closely the algorithm actually tracks the homotopy curve. The details of this algorithm are presented in Ref. 13.

We now present some remarks concerning the choice of matrices $(\ )_0$ used to initialize the homotopy algorithm. In practice we have found it sufficient to choose $B_0 = B$ and $C_0 = C$ [i.e., $B(\alpha) = B$ and $C(\alpha) = C$ for $\alpha \in [0,1]$], whereas for systems with lightly damped modes we have sometimes used $A_0 = A - \sigma D$ for some positive diagonal D and $\sigma \geqslant 0$. The

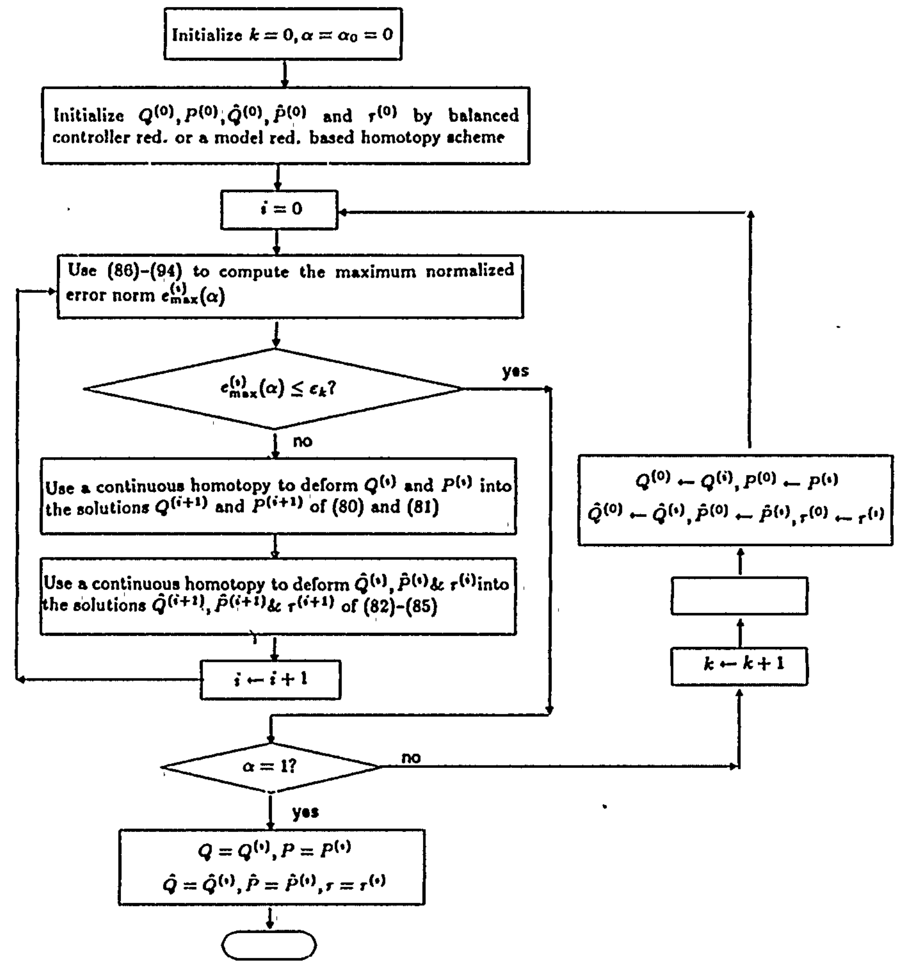

Fig. 1 The homotopy algorithm for solving the optimal projection equations uses both discrete homotopy and continuous homotopy techniques.

initial weighting matrices are usually selected by one of the following two schemes:

Scheme 1. Choose $(R_2^{-1})_0 = \rho_R R_2^{-1}$ and $(V_2^{-1})_0 = \rho_V V_2^{-1}$ for some $\rho_R < 1$ and $\rho_V < 1$, and let $(R_1)_0 = R_1$ and $(V_1)_0 = V_1$ (i.e., $R_1(\alpha) = R_1$ and $V_1(\alpha) = V_1$ for $\alpha \in [0,1]$).

Scheme 2. Choose $(R_1)_0 = \gamma_R R_1$ and $(V_1)_0 = \gamma_V V_1$ for some $\gamma_R < 1$ and $\gamma_V < 1$, and let $(R_2^{-1})_0 = R_2^{-1}$ and $(V_2^{-1})_0 = V_2^{-1}$ (i.e., $R_2^{-1}(\alpha) = R_2^{-1}$ and $V_2^{-1}(\alpha) = V_2^{-1}$ for $\alpha \in [0,1]$).

Relationships Between the Optimal Projection Approach and Model/Controller Reduction Techniques

Because of widespread interest in designing low-order controllers for high-order systems, numerous reduction methods have been proposed.[102–146] These methods usually involve either plant approximation prior to controller design or controller reduction once a high-order controller has been obtained (Fig. 2). The purpose of this section is to shed some light on the relationship between optimal projection theory and controller reduction methods.

In many applications, such as finite-element modeling of flexible structures, the models possess far too many degrees of freedom to be tractable for realistic control design. Hence, at least some form of model reduction is usually required to reduce the plant model to a tractable, perhaps "Riccati solvable" dimension. However, it is important to stress that reduction of the open-loop plant model will generally result in the loss of modeling information. Hence, it is desirable that this reduction step be limited as much as possible. Once a model of tractable order is obtained, it is still often necessary to design a controller of significantly lower order than the model. Controller reduction methods perform this step by first designing a high-order controller and then applying model reduction techniques to the controller itself. The principal drawback of this two-step approach is that guarantees of stability, optimality, robustness, and other desirable design objectives are often lacking.

However, it is possible to argue that such *indirect* controller reduction methods are still useful in practice. For example, reduction methods may be computationally simpler than direct methods and may give the user insight into the reduction procedure. However, such features are not surprising, since, as will be seen, at least some controller reduction methods are, in fact, suboptimal approximations to the optimal projection equations.

To demonstrate these connections, we shall consider three indirect methods of reduced-order controller design (for details see Ref. 6). The first method is the component cost analysis (CCA) approach:[134] In the notation of the section containing a brief overview of basic optimal projection theory, CCA can be summarized by the equations

$$0 = AQ + QA^T + V_1 - Q\Sigma Q \tag{95}$$

$$0 = A^TP + PA + R_1 - P\Sigma P \tag{96}$$

$$0 = (A - \Sigma P)^T\hat{Q} + \hat{Q}(A - \Sigma P) + Q\Sigma Q \tag{97}$$

$$\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}P\Sigma P] \tag{98}$$

As in optimal projection theory, the reduced-order controller gains are

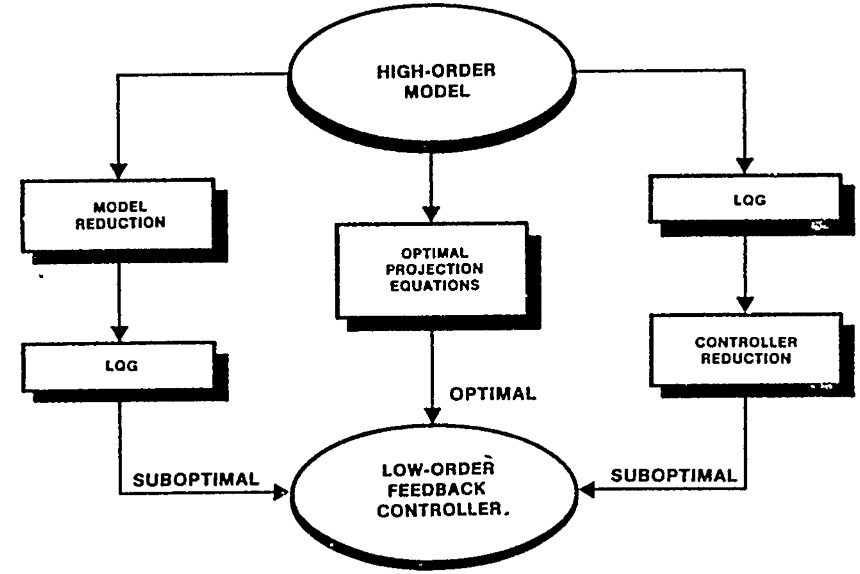

Fig. 2 There are three principal approaches to reduced-order controller design: model reduction followed by low-order controller design, high-order controller design followed by controller reduction, and direct optimization of the fixed-order compensator gains.

formed by factoring the projection τ by means of $\tau = G^T\Gamma$, $\Gamma G^T = I_{n_c}$, and forming the gains [Eqs. (40–42)].

Within this common framework it can immediately be seen that CCA lacks the fourth equation for $\hat{P}$ present in optimal projection theory; furthermore, the remaining three equations lack coupling terms involving τ. Note also that CCA forms the projection by using $P\Sigma P$ in place of $\hat{P}$. This is not altogether unreasonable since $P\Sigma P$ is essentially the nonhomogeneous driving term in the $\hat{P}$ equation [Eq. (46)].

Next we turn to the balanced controller reduction algorithm (BCRA), which corresponds to the application of internal balancing model reduction[122] to an LQG controller. Again, in the previous notation, this method can be written as

$$0 = AQ + QA^T + V_1 - Q\Sigma Q \tag{99}$$

$$0 = A^TP + PA + R_1 - P\Sigma P \tag{100}$$

$$0 = (A - Q\Sigma - \Sigma P)\hat{Q} + \hat{Q}(A - Q\Sigma - \Sigma P)^T + Q\Sigma Q \tag{101}$$

$$0 = (A - Q\Sigma - \Sigma P)^T\hat{P} + \hat{P}(A - Q\Sigma - \Sigma P) + P\Sigma P \tag{102}$$

$$\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}\hat{P}] \tag{103}$$

where τ is again factored as $\tau = G^T\Gamma$, $\Gamma G^T = I_{n_c}$, and the reduced-order controller gains are given by Eqs. (40–42). Note that this method, in contrast to CCA, involves four equations as in optimal projection theory. Again, however, the coupling terms present in optimal projection theory are absent. Note also that $\hat{Q}$ and $\hat{P}$ in BCRA are essentially the controllability and observability Gramians for the LQG compensator since this coefficient is $A + BC_c - B_cC$. In contrast, the coefficients of $\hat{Q}$ and $\hat{P}$ in optimal projection theory are $A - B_cC$ and $A + BC_c$, respectively, which correspond to estimator and regulator dynamics in LQG theory.

Finally, we consider the balanced controller reduction algorithm-modified (BCRAM).[140] This method corrects an obvious drawback associated with BCRA in that the coefficient $A - Q\Sigma - \Sigma P$ of $\hat{Q}$ and $\hat{P}$ in Eqs. (101) and (102) may not be asymptotically stable. In our notation this method becomes

$$0 = AQ + QA^T + V_1 - Q\Sigma Q \tag{104}$$

$$0 = A^TP + PA + R_1 - P\Sigma P \tag{105}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\Sigma Q \tag{106}$$

$$0 = (A - Q\Sigma)^T\hat{P} + \hat{P}(A - Q\Sigma) + P\Sigma P \tag{107}$$

$$\tau = \sum_{i=1}^{n_c} \Pi_i[\hat{Q}\hat{P}] \tag{108}$$

with the usual factorization of τ and gain expressions. Note that the coefficients of $\hat{Q}$ and $\hat{P}$, namely, $A - \Sigma P$ and $A - Q\Sigma$, are now guaranteed to be asymptotically stable since they correspond to regulator and observer dynamics of LQG theory. Clearly, this method is most similar to optimal projection theory since it lacks only the coupling terms. Indeed, if the optimal projection equations are solved by means of an iterative substitution method with $\tau = I_n$ as the starting projection (see the previous section), then the first iteration of this algorithm corresponds to BCRAM. Of course, the result of this first iteration cannot be expected to be optimal, or even stabilizing. However, further iterations can lead to progressive refinement of this initial BCRAM design.

Hence, we have seen that three controller reduction methods can be viewed as suboptimal approximations to the optimal projection equations. Each method involves the construction of a projection matrix whose factorization leads to a projection of the LQG controller. However, it should be stressed that the optimal projection controller is not a projection of the full-order controller but rather is obtained by solving a system of four matrix equations with additional coupling terms not present in the suboptimal methods.

An alternative indirect approach to controller reduction is the LQG-balancing method.[124,133] This method was discussed in Ref. 140, where it was noted that it may fail to yield a minimal order compensator when one exists. Although this property was not, in fact, claimed in Refs. 124 and 133, this fact serves as a further illustration of the drawbacks inherent in indirect methods. For optimal projection theory, *Theorem* 2 shows that the rank conditions [Eq. (47)] on $\hat{Q}$ and $\hat{P}$ are directly related to the minimality of (A_c, B_c, C_c). To illustrate this point, consider the example of Ref. 140:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [1 \quad -1]$$

$$R_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R_2 = 1, \qquad V_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad V_2 = 1$$

Then the LQG controller is found to be

$$A_c = \begin{bmatrix} -3 & 0 \\ 5 & -4 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \qquad C_c = [-2 \quad 0]$$

However, this realization is not observable, and a minimal realization is given by

$$A_c = -3, \qquad B_c = 1, \qquad C_c = -2$$

This nonminimality is not surprising since letting $n_c = 2$ and $\tau = I_2$ yields

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} \frac{17}{4} & -8 \\ -8 & \frac{65}{4} \end{bmatrix}, \qquad \hat{P} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Note that since rank $\hat{P} = 1$ the rank conditions [Eq. (47)] are not satisfied. However, if $n_c = 1$, then

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\hat{Q} = \frac{17}{4}\begin{bmatrix} 1 & \beta \\ \beta & \beta^2 \end{bmatrix}, \qquad \hat{P}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix}.$$

where

$$\alpha = \frac{17}{4} + \frac{16}{17}, \qquad \beta = \frac{-32}{17}$$

Now rank $\hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = 1$ as required by Eq. (47).

Numerical Comparisons

We now consider some numerical comparisons for several additional LQG reduction methods. A convenient starting point for such comparisons is the study carried out in Ref. 144. The problem considered in Ref. 144, which originated in Ref. 147, involves multiple disks mounted on a flexible shaft with a noncollocated sensor/actuator pair. Because of noncollocation, the system has a complex pair of nonminimum phase zeros. The problem data are

$$A = \begin{bmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$R_1 = (1.0 \times 10^{-6}) E_1^T E_1,$$

$$E_1 = [0 \quad 0 \quad 0 \quad 0 \quad 0.55 \quad 11 \quad 1.32 \quad 18.0],$$

$$R_2 = 1$$

$$V_1 = q_2 BB^T, \qquad V_2 = 1$$

The parameter q_2 is a scale factor for the plant disturbance noise used to adjust controller authority. The methods compared in Ref. 144 include the following:

1) Enns: a frequency-weighted balancing technique[137];
2) Glover: an optimal Hankel norm approximation method[138];
3) Davis and Skelton: an extension of balancing to address unstable controllers[139];
4) Yousuff and Skelton: the BCRAM method discussed earlier[140]; and
5) Liu and Anderson: an approximation technique applied to the component parts of a fractional representation of the controller.[144]

Table 1 summarizes the results reported in Ref. 144 along with the results of an optimal projection study performed in Ref. 11. Clearly, all of the LQG reduction methods experience increasing difficulty as controller order decreases and controller authority increases (Table 2). This is not unexpected, since, as discussed in the overview, the coupling terms involving $\tau_\perp$ play an increasingly important role. However, these terms have no counterpart in indirect methods.

Note that, although the indirect methods failed to reliably stabilize the plant with reduced-order controllers, the optimal projection approach succeeded in every case considered. This is not surprising since *Theorem* 2 *guarantees* closed-loop stability in the presence of a generically satisfied disturbability condition. Alternatively, stability is guaranteed by the fact that the numerical solution of the optimal projection equations effectively corresponds to the construction of the Liapunov function $V(\tilde{x}) = \tilde{x}^T \tilde{P} \tilde{x}$, where $\tilde{x}$ is the closed-loop state and $\tilde{P}$ satisfies Eq. (12). In addition, optimal projection theory chooses compensator gains to minimize the quadratic cost subject to the satisfaction of the Liapunov constraint [Eq. (12)]. These observations thus clarify the guarantees inherent in optimal projection theory vs the lack thereof in indirect methods.

More recently, an additional improved indirect method involving controller canonical correlation coefficients C^4 was proposed in Ref. 145. This method was also applied to the example of Ref. 147 considered in Ref. 144 with q_2 now as large as 10^6. This new method proved to be more reliable with regard to stability even with larger values of q_2. A corresponding study in Ref. 12 yielded uniform improvement in quadratic costs—not a surprising result since optimal projection theory is based on quadratic optimality (see Table 3).

The differences between optimal projection theory and the C^4 method can be seen more clearly by considering the results for the lightly damped beam example addressed in Ref. 145. The data for this problem are

$$A = \text{block diag}_{i=1,\ldots,5} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}$$

$$\zeta = 0.005, \qquad \omega_i = i^2, \qquad i = 1, \ldots, 5$$

$$B = [0 \quad 0.9877 \quad 0 \quad -0.3090 \quad 0 \quad -0.8910 \quad 0 \quad -0.5878 \quad 0 \quad 0.7071]^T$$

$$C = [0.9877 \quad 0 \quad 0.3090 \quad 0 \quad -0.8910 \quad 0 \quad -0.5878 \quad 0 \quad 0.7071 \quad 0]$$

$$R_1 = C^T C, \qquad R_2 = \rho, \qquad V_1 = BB^T, \qquad V_2 = 0.1$$

where ρ is the design parameter. Results for both the C^4 method and optimal projection theory are shown in Fig. 3. Note that at higher authority levels the suboptimal C^4 method exhibits degraded performance

Table 1 LQG reduction methods applied to example problems to determine the stability of the resulting closed-loop system

Method	n_c	$q_2=$ 0.01	0.1	1	10	100	1000	2000
Enns	7	S	S	S	S	S	S	S
	6	S	S	S	S	S	S	S
	5	S	S	S	S	S	S	S
	4	S	S	S	S	S	S	U
	3	S	S	S	S	S	S	S
	2	S	S	S	S	U	U	U
Glover	7	S	S	S	S	S	U	S
	6	S	S	S	S	U	U	U
	5	S	S	S	S	U	U	U
	4	S	S	S	S	U	U	U
	3	S	S	U	S	U	U	U
	2	S	U	S	U	S	U	U
Davis and Skelton	7	S	U	U	S	S	S	S
	6	S	S	S	S	S	S	S
	5	S	U	S	S	S	U	U
	4	S	S	U	S	S	U	U
	3	U	U	U	U	U	U	U
	2	S	U	S	U	U	U	U
Yousuff and Skelton	7	S	S	S	S	U	U	U
	6	S	S	S	S	U	U	U
	5	S	S	S	U	U	U	U
	4	S	S	S	U	U	U	U
	3	S	U	U	U	U	U	U
	2	S	S	S	U	U	U	U
Liu and Anderson	7	S	S	S	S	S	S	U
	6	S	S	S	S	S	S	U
	5	S	S	S	S	S	S	S
	4	S	S	S	S	S	S	S
	3	S	S	S	S	S	U	U
	2	S	S	S	S	S	S	S
Optimal projection	7	S	S	S	S	S	S	S
	6	S	S	S	S	S	S	S
	5	S	S	S	S	S	S	S
	4	S	S	S	S	S	S	S
	3	S	S	S	S	S	S	S
	2	S	S	S	S	S	S	S

S, the closed-loop system is stable; U, the closed-loop system is unstable.
Only the optimal projection equations yielded stabilizing controllers in every case, even at high authority.

Table 2 Percentage totals of the controller reduction results reveal increasing instability at high authority and low controller order, whereas optimal projection theory guarantees closed-loop asymptotic stability

Method	$q_2=$ 0.01	0.1	1	10	100	1000	2000	Total % for all cases
Enns	100	100	100	100	83.3	83.3	66.7	90.5
Glover	100	83.3	83.3	83.3	33.3	0	16.7	57.1
Davis and Skelton	83.3	33.3	50.0	66.7	66.7	33.3	33.3	52.4
Yousuff and Skelton	100	83.3	83.3	33.3	0	0	0	42.9
Liu and Anderson	100	100	100	100	100	83.3	50.0	90.5
Optimal projection	100	100	100	100	100	100	100	100.0

Table 3 Optimal projection theory yielded improved performance compared to the C^4 method

q_2	Controller order	Method	State cost (rms)	Control cost (rms)	Total cost (rms)
2000	8	LQG	8.45	12.7	15.3
	4	C^4	15.6	14.7	21.0
		OP	9.28	13.3	16.2
	2	C^4	15.1	14.7	21.0
		OP	9.83	13.5	16.8
10^6	8	LQG	174	277	327
	6	C^4	214	288	359
		OP	195	288	348
	5	C^4	363	376	522
		OP	203	295	358
	4	C^4	355	355	502
		OP	209	297	362
	3	C^4	986	857	1310
		OP	214	299	368
	2	C^4	349	343	489
		OP	220	303	375

tending toward instability. However, the optimal projection designs yield monotonically improving performance at higher authority levels. Note that for $n_c = 2,4$ the optimal projection designs exhibit an upper bound on allowable controller effort. Hence, these plots do not extend indefinitely to the right as in the $n_c = 6$ and full-order cases. These results thus reveal an inherent limitation on the ability of the controller to further improve the state cost as imposed by the controller order. This is precisely the source of difficulty encountered by the C^4 method: By attempting to reduce a high-authority LQG controller, it cannot appropriately curtail the controller effort in accordance with the order-imposed bounds.

Finally, it should be noted that comparisons between direct and indirect design approaches have also been carried out experimentally, for a truss structure with numerous flexible modes. LQG truncation was compared to optimal projection theory.[148,149] The experimental results obtained confirm qualitatively the differences discussed in this section between indirect methods and optimal projection theory.

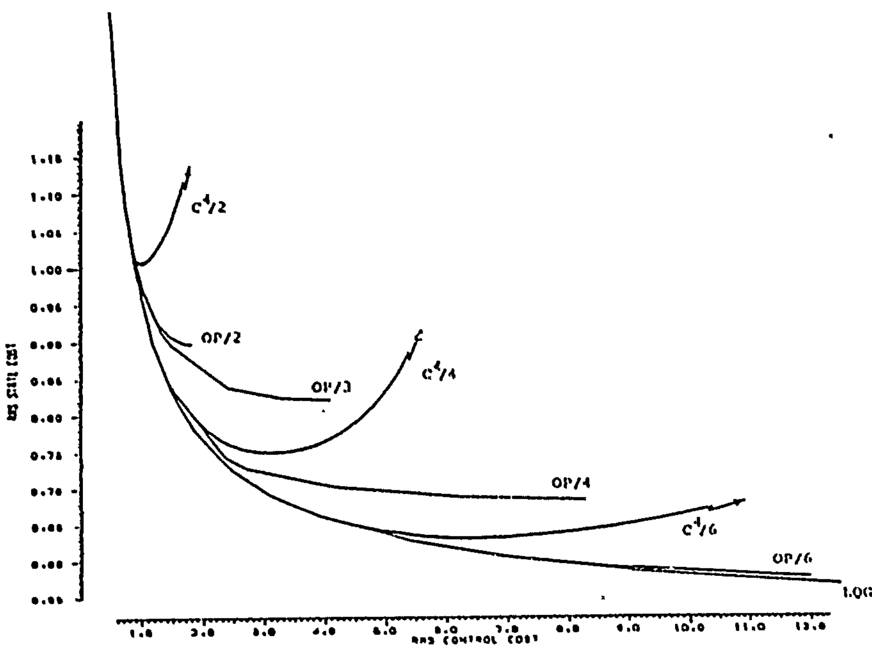

Fig. 3 A study of rms state error vs control effort reveals the tendency of C^4 controllers to destabilize at high authority, whereas optimal projection controllers exhibit monotonically improving performance.

Optimal Projection/H_∞ Theory: Frequency Domain Extensions

Although the development of LQG theory in the 1960s promised a truly multivariable design theory, it eventually became clear that LQG theory had numerous deficiencies in practical application. For example, LQG controllers lacked rudimentary forms of robustness such as gain and phase margins[150] and could be arbitrarily sensitive to parameter variations.[151,152] However, such deficiencies were not to be unexpected, since an LQG controller is constructed to optimize a single, narrowly defined performance measure, namely, a quadratic cost. Although a quadratic performance measure is physically relevant to many applications such as regulation with rms error specifications in the presence of persistent disturbances, the deficiencies of LQG design largely precluded its practical application. Hence, classical single-loop synthesis techniques (e.g., root locus, Bode, Nyquist, and Nichols) were in no danger of being dislodged by the newfangled quadratic machinery.

However, some relief was offered when it was shown that a quadratically optimal full-state feedback (LQR) controller has 6-dB/60 deg gain and phase margins.[153] However, full-state feedback controllers are only implementable in the case $C = I_n$. Thus, such regulators are inapplicable even to single-input single-output systems.

However, such restrictions notwithstanding, these LQR features suggest a remedy to the LQG defects: speed up the regulator and/or estimator dynamics to "recover" the LQR properties.[154–156] This approach is a specific application of a more general class of "loop shaping" procedures ultimately aimed at allowing performance/robustness tradeoffs.[157–159] By minimizing sensitivity at low frequency and complementary sensitivity at high frequency, it becomes possible to achieve both performance specifications and robustness to unmodeled dynamics.

However, such loop shaping procedures were predicated on a quadratic performance criterion. Thus, there were two drawbacks: 1) since the quadratic criterion concerns only the H_2 norm of the frequency response, the designer has only limited control over the loop shape; and 2) since the description of plant uncertainty is not consistent with the performance metric, it is difficult to derive robust performance bounds. Both difficulties were eventually overcome with the development of H_∞ control theory.[160–164] Since the H_∞ performance measure corresponds to worst-case frequency attenuation, it permits precise loop shaping and uncertainty characterization by means of dynamic weights.

Although H_∞ theory addressed "classical" control concerns and filled the gaps of H_2/loop shaping, it had one major drawback: it lacked the elegance of LQG synthesis via algebraic matrix Riccati equations. Until now.

Optimal projection theory was unified with H_∞ theory in Refs. 25 and 26. The technique by which this was done is quite simple and can be described as follows: Optimal projection design is based on minimizing

$$J(A_c,B_c,C_c) = \operatorname{tr} \tilde{Q}\tilde{R} \tag{109}$$

where $\tilde{Q}$ satisfies

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} \tag{110}$$

But suppose Eq. (110) is replaced by

$$0 = \tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^T + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty\mathscr{Q} + \tilde{V} \tag{111}$$

where $\gamma > 0$ is a given constant, and $\tilde{R}_\infty \geqslant 0$. Then it is easy to show that

$$\tilde{Q} \leqslant \mathscr{Q} \tag{112}$$

Thus,

$$J(A_c,B_c,C_c) \leqslant \mathscr{J}(A_c,B_c,C_c) \tag{113}$$

where

$$\mathscr{J}(A_c,B_c,C_c) \triangleq \operatorname{tr}\mathscr{Q}\tilde{R} \tag{114}$$

The motivation for replacing Eq. (110) by Eq. (111) stems from the fact that, if Eq. (111) has a solution, then

$$\|\tilde{G}(s)\|_\infty \leqslant \gamma \tag{115}$$

where

$$\tilde{G}(s) \triangleq \tilde{E}_\infty(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D}, \qquad \tilde{E}_\infty^T\tilde{E}_\infty = \tilde{R}_\infty, \qquad \tilde{D}\tilde{D}^T = \tilde{V} \tag{116}$$

The constraint (115) can then be used to enforce loop shaping, robustness, or frequency domain performance specifications.

To enforce Eq. (115), we thus need only rederive optimal projection theory with Eqs. (111) and (114) in place of Eqs. (110) and (109). In place of Eqs. (43–49) one then obtains[25,26]

$$0 = AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\Sigma Q + \tau_\perp Q\Sigma Q\tau_\perp^T \tag{117}$$

$$0 = (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^TP + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1$$
$$- S^TP\Sigma PS + \tau_\perp^TS^TP\Sigma PS\tau_\perp \tag{118}$$

$$0 = (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^T$$
$$+ \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2S^TP\Sigma PS)\hat{Q} + Q\Sigma Q - \tau_\perp Q\Sigma Q\tau_\perp^T \tag{119}$$

$$0 = (A - Q\Sigma + \gamma^{-2}QR_{1\infty})^T\hat{P} + \hat{P}(A - Q\Sigma + \gamma^{-2}QR_{1\infty})$$
$$+ S^TP\Sigma PS - \tau_\perp^TS^TP\Sigma PS\tau_\perp \tag{120}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{121}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\# \tag{122}$$

where the compensator gains are given by

$$A_c = \Gamma(A - Q\Sigma - \Sigma PS + \gamma^{-2}QR_{1\infty})G^T \tag{123}$$

$$B_c = \Gamma QC^TV_2^{-1} \tag{124}$$

$$C_c = -R_2^{-1}B^TPSG^T \tag{125}$$

the H_2 performance is bounded by

$$J(A_c,B_c,C_c) \le \operatorname{tr}[(Q + \hat{Q})R_1 + \hat{Q}S^TP\Sigma PS] \tag{126}$$

and

$$R_{1\infty} \triangleq E_{1\infty}^TE_{1\infty}, \qquad R_{2\infty} \triangleq E_{2\infty}^TE_{2\infty} = \beta^2R_2, \qquad \tilde{E} \triangleq E_{1\infty} + E_{2\infty}C_c \tag{127}$$

$$S \triangleq (I_n + \beta^2\gamma^{-2}\hat{Q}P)^{-1} \tag{128}$$

The additional terms in Eqs. (117–120) guarantee that, if these equations have a solution, then the transfer function $\tilde{G}(s)$ from disturbances $w(t)$ to H_∞ performance variables $z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t)$ has H_∞ norm bounded by γ. Note formally that, if $\gamma \to \infty$, then all terms multiplied by γ^{-2} drop out and the usual H_2 optimal projection equations are recovered. Also, note that this result addresses a mixed-norm problem, since both H_2 and H_∞ objectives are treated.

In the full-order case $n_c = n$ the four optimal projection/H_∞ equations specialize to the three equations:

$$0 = AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\Sigma Q \tag{129}$$

$$0 = (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^TP + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1 - S^TP\Sigma PS \tag{130}$$

$$0 = (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^T$$
$$+ \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2S^TP\Sigma PS)\hat{Q} + Q\Sigma Q \tag{131}$$

with gains

$$A_c = A - Q\Sigma - \Sigma PS + \gamma^{-2} Q R_{1\infty} \tag{132}$$

$$B_c = QC^T V_2^{-1} \tag{133}$$

$$C_c = -R_2^{-1} B^T PS \tag{134}$$

Specializing even further to the equalized weight case $R_1 = R_{1\infty}$, $R_2 = R_{2\infty}$, and transforming to a new variable Y_∞ yields the pair of equations:

$$0 = AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q\Sigma Q \tag{135}$$

$$0 = A^T Y_\infty + Y_\infty A + R_{1\infty} + \gamma^{-2} Y_\infty V_1 Y_\infty - Y_\infty \Sigma Y_\infty \tag{136}$$

where Q and Y_∞ satisfy

$$\rho(Q Y_\infty) < \gamma^2 \tag{137}$$

and with gains given by

$$A_c = A - Q\Sigma - \Sigma(Y_\infty^{-1} - \gamma^{-2} Q)^{-1} + \gamma^{-2} Q R_{1\infty} \tag{138}$$

$$B_c = QC^T V_2^{-1} \tag{139}$$

$$C_c = -R_{2\infty}^{-1} B^T (Y_\infty^{-1} - \gamma^{-2} Q)^{-1} \tag{140}$$

The special case of full-state feedback $u = Kx$ is given by

$$0 = A^T P + PA + R_{1\infty} + \gamma^{-2} P V_1 P - P\Sigma P \tag{141}$$

with gain

$$K = -R_{2\infty}^{-1} B^T P \tag{142}$$

The Riccati equation approach to H_∞ control can be traced to Refs. 165 and 166. Synthesis results for full-state feedback first appeared in Ref. 167 with extensions in Refs. 168 and 169. The full-order dynamic-compensation problem with equalized weights was treated in Refs. 170 and 171. We also note interesting connections with the exponential-of-quadratic cost problem,[172–174] whose solution involves essentially the same Riccati equations. Finally, connections between the H_2 cost bound $\mathcal{J}(A_c, B_c, C_c)$ and a frequency domain entropy functional are explored in Refs. 30 and 175–177.

The unification of optimal projection theory and H_∞ theory represents a potentially useful and flexible combination of approaches to controller synthesis. Clearly, although optimal projection theory alone lacks the means to meet frequency domain specifications, it brings to H_∞ theory the means to design low-order H_∞ controllers *directly*. Besides the plant order itself, low-order design is desirable within H_∞ theory to counteract the net increase in plant dimension due to the inclusion of dynamic weights for performing loop shaping and for bounding plant uncertainties.

Optimal Projection/Guaranteed Cost Theory: Parameter-Robust Extensions

H_∞ theory provides the means for guaranteeing robust stability and performance in the presence of unstructured modeling uncertainty, that is, uncertainty modeled by bounds on transfer function perturbations. However, if the uncertainty is more naturally expressed as matrix perturbations to a state space plant realization, then alternative techniques are required. In this section we discuss extensions of optimal projection theory to this problem involving Liapunov functions.

To illustrate the basis for our approach, consider the system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \qquad t \in [0, \infty) \tag{143}$$

$$z(t) = Ex(t) \tag{144}$$

where $x(t)$ is an n vector, A is an $n \times n$ matrix denoting the nominal dynamics matrix, ΔA denotes an uncertain perturbation of A belonging to a specified set $\mathcal{U}$, $Dw(t)$ is a white noise signal of intensity $V \triangleq DD^T$, and $z(t)$ is a vector of outputs. The initial condition $x(0)$ is unspecified, since it plays no role in the analysis of steady-state performance criteria.

For the system (143) the performance measure involves the steady-state second moment of the variables $z(t)$. In practice, the diagonal elements of the second moment are measures of the ability of the external disturbances $Dw(t)$ to excite specified states. In the presence of uncertainties ΔA, it is of interest to determine the *worst-case* steady-state values of the second moments of selected states. Thus, we define the scalar performance criterion

$$J(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \lim_{t \to \infty} \mathbb{E}[z^T(t) z(t)] \tag{145}$$

To evaluate Eq. (145), define the second-moment matrix

$$Q_{\Delta A}(t) \triangleq \mathbb{E}[x(t) x^T(t)] \tag{146}$$

which satisfies the Liapunov differential equation

$$\dot{Q}_{\Delta A}(t) = (A + \Delta A) Q_{\Delta A}(t) + Q_{\Delta A}(t)(A + \Delta A)^T + V \tag{147}$$

so that Eq. (145) becomes

$$J(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \lim_{t \to \infty} \operatorname{tr} Q_{\Delta A}(t) R \tag{148}$$

where $R \triangleq E^T E$. To guarantee both robust stability and performance, we consider modified algebraic Liapunov equations of the form

$$0 = AQ + QA^T + \Omega(Q) + V \tag{149}$$

where $\Omega(\ )$ is a matrix operator satisfying

$$\Delta A Q + Q \Delta A^T \leqslant \Omega(Q) \tag{150}$$

for all $\Delta A \in \mathscr{U}$ and all nonnegative-definite matrices Q. The ordering in Eq. (150) is defined with respect to the cone of nonnegative-definite matrices. Our results are based on the following robust stability and performance result. (For convenience, assume that V is positive-definite.) If there exists a positive-definite solution Q to Eq. (149), where $\Omega(\ )$ satisfies Eq. (150), then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}$. Furthermore, the worst-case performance can be bounded according to

$$J(\mathscr{U}) \leqslant \operatorname{tr} QR \tag{151}$$

The robust stability result is a direct consequence of Liapunov theory, whereas the performance bound (151) follows from the fact that since $A + \Delta A$ is asymptotically stable, $Q_{\Delta A} \triangleq \lim_{t \to \infty} Q_{\Delta A}(t)$ exists, is independent of $Q_{\Delta A}(0)$, and satisfies

$$0 = (A + \Delta A) Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V \tag{152}$$

Now subtracting Eq. (152) from Eq. (149) yields

$$\begin{aligned} 0 = {} & (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^T \\ & + \Omega(Q) - (\Delta A Q + Q \Delta A^T) + V \end{aligned} \tag{153}$$

which, by Eq. (150) and the fact that $A + \Delta A$ is stable, implies

$$Q_{\Delta A} \leqslant Q \tag{154}$$

Now Eqs. (148) and (154) yield the bound (151).

Since the ordering induced by the cone of nonnegative-definite matrices in only a partial ordering, it should not be expected that there exists an operator $\Omega(\ )$ satisfying Eq. (150) that is a least upper bound. Indeed, there

are many alternative definitions for the bound $\Omega(\ )$. To illustrate some of these alternatives, assume for convenience that ΔA is of the form

$$\Delta A = \sigma_1 A_1, \qquad |\sigma_1| \leqslant \Delta_1 \tag{155}$$

where σ_1 is an uncertain real scalar parameter assumed only to satisfy the stated bounds, and A_1 is a known matrix denoting the structure of the parametric uncertainty. The bound $\Omega(\ )$ utilized in Refs. 179 and 180 for full-state-feedback design was chosen to be

$$\Omega(Q) = \Delta_1 |A_1 Q + Q A_1^T| \tag{156}$$

where $|\ |$ denotes the nonnegative-definite matrix obtained by replacing each eigenvalue by its absolute value. More recently, the *quadratic* (in Q) bound

$$\Omega(Q) = \Delta_1 [A_L A_L^T + Q A_R^T A_R Q] \tag{157}$$

has been considered where A_L, A_R are a factorization of A_1 of the form $A_1 = A_L A_R$. A third bound that has been considered is the *linear* (in Q) bound

$$\Omega(Q) = \Delta_1 [\alpha Q + \alpha^{-1} A_1 Q A_1^T] \tag{158}$$

where α is an arbitrary positive scalar. A thorough development of bounds (156–158) as well as additional bounds for robustness analysis can be found in Ref. 55.

To make connections with the standard literature, we note that the procedure outlined earlier is closely related to the usual quadratic Liapunov function

$$V(x) = x^T P x \tag{159}$$

Since the matrix P can be characterized by means of an equation of the form

$$0 = A^T P + PA + R \tag{160}$$

it follows that the covariance Q can be viewed as providing a Liapunov function

$$\tilde{V}(x) = x^T Q x \tag{161}$$

for the dual system, i.e., with A replaced by A^T. The reason for working with the dual setting is that it leads more directly to robust performance bounds.

Guaranteed Cost Synthesis

The development of Liapunov techniques for robust analysis and synthesis is quite extensive.[178-196] The bounding technique described earlier is useful for developing a robust optimal projection theory for two main reasons. First, the bounding technique yields an upper bound (guaranteed cost) for the worst-case performance and guarantees robust stability over the given uncertainty set $\mathcal{U}$. Second, the uncertainty bound can be directly incorporated into the design procedure by predicating the design optimization on the modified Liapunov equation [Eq. (149)] in place of the standard Liapunov equation.

Although the guaranteed cost approach originated in Refs. 178 and 180, it turns out that the bound (156) utilized there is unsuitable for optimal projection synthesis for the simple reason that $\Omega(\ )$ is not differentiable. However, both the linear and the quadratic bounds can be incorporated directly within optimal projection theory to characterize robust fixed-order controllers. The key to this development is to derive the optimality conditions for the cost bound (151). Controllers that minimize this "auxiliary cost" thus minimize a bound on worst-case performance and guarantee robust stability. We now briefly review this development for both the linear and quadratic bounds.

Linear Bound

The origins of the linear bound can be traced back to the stochastic control literature. To see the connections, suppose that the uncertainty ΔA is modeled as $v_1(t)A_1$, where $v_1(t)$ is a scalar white noise process. Then $v_1(t)$ multiplies x and hence is a multiplicative rather than additive noise input disturbance. The idea of representing an uncertain parameter by multiplicative noise is certainly heuristic since uncertain parameters rarely behave in this manner. However, we shall show that such an approach is indeed sound and, consequently, that such a heuristic, nonphysical model is, in a sense, equivalent to Liapunov bounds.

If $\Delta A = v_1(t)A_1$, where $v_1(t)$ is white noise, then it can be shown that under a second-moment stability condition the steady-state covariance equation is given by[197]

$$0 = AQ + QA^T + A_1QA_1^T + V \tag{162}$$

where V is the intensity of the additive white noise. To bridge the gap to robust stability and performance, we make one additional modification to Eq. (162); namely, we replace A by the right-shifted dynamics $A + \tfrac{1}{2}I_n$ to obtain

$$0 = (A + \tfrac{1}{2}I_n)Q + Q(A + \tfrac{1}{2}I_n)^T + A_1QA_1^T + V \tag{163}$$

It is now easy to show that the existence of a solution to Eq. (163) guarantees *deterministic* robust stability of $A + \Delta A$ for all ΔA of the form $\sigma_1 A_1$, where $\sigma_1 \in [-1,1]$ is an uncertain constant. Hence, the stochastic

result provides a critical part of a deterministic bound that corresponds to a quadratic Liapunov function so that the stochastic interpretation of the bound can be disregarded in favor of a physically meaningful deterministic model. Consequently, this provides a rigorous robustness interpretation for controller synthesis methods based on multiplicative noise.[198-207] An extensive discussion of these issues can be found in Refs. 44 and 48. The linear bound was also considered in Ref. 195.

Optimal projection/guaranteed cost theory with the linear bound can now be summarized. In place of the plant and measurement model (1), (2), consider the perturbed model

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + D_1w(t), \qquad t \in [0,\infty) \tag{164}$$

$$y(t) = (C + \Delta C)x(t) + Du(t) + D_2w(t) \tag{165}$$

with the worst-case performance criterion

$$J(A_c,B_c,C_c) \triangleq \sup_{(\Delta A,\Delta B,\Delta C)\in\mathcal{U}} \lim_{t\to\infty} \mathbb{E}[x^T(t)R_1x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2u(t)] \tag{166}$$

For the linear bound the uncertainty set is given by

$$\mathcal{U}_1 = \left\{(\Delta A,\Delta B,\Delta C): \Delta A = \sum_{i=1}^{p}\sigma_iA_i, \quad \Delta B = \sum_{i=1}^{p}\sigma_iB_i, \quad \Delta C = \sum_{i=1}^{p}\sigma_iC_i, \quad \sum_{i=1}^{p}\sigma_i^2/\alpha_i^2 \leq 1\right\} \tag{167}$$

where, for $i = 1, \ldots, p$: $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, and $C_i \in \mathbb{R}^{\ell\times n}$ are fixed matrices denoting the structure of the parametric uncertainty; α_i is a given positive number; and σ_i is an uncertain real parameter. Note that the uncertain parameters σ_i are assumed to lie in a specified ellipsoidal region in $\mathbb{R}^p$. Note also that the uncertainties $\Delta A,\Delta B,\Delta C$ can be correlated in the sense that a given uncertain parameter σ_i may appear in, say, both ΔA and ΔB. Of course, one can set $A_i = 0$, $B_i = 0$, or $C_i = 0$ if there is no correlation.

With this uncertainty model the closed-loop system is given by

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \qquad t \in [0,\infty) \tag{168}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c + B_cDC_c \end{bmatrix}, \qquad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix} \tag{169}$$

$$\Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\,\Delta C & 0 \end{bmatrix} = \sum_{i=1}^{p}\sigma_i\tilde{A}_i, \qquad \tilde{A}_i = \begin{bmatrix} A_i & B_iC_c \\ B_cC_i & 0 \end{bmatrix}, \quad i = 1, \ldots, p \tag{170}$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has the intensity

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & B_c V_{12} \\ V_{12}^T B_c^T & B_c V_2 B_c^T \end{bmatrix} \tag{171}$$

To state the optimal projection/guaranteed cost synthesis result for the linear bound, let $\alpha > 0$ and define the notation

$$\gamma_i \triangleq \alpha_i^2/\alpha, \qquad A_\alpha \triangleq A + \frac{\alpha}{2} I_n \tag{172}$$

$$V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i C_i (Q + \hat{Q}) C_i^T, \qquad R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i B_i^T (P + \hat{P}) B_i \tag{173}$$

$$Q_s \triangleq QC^T + V_{12} + \sum_{i=1}^{p} \gamma_i A_i (Q + \hat{Q}) C_i^T,$$

$$P_s \triangleq B^T P + R_{12}^T + \sum_{i=1}^{p} \gamma_i B_i^T (P + \hat{P}) A_i \tag{174}$$

Theorem 3. Let $\alpha > 0$ and suppose there exist $Q, P, \hat{Q}, \hat{P} \geq 0$ satisfying

$$0 = A_\alpha Q + Q A_\alpha^T + V_1 + \sum_{i=1}^{p} \gamma_i [A_i Q A_i^T + (A_i - B_i R_{2s}^{-1} P_s) \times \hat{Q} (A_i - B_i R_{2s}^{-1} P_s)^T] - Q_s V_{2s}^{-1} Q_s^T + \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T \tag{175}$$

$$0 = A_\alpha^T P + P A_\alpha + R_1 + \sum_{i=1}^{p} \gamma_i [A_i^T P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^T \times \hat{P} (A_i - Q_s V_{2s}^{-1} C_i)] - P_s^T R_{2s}^{-1} P_s + \tau_\perp^T P_s^T R_{2s}^{-1} P_s \tau_\perp \tag{176}$$

$$0 = (A_\alpha - B R_{2s}^{-1} P_s) \hat{Q} + \hat{Q} (A_\alpha - B R_{2s}^{-1} P_s)^T + Q_s V_{2s}^{-1} Q_s^T - \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T \tag{177}$$

$$0 = (A_\alpha - Q_s V_{2s}^{-1} C)^T \hat{P} + \hat{P} (A_\alpha - Q_s V_{2s}^{-1} C) + P_s^T R_{2s}^{-1} P_s - \tau_\perp^T P_s^T R_{2s}^{-1} P_s \tau_\perp \tag{178}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{179}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{180}$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma (A - B R_{2s}^{-1} P_s - Q_s V_{2s}^{-1} C + Q_s V_{2s}^{-1} D R_{2s}^{-1} P_s) G^T \tag{181}$$

$$B_c = \Gamma Q_s V_{2s}^{-1} \tag{182}$$

$$C_c = - R_{2s}^{-1} P_s G^T \tag{183}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}_1$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}_1$. In this case the worst-case closed-loop performance [Eq. (166)] satisfies the guaranteed cost bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q}) R_1 - 2 R_{12} R_{2s}^{-1} P_s \hat{Q} + P_s^T R_{2s}^{-1} R_2 R_{2s}^{-1} P_s \hat{Q}] \tag{184}$$

It is easy to see that Eqs. (175–178) differ from the usual optimal projection equations [Eqs. (43–46)] precisely because of the bounding terms. If $\gamma_i = 0$, $i = 1, \ldots, p$, then these terms are absent and Eqs. (43–46) are recovered with a slight generalization due to R_{12} and V_{12}. It is also interesting to specialize Eqs. (175–178) to the full-order case. Although the projection no longer plays a role in coupling the equations, there is still coupling due to the uncertainty bounds since $\hat{Q}$ and $\hat{P}$ appear in Eqs. (175) and (176), respectively. Thus, we see that the guaranteed cost generalization of LQG theory now involves four equations as in optimal projection theory.

Finally, we note that the form of the uncertainty terms in Eqs. (175) and (176) can be interpreted in terms of their effect on control authority. For example, the uncertainty bounds corresponding to ΔB effectively lead to the replacement of R_2 by R_{2s}. Since $R_{2s} \geq R_2$, this replacement implies a corresponding *decrease* in regulator authority due to uncertainty ΔB. On the other hand, Eq. (175) for Q now involves a term $B_i R_{2s}^{-1} P_s \hat{Q} (B_i R_{2s}^{-1} P_s)^T$, which corresponds to an augmentation of V_1 and hence to a net *increase* in estimator authority. These phenomena are quite reasonable in the face of uncertainty in the control input matrix B. A dual situation clearly arises when uncertainty ΔC is present. However, uncertainty in the A matrix leads to additional nonnegative terms in both the Q and P equations and hence to both higher regulator and estimator authority. In the next section we shall consider a variant of the linear bound that treats the uncertainty in A somewhat differently.

Quadratic Bound

The quadratic bound appears to have originated more recently than the linear bound.[49,185,188,191,194] The interesting feature of this bound is its similarity to the terms that enforce the H_∞ norm constraint. Indeed, it is this similarity that led to the pivotal paper of Petersen[167] (P^3) on the synthesis of H_∞ controllers.

For the quadratic bound the uncertainty model is given by

$$\mathcal{U}_2 = \left\{ (\Delta A, \Delta C): \quad \Delta A = \sum_{i=1}^{p} D_i M_i N_i E_i, \quad \Delta C = \sum_{i=1}^{p} F_i M_i N_i E_i \right.$$
$$\left. M_i M_i^T \leq \bar{M}_i, \quad N_i^T N_i \leq \bar{N}_i, \quad i = 1, \ldots, p \right\} \tag{185}$$

where, for $i = 1, \ldots, p$: $D_i \in \mathbb{R}^{n \times r_i}$, $E_i \in \mathbb{R}^{t_i \times n}$, and $F_i \in \mathbb{R}^{l \times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\bar{M}_i \geq 0$ and $\bar{N}_i \geq 0$ are given uncertainty bounds; and $M_i \in \mathbb{R}^{r_i \times s_i}$ and $N_i \in \mathbb{R}^{s_i \times t_i}$ are uncertain matrices. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{p} \tilde{D}_i M_i N_i \tilde{E}_i \tag{186}$$

where

$$\tilde{D}_i \triangleq \begin{bmatrix} D_i \\ B_c F_i \end{bmatrix}, \qquad \tilde{E} \triangleq [E_i \quad 0] \tag{187}$$

The case $\Delta B \neq 0$ and $\Delta C = 0$ can be handled in a dual fashion, whereas the general case $\Delta B \neq 0$, $\Delta C \neq 0$ can also be handled but with increased complexity. To state the optimal projection/guaranteed cost synthesis result for the quadratic bound define the notation

$$D \triangleq \sum_{i=1}^{p} D_i \bar{M}_i D_i^T, \qquad E \triangleq \sum_{i=1}^{p} E_i^T \bar{N}_i E_i \tag{188}$$

$$P_c \triangleq B^T P + R_{12}^T, \qquad Q_a \triangleq QC^T + V_{12} + \sum_{i=1}^{p} D_i \bar{M}_i F_i^T \tag{189}$$

$$A_P \triangleq A - BR_2^{-1}P_a, \qquad A_Q \triangleq A - Q_a V_{2a}^{-1} C, \qquad V_{2a} \triangleq V_2 + \sum_{i=1}^{p} F_i \bar{M}_i F_i^T \tag{190}$$

Theorem 4. Suppose there exist $Q, P, \hat{Q}, \hat{P} \geq 0$ satisfying

$$0 = AQ + QA^T + V_1 + D + QEQ - Q_a V_{2a}^{-1} Q_a^T + \tau_\perp Q_a V_{2a}^{-1} Q_a^T \tau_\perp^T \tag{191}$$

$$0 = [A + (Q + \hat{Q})E]^T P + P[A + (Q + \hat{Q})E] + R_1 - P_a^T R_2^{-1} P_a + \tau_\perp^T P_a^T R_2^{-1} P_a \tau_\perp \tag{192}$$

$$0 = (A_P + QE)\hat{Q} + \hat{Q}(A_P - QE)^T + \hat{Q}E\hat{Q} + Q_a V_{2a}^{-1} Q_a^T - \tau_\perp Q_a V_{2a}^{-1} Q_a^T \tau_\perp^T \tag{193}$$

$$0 = (A_Q + QE)^T \hat{P} + \hat{P}(A_Q + QE) + P_a^T R_2^{-1} P_a - \tau_\perp^T P_a^T R_2^{-1} P_a \tau_\perp \tag{194}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{195}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{196}$$

and let

$$A_c = \Gamma(A - BR_2^{-1}P_a - Q_a V_{2a}^{-1} C + QE)G^T \tag{197}$$

$$B_c = \Gamma Q_a V_{2a}^{-1} \tag{198}$$

$$C_c = -R_2^{-1} P_a G^T \tag{199}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathscr{U}_2$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathscr{U}_2$. In this case the closed-loop performance satisfies the guaranteed cost bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q})R_1 + \hat{Q}PBR_2^{-1}P_a - R_{12}R_2^{-1}P_a\hat{Q}] \tag{200}$$

Finally, we show how optimal projection/guaranteed cost theory can be merged with optimal projection/H_∞ theory to form optimal projection/guaranteed cost/H_∞ theory. To illustrate such a combined theory, consider the uncertainty set $\mathscr{U}_1$, where for simplicity we assume $\Delta B = 0$ (i.e., $B_i = 0$, $i = 1, \ldots, p$). The principal feature of this result is that the H_∞ bound on the transfer function $\tilde{G}_{\Delta\tilde{A}}(s)$ holds for *all* uncertainties in $\mathscr{U}_1$. The following results can be found in Ref. 27; for a related approach see Ref. 208.

Theorem 5. Suppose there exist $Q, P, \hat{Q}, \hat{P} \geq 0$ satisfying

$$0 = A_\alpha Q + QA_\alpha^T + \gamma^{-2} QR_{1\infty} Q + V_1 + \sum_{i=1}^{p} \gamma_i A_i (Q + \hat{Q}) A_i^T - Q_s V_{2s}^{-1} Q_s^T + \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T \tag{201}$$

$$0 = (A_\alpha + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^T P + P(A_\alpha + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1 + \sum_{i=1}^{p} \gamma_i [A_i^T P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^T \hat{P} (A_i - Q_s V_{2s}^{-1} C_i)] - S^T P_s^T R_2^{-1} P_s S + \tau_\perp^T S^T P_s^T R_2^{-1} P_s S \tau_\perp \tag{202}$$

$$0 = (A_\alpha - BR_2^{-1}P_s S + \gamma^{-2} QR_{1\infty})\hat{Q} + \hat{Q}(A_\alpha - BR_2^{-1}P_s S + \gamma^{-2} QR_{1\infty})^T + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2 S^T P_s R_2^{-1} P_s S)\hat{Q} + Q_s V_{2s}^{-1} Q_s^T - \tau_\perp Q_s V_{2s}^{-1} Q_s^T \tau_\perp^T \tag{203}$$

$$0 = (A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2} QR_{1\infty})^T \hat{P} + \hat{P}(A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2} QR_{1\infty}) + S^T P_s R_2^{-1} P_s S - \tau_\perp^T S^T P_s R_2^{-1} P_s S \tau_\perp \tag{204}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c \tag{205}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{206}$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma(A - BR_2^{-1}P_sS - Q_sV_{2s}^{-1}C + \gamma^{-2}QR_{1\infty})G^T \tag{207}$$

$$B_c = \Gamma Q_s V_{2s}^{-1} \tag{208}$$

$$C_c = -R_2^{-1}P_sSG^T \tag{209}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathscr{U}_1$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathscr{U}_1$. In this case, the closed-loop transfer function $G_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|G_{\Delta\tilde{A}}(s)\|_\infty \leqslant \gamma \qquad \text{for all} \qquad (\Delta A, \Delta C) \in \mathscr{U}_1 \tag{210}$$

and the worst-case H_2 performance criterion satisfies the bound

$$J(A_c, B_c, C_c) \leqslant \mathrm{tr}[(Q + \hat{Q})R_1 + \hat{Q}S^TP_sR_2^{-1}P_sS] \tag{211}$$

Optimal Projection/Positive Real Theory

We now turn to an extension of optimal projection theory that was developed specifically for controlling flexible structures with numerous lightly damped modes. Although this extension is similar in form to the optimal projection/guaranteed cost theory with the linear bound, it operates via wholly different principles. Specifically, optimal projection/positive real theory does not achieve robustness by bounding the worst-case value of the performance functional over a class of uncertainties. Rather, it seeks to configure the controller to exploit the dissipative nature of the structural system. In classical terminology we are thus seeking a theory of phase stabilization in contrast to the more standard approach of gain stabilization. The goal of this section is to demonstrate how optimal projection theory can be extended to achieve this goal.

To begin, we consider a damped oscillator with force input and velocity measurement. For such a system the transfer function is known to be positive real, i.e., to have phase shift less than 90 deg. In a negative feedback configuration, a controller that is strictly positive real cannot destabilize the system since the loop transfer function has phase less than 180 deg. Hence, such a control system will be unconditionally robust to natural frequency and damping uncertainties. In controlling structural vibrations a common application of this principle is the rate feedback damper with a colocated sensor/actuator pair. Of course, these observations assume perfect sensors and actuators in order to avoid introducing additional phase shift. If the sensors and actuators do have significant dynamics, then the feedback law must be chosen so that the transfer function from sensor input to actuator is strictly positive real. If in practice positive realness can only be enforced over a limited frequency band, then loop gain rolloff is required when unknown phase shifts become significant.

Optimal projection/positive real theory seeks to embed these design principles within a compensator synthesis theory for lightly damped structures. This extension was developed in a series of papers predating optimal projection theory itself.[31–34] The original motivation arose from a branch of structural analysis known as statistical energy analysis (SEA).[209–214] SEA is a structural analysis theory developed to analyze systems with numerous lightly damped modes. It has found considerable practical applications in analyzing acoustic vibrations in cases in which very little information is available concerning modal frequency and mode shapes. Often the only available knowledge is the modal density, i.e., the average number of modes per frequency band.

The basic concept of SEA is power flow among coupled modes. Estimates of modal density provide the means for computing the energy distribution of modes in steady-state excitation. An important phenomenon in this regard is the *equipartition* of energy, that is, the tendency of modal energy (rms vibration levels) to become equalized among modes due to power flow. Equipartitioning leads to *isotropy*, the state of uniform energy distribution.

For control system analysis and design the modal phenomena of SEA can be reinterpreted in terms of model uncertainty. Specifically, the covariance matrix of a modal system may represent mean square response due to both external excitation and modal uncertainty. Highly uncertain lightly damped modes will thus give rise to a diagonal covariance matrix with equal diagonal elements denoting equipartition in the isotropic state. Off-diagonal elements of the covariance matrix are essentially zero due to the absence of correlation (called "decorrelation") caused by modal uncertainty. At slightly lower frequencies, the diagonal elements may be different because of greater modeling fidelity, although diagonal dominance may still occur. At low frequencies, a reasonably accurate model will, of course, result in a fully dense covariance matrix.

The contribution of Refs. 31–41 was the construction of an SEA covariance equation that captures the effects of modal uncertainty. As we shall see, optimal projection synthesis with the SEA covariance equation induces a robust controller with positive real characteristics that robustify against modal uncertainty. The SEA covariance equation is given by

$$0 = A_sQ + QA_s^T + \sum_{i=1}^{r} A_iQA_i^T + V \tag{212}$$

where

$$A = \text{block diag}\left(\begin{bmatrix} -\zeta_1 & -\omega_1 \\ \omega_1 & -\zeta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\zeta_r & -\omega_r \\ -\omega_r & -\zeta_r \end{bmatrix}\right) \tag{213}$$

$$A_i = \text{block diag}\left(0, \ldots, 0, \begin{bmatrix} 0 & -\beta_i \\ \beta_i & 0 \end{bmatrix}, 0, \ldots, 0\right) \tag{214}$$

$$A_s \triangleq A + \frac{1}{2}\sum_{i=1}^{r} A_i^2 \tag{215}$$

Several observations are in order. First, note that the SEA covariance equation is a modified Liapunov equation of the form of Eq. (150) with an uncertainty operator $\Omega_s(Q)$ defined by

$$\Omega_s(Q) = \sum_{i=1}^{r} [\tfrac{1}{2}A_i^2 Q + A_i Q A_i^T + \tfrac{1}{2} Q A_i^{2T}] \tag{216}$$

Although this operator has the term $A_i Q A_i^T$ in common with the linear bound, it differs in one crucial respect. Whereas the linear bound (158) has a uniform right-shift term $A + (\alpha/2)I_n$, the operator (216) invokes the term

$$A + \frac{1}{2}\sum_{i=1}^{r} A_i^2$$

which is a variable left shift. To see this, note that

$$\tfrac{1}{2}A_i^2 = \text{block diag}(0, ..., 0, -\tfrac{1}{2}\beta_i^2 I_2, 0, ..., 0) \tag{217}$$

which effectively shifts the ith mode to the left by an amount $-\tfrac{1}{2}\beta_i^2$. Thus, β_i can be increased at a higher frequency to represent greater uncertainty.

An interesting observation concerning $\Omega_s(Q)$ is that its form coincides precisely with the covariance equation arising from a multiplicative noise model interpreted in the sense of Stratonovich.[215–217] The "Stratonovich correction" term

$$\frac{1}{2}\sum_{i=1}^{r} A_i^2$$

arises due to the conversion from Stratonovich stochastic calculus to the more common Ito calculus. Justification for the Stratonovich interpretation was originally given in terms of maximum entropy principles.[218–221]

Next note that the uncertainty matrices A_i correspond to uncertainty in the modal frequencies. In contrast to damping uncertainty, modal frequency uncertainty is nondestabilizing. However, it does have a significant effect on the state covariance. To see this, let $r = 2$ in Eq. (213) for convenience and note that $\Omega_s(Q)$ is given by

$$\Omega_s(Q) =$$

$$\begin{bmatrix} \beta_1^2[Q_{22} - Q_{11}] & -2\beta_1^2 Q_{12} & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{13} & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{14} \\ -2\beta_1^2 Q_{21} & \beta_1^2[Q_{11} - Q_{22}] & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{23} & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{24} \\ -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{31} & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{32} & \beta_2^2[Q_{44} - Q_{33}] & -2\beta_2^2 Q_{34} \\ -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{41} & -\tfrac{1}{2}(\beta_1^2+\beta_2^2)Q_{42} & -2\beta_2^2 Q_{43} & \beta_2^2[Q_{33} - Q_{44}] \end{bmatrix} \tag{218}$$

Brief inspection of $\Omega_s(Q)$ reveals two things. First, the off-diagonal terms of $\Omega_s(Q)$ are proportional to the corresponding terms of Q with a negative constant of proportionality. By viewing the covariance equation as a differential equation, it follows that these terms cause the off-diagonal terms to decay (i.e., each is of the form $\dot{x} = -\beta x$). This is the phenomenon of decorrelation due to frequency uncertainty. Next, note that the diagonal terms of $\Omega_s(Q)$ are sign variable. If, for example, $Q_{11} > Q_{22}$, then the (1,1) term of $\Omega_s(Q)$ is negative and thus causes a corresponding decrease in Q_{11}, whereas the (2,2) term is positive and causes Q_{22} to increase. However, if $Q_{11} < Q_{22}$, then the (2,2) element is negative and the situation is reversed. The overall effect of such modal interaction is *equilibration* among modes. Thus, the SEA/Stratonovich model captures the phenomena of decorrelation and equilibration.

However, the uncertainty operator $\Omega_s(Q)$ is not a bound on the effects of frequency uncertainty. In fact, it can be shown that $\Omega_s(Q)$ is actually indefinite. Hence, the role of $\Omega_s(Q)$ cannot be simply explained via guaranteed cost theory. To examine the control-design ramifications of $\Omega_s(Q)$ with optimal projection theory, we can derive the appropriate design equations[41,42]:

$$0 = A_s(Q) + QA_s^T + V_1 + \sum_{i=1}^{r} A_i(Q + \hat{Q})A_i^T - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{219}$$

$$0 = A_s^T P + PA_s + R_1 + \sum_{i=1}^{r} A_i^T(P + \hat{P})A_i - P\Sigma P + \tau_\perp^T P\Sigma P\tau_\perp \tag{220}$$

$$0 = (A_s - \Sigma P)\hat{Q} + \hat{Q}(A_s - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{221}$$

$$0 = (A_s - Q\bar{\Sigma})^T\hat{P} + \hat{P}(A_s - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P\tau_\perp \tag{222}$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c \tag{223}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{224}$$

with gains

$$A_c = \Gamma(A_s - Q\bar{\Sigma} - \Sigma P)G^T \tag{225}$$

$$B_c = \Gamma Q C^T V_2^{-1} \tag{226}$$

$$C_c = -R_2^{-1}B^T P G^T \tag{227}$$

To illustrate the consequences of the SEA/Stratonovich model, we consider a simply supported beam with a *single* collocated force actuator/velocity sensor pair. The modal frequencies are assumed to be uncertain with uncertainty increasing linearly with mode number. A design study carried out in Refs. 36 and 40 yielded the results shown in Fig. 4. The principal feature of this result is the tendency of the controller to become

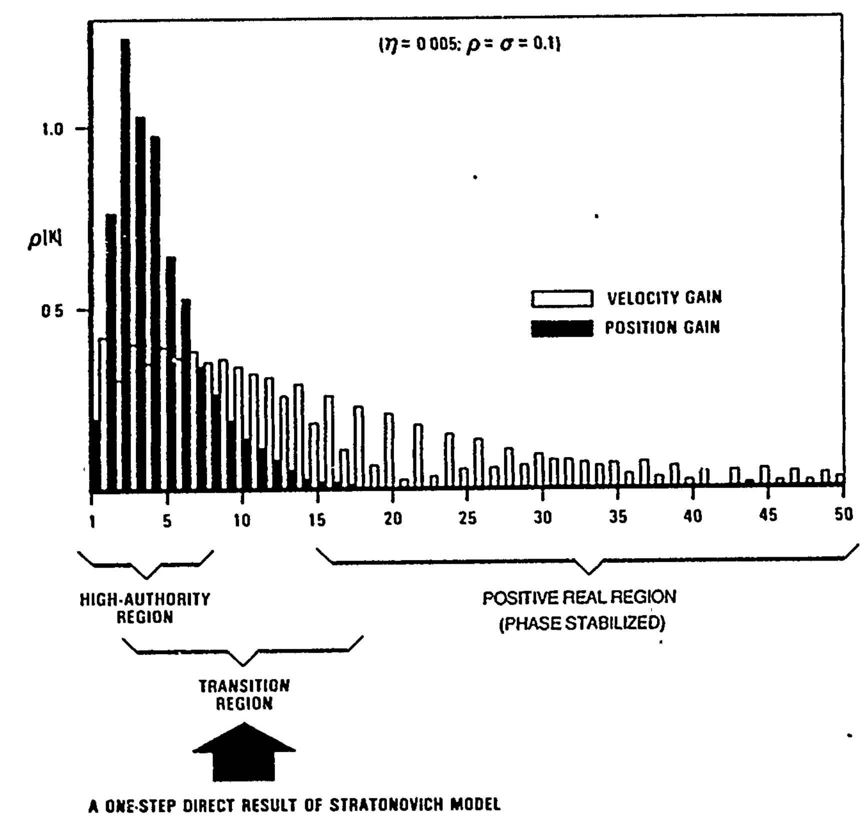

Fig. 4 The optimal projection/positive real controllers achieve robust phase stabilization by means of dissipative rate feedback for highly uncertain modes.

increasingly positive real at higher frequency. That is, in the low-frequency region, the controller assigns high gains to both position and velocity, whereas at high frequency only velocity is fed back, indicating a phase stabilizing, rate dissipative controller. Related results obtained using this approach were reported in Refs. 222–224.

Open Problems and Extensions

Open problems represent a source of vitality and motivation for future development. We close by describing a collection of research topics that are, at the same time, practically motivated and theoretically meaningful. Although all of these problems relate in some way to optimal projection theory, many have much broader significance.

Stabilizability

Optimal projection theory notwithstanding, one of the most basic unsolved problems in control theory is the problem of output feedback

stabilizability: Given A,B,C determine necessary and sufficient conditions under which there exists K such that $A+BKC$ is asymptotically stable. If either B or C is the identity matrix, then the solution is well known, whereas in the general case partial results are given in Refs. 225 and 226. The decentralized generalization of this problem is also of interest: Given $A, B_1, \ldots, B_r, C_1, \ldots, C_r$, determine necessary and sufficient conditions under which there exist $K_1, \ldots, K_r$ such that

$$A + \sum_{i=1}^{r} B_i K_i C_i$$

is asymptotically stable. This problem is of interest since it includes stabilizability by reduced-order compensators as a special case. To see this, write the closed-loop dynamics as

$$\begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} A_c [0 \quad I] + \begin{bmatrix} 0 \\ I \end{bmatrix} B_c [C \quad 0] + \begin{bmatrix} B \\ 0 \end{bmatrix} C_c [0 \quad I]$$

which has decentralized structure with $r=3$. If $n_c = n$, then necessary and sufficient conditions for stabilizability are well known from LQG theory. The reduced-order case $n_c < n$ is unsolved. Finally, a further extension is to require repetition of certain feedback gains; i.e., $K_j = K_\ell$ for selected indices. It can be shown using the techniques of Refs. 57 and 58 that this feedback structure encompasses all affinely parameterized static and dynamic feedback structures.

Existence Theory

For each optimization problem considered within optimal projection theory, it is of interest to know whether a global minimum exists. In practice such questions are largely academic, since necessary conditions serve as useful tools for computing extremals when they do exist. Nevertheless, in many cases existence theory is related to the achievability of design goals. For example, in the H_∞ problem it is of interest to know whether optimal projection/H_∞ theory is effective in characterizing H_∞-constrained controllers when they exist. Such questions relate directly to the existence of solutions to the constrained optimization problem. Historically, existence problems in nonconvex optimization theory have proven to be difficult to treat mathematically. For fixed-structure optimization problems, it appears that a combination of compactness and monotonicity arguments may be effective.

Analysis of the Design Equations

Analysis of the standard matrix Riccati equation has been ongoing for more than three decades. Such efforts have been extremely fruitful with ramifications in mathematics and other branches of engineering. However, there can be no doubt that the systems of coupled modified Riccati and Liapunov equations arising in optimal projection theory lie beyond the

scope of existing Riccati equation theory. Investigation is needed to address questions of existence, multiplicity, monotonicity, stabilizability, and optimality. For example, we would expect that conditions guaranteeing the existence of solutions to the optimal projection equations [Eqs. (44–46)] would be closely related to stabilizability by means of fixed-order compensators. A thorough analysis of these equations represents a significant mathematical challenge for the next decade.

Quadratic Liapunov Bounds

Numerous questions remain unanswered concerning the linear and quadratic Liapunov bounds. For example, the conservatism of such bounds requires further examination. In addition, the gap between robust stability and quadratic robust stability (robustness guaranteed via a fixed quadratic Liapunov function) is not well understood. In this direction it can be seen that the four-polynomial robust stability test of Kharitonov must correspond to a four-Liapunov function robust stability test for an interval matrix in companion form. This observation suggests that a reduction in conservatism may be possible by adopting a multi-Liapunov function approach in place of the single Liapunov function technique.

Positive Real Theory

In the previous section it was argued heuristically and demonstrated numerically that the SEA/Stratonovich uncertainty operator induces robustness by exploiting positive real properties. On the one hand, it would appear to be a straightforward task to prove this mathematically by applying positive real system criteria.[227–232] However, in fact, the situation is much more subtle since the SEA/Stratonovich operator induces positive realness only to the extent that phase shift is needed to robustify with respect to a particular level. Hence, a much finer characterization of phase properties appears to be necessary. The H_∞ approach to positive real design[233] may be a useful starting point in this regard.

Optimal Projection/μ Theory

Structured singular value theory[234,235] was developed to address problems in which direct application of H_∞ bounds leads to overly conservative results. For example, if unstructured uncertainty appears at multiple points within the plant, then the overall uncertainty has a block-diagonal structure. The same situation arises in bounding the worst-case H_∞ performance in the presence of unstructured uncertainty. To address such problems the μ function was developed to generalize the small gain theorem to block-diagonal uncertainties. The extension of optimal projection theory to the μ-synthesis problem[236,237] remains a potentially fruitful research area. Alternative approaches to unstructured uncertainty such as the all-pass characterization[238–240] are also promising in this regard.

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Application of Maximum Entropy/Optimal Projection Design Synthesis to a Benchmark Problem

Emmanuel G. Collins*, James A. King†, and Dennis S. Bernstein‡
Harris Corporation, Melbourne, Florida, 32901

Maximum Entropy/Optimal Projection (ME/OP) design synthesis is a methodology for designing robust, fixed-order controllers for flexible structures. This paper reviews the theoretical basis for ME/OP and illustrates the approach using a benchmark problem. The benchmark problem involves two masses with spring coupling, an uncertain spring constant, and a sensor and actuator which are noncolocated.

* Associate Principal Engineer, Government Aerospace Systems Division
† Senior Engineer, Government Aerospace Systems Division
‡ Staff Engineer, Government Aerospace Systems Division

I. Introduction

Active feedback control for vibration suppression in lightly damped structures continues to be a challenging area of aerospace engineering[1]. Typically, such problems involve multi-input/multi-output systems in noncolocated sensor/actuator configurations. The task of designing disturbance attenuation feedback controllers for such systems is further exacerbated by constraints on real-time processing capacity for feedback control law implementation as well as modeling uncertainty associated with complex structures. Among the numerous methodologies proposed for addressing the structural control problem, this paper focusses on the maximum entropy/optimal projection (ME/OP) approach. This approach was originally developed by Hyland in a series of technical reports and conference papers (Refs. 2–6). The ME/OP approach has been applied experimentally to various structural control testbeds[7–9] and has been evaluated by other researchers in Refs. 10–12. Subsequently, numerous extensions and variations of this approach have been developed to address a variety of problems in robust, fixed-structure controller synthesis[13–17]. A detailed review of this work with extensive references to related literature is given in Ref. 18.

The purpose of this paper is twofold. First we provide (in Sections II and III) a brief review of robust fixed-order controller synthesis, in particular, robust optimal projection controller synthesis. Although a detailed review is given in Ref. 13, the brief review given here focusses more directly on the ME/OP technique for robust controller synthesis, which is described in Section III. After reviewing ME/OP, we then turn our attention in Section IV to problems #1 and #3 of the benchmark example given in Ref. 19. The ME/OP design approach is applied to the benchmark example in Section V and the design results are discussed in Section VI.

II. Robust Fixed-Order Controller Synthesis

Optimal projection theory[13] generalizes LQG theory to the case of reduced-order controllers. While LQG theory provides quadratically (H_2) optimal full-order dynamic compensators by means of two uncoupled Riccati equations, optimal projection theory characterizes quadratically optimal reduced-order (i.e., fixed-order) controllers via a coupled system consisting of two modified Riccati equations and two modified Lyapunov equations. When the controller order is set equal to the plant order, the idempotent matrix responsible for the coupling (the so-called "optimal projection") becomes the identity, the coupling terms vanish, the Lyapunov equations are rendered superfluous, and the LQG Riccati equations are recovered. Numerical algorithms for solving the optimal projection equations via iterative and homotopy techniques have been developed in Refs.

20–22.

Robust optimal projection theory (as well as robust LQG theory) has been developed by incorporating uncertainty bounds within the design optimization procedure. The idea behind this approach can most clearly be illustrated within the context of robust *analysis*, while its application to synthesis is a fairly straightforward extension of fixed-structure optimization. The description given here follows the development of uncertainty bounds given in Ref. 23.

For the asymptotically stable linear system

$$\dot{x}(t) = Ax(t), \tag{1}$$

we consider a quadratic Lyapunov function of the form

$$\mathcal{V}(x) = x^{\mathrm{T}}Px, \tag{2}$$

where the positive-definite matrix P is given by the Lyapunov equation

$$0 = A^{\mathrm{T}}P + PA + R, \tag{3}$$

where R is positive definite. In order to address additive disturbances for a system of the form

$$\dot{x}(t) = Ax(t) + w(t), \tag{4}$$

it is convenient to utilize the dual equation

$$0 = AQ + QA^{\mathrm{T}} + V \tag{5}$$

in which A is replaced by A^{T} (which has the same spectrum as A) and where V is interpreted as the intensity of the white noise disturbance $w(t)$. In (5), the matrix Q can be viewed as a controllability Gramian or covariance matrix with associated quadratic (H_2) performance measure

$$J = \operatorname{tr} QR = \operatorname{tr} PV. \tag{6}$$

If A is uncertain so that (1) is replaced by

$$\dot{x}(t) = (A + \Delta A)x(t), \tag{7}$$

where $\Delta A \in \mathcal{U}$, a set of perturbations, then we wish to determine whether $A + \Delta A$ remains stable for all $\Delta A \in \mathcal{U}$. One approach to this problem involves replacing (5) by

$$0 = AQ + QA^{\mathrm{T}} + \Omega + V, \tag{8}$$

where Ω is a constant positive-definite matrix. Rewriting (8) as

$$0 = (A + \Delta A)Q + Q(A + \Delta A)^{\mathrm{T}} + \Omega - (\Delta A Q + Q \Delta A^{\mathrm{T}}) + V, \tag{9}$$

it follows that $A + \Delta A$ is stable so long as ΔA satisfies

$$\Delta A Q + Q \Delta A^{\mathrm{T}} \leq \Omega, \tag{10}$$

where Q is the solution to (8).

A variation on equation (8) involves allowing Ω to be a function of Q. Thus we consider the modified Lyapunov equation

$$0 = AQ + QA^{\mathrm{T}} + \Omega(Q) + V, \tag{11}$$

where $\Omega(\cdot)$ satisfies

$$\Delta A Q + Q \Delta A^{\mathrm{T}} \leq \Omega(Q), \text{ for all } \Delta A \in \mathcal{U}, \tag{12}$$

and for all nonnegative-definite Q. It then follows by rewriting (11) as

$$0 = (A + \Delta A)Q + Q(A + \Delta A)^{\mathrm{T}} + \Omega(Q) - (\Delta A Q + Q \Delta A^{\mathrm{T}}) + V \tag{13}$$

that $A + \Delta A$ is stable. Furthermore, letting $Q_{\Delta A}$ satisfy

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^{\mathrm{T}} + V, \tag{14}$$

and subtracting (14) from (13) yields

$$0 = (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^{\mathrm{T}} + \Omega(Q) - (\Delta A Q + Q \Delta A^{\mathrm{T}}), \tag{15}$$

which implies that

$$Q_{\Delta A} \leq Q, \quad \Delta A \in \mathcal{U}. \tag{16}$$

Thus tr QR provides a worst-case bound for the *actual* H_2 performance tr $Q_{\Delta A}R$.

Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, it should not be expected that there exists an operator $\Omega(\cdot)$ satisfying (12) that is a least upper bound. Indeed, there are many alternative definitions for the bound $\Omega(\cdot)$. To illustrate some of these alternatives, assume for convenience that ΔA is of the form

$$\Delta A = \sigma_1 A_1, \quad |\sigma_1| \leq \delta_1, \tag{17}$$

where σ_1 is an uncertain real scalar parameter assumed only to satisfy the stated bounds and A_1 is a known matrix denoting the structure of the parametric uncertainty. The bound $\Omega(\cdot)$ utilized in Ref. 24 for full-state-feedback design was chosen to be the *absolute value* bound

$$\Omega(Q) = \delta_1 |A_1 Q + Q A_1^{\mathrm{T}}|, \tag{18}$$

where $|\cdot|$ denotes the nonnegative-definite matrix obtained by replacing each eigenvalue by its absolute value. Since the bound defined in (18) is not differentiable with respect to Q, it has limited usefulness in fixed-structure controller synthesis. A more useful bound is the *quadratic* (in Q) bound

$$\Omega(Q) = \delta_1 [A_L A_L^{\mathrm{T}} + Q A_R^{\mathrm{T}} A_R Q], \tag{19}$$

which has been considered in Refs. 25, 26. In (19), A_L, A_R are a factorization of A_1 of the form $A_1 = A_L A_R$. A third bound that has been considered is the *linear* (in Q) bound

$$\Omega(Q) = \delta_1 [\alpha Q + \alpha^{-1} A_1 Q A_1^{\mathrm{T}}], \tag{20}$$

where α is an arbitrary positive scalar. As discussed in Ref. 14, the linear bound is closely related to a multiplicative white noise model[27]. Extensions of LQG theory to include such effects are given in Refs. 28, 29.

Within the context of fixed-structure controller synthesis, the linear and quadratic bounds have been merged with optimal projection theory in Refs. 15 and 16, respectively. The quadratic bound also has the useful property that it enforces an H_∞ (bounded real) constraint. This extension has been incorporated within optimal projection theory in Ref. 17.

In summary, it can be seen that both the linear and quadratic bounds guarantee robust stability and performance with respect to parameter uncertainty and lead to generalizations of LQG and optimal projection theory. As discussed in Ref. 30, however, these bounds actually guarantee robustness with respect to time-varying parameter variations, which may lead to conservatism when the parameter variations are known to be constant. Viewed in the frequency domain, such bounds correspond to small-gain-type conditions which enforce robust stability with respect to complex, frequency-dependent uncertainty, which is conservative if the uncertain parameters are known to be real and constant. Such conservatism may have serious consequences in controlling flexible structures with stiffness uncertainty, which is a highly structured, inherently real form of parameter uncertainty. Consequently, we now turn our attention to the maximum entropy/optimal

projection approach to robust controller synthesis which seeks to overcome these difficulties for a particular form of parametric uncertainty.

III. Maximum Entropy/Optimal Projection Design Synthesis

As discussed in Section I, the ME/OP approach was originally developed in Refs. 2–6. In brief, the basis of the ME/OP idea is to choose the operator $\Omega(Q)$ in the modified Lyapunov equation (8) to be of the form

$$\Omega(Q) = \sum_{i=1}^{r} \delta_i [\frac{1}{2} A_i^2 Q + A_i Q A_i^{\mathrm{T}} + \frac{1}{2} Q A_i^{2\mathrm{T}}], \tag{21}$$

where the summation corresponds to an uncertainty model of the form $A + \Delta A$, where

$$\Delta A = \sum_{i=1}^{r} \sigma_i A_i, \tag{22}$$

$\sigma_1, \ldots, \sigma_r$ are uncertain real parameters, and $\delta_1, \ldots, \delta_r \geq 0$ are uncertainty scalings. Note that in (17) we set $r = 1$ for convenience although (17)–(20) could readily be generalized to the case $r > 1$.

The unusual feature of (21) is that (as will be seen shortly) $\Omega(Q)$ is not a bound in the sense of (12) as are (18)–(20). Thus we do not stipulate a precise uncertainty range for the uncertain parameters σ_i as in (17). Rather, the constants $\delta_1, \ldots, \delta_r$ should only be viewed as scalings. Indeed, whereas the bounds (18)–(20) are valid for arbitrary choices of A_1, the operator (21) will only be used (in this paper) under restrictive, but practically useful, assumptions. Specifically, we now assume that the nominal dynamics matrix A is of the form

$$A = \text{block-diag}\left(\begin{bmatrix} -\eta_1 & \omega_1 \\ -\omega_1 & -\eta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\eta_r & \omega_r \\ -\omega_r & -\eta_r \end{bmatrix} \right) \tag{23}$$

which is representative of a lightly damped structure in a modal basis, while the uncertainty matrix A_i is of the form

$$A_i = \text{block-diag}(0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0), \tag{24}$$

where the position of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ corresponds to the ith diagonal block of A. Since the poles of $A + \sum_{i=1}^{r} \sigma_i A_i$ are of the form $-\eta + j(\omega_i + \sigma_i)$, each term $\sigma_i A_i$ represents uncertainty in the imaginary part of a pole location.

To further illustrate the structure of $\Omega(Q)$ given by (21), define

$$S \triangleq \sum_{i=1}^{r} \frac{1}{2} \delta_i A_i^2, \tag{25}$$

so that (21) becomes

$$\Omega(Q) = SQ + QS + \sum_{i=1}^{r} \delta_i A_i Q A_i^{\mathrm{T}} \tag{26}$$

and the modified Lyapunov equation (11) is of the form

$$0 = (A+S)Q + Q(A+S)^{\mathrm{T}} + \sum_{i=1}^{r} \delta_i A_i Q A_i^{\mathrm{T}} + V. \tag{27}$$

Using (24) and the fact that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, S can be written as

$$S = \text{block-diag}\left(\begin{bmatrix} -\frac{1}{2}\delta_1 & 0 \\ 0 & -\frac{1}{2}\delta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\frac{1}{2}\delta_r & 0 \\ 0 & -\frac{1}{2}\delta_r \end{bmatrix} \right). \tag{28}$$

It is interesting to contrast (26) with the linear bound (20) in light of the structure of S. To do this, generalize (20) to the case $r > 1$ (but setting $\alpha = 1$) which now has the form

$$\Omega(Q) = \sum_{i=1}^{r} \delta_i [Q + A_i Q A_i^{\mathrm{T}}]. \tag{29}$$

Now rewrite (29) as

$$\Omega(Q) = \hat{S}Q + Q\hat{S} + \sum_{i=1}^{r} \delta_i A_i Q A_i^{\mathrm{T}}, \tag{30}$$

where

$$\hat{S} \triangleq \frac{1}{2} \sum_{i=1}^{r} \delta_i I,$$

which yields a modified Lyapunov equation of the form

$$0 = (A+\hat{S})Q + Q(A+\hat{S})^{\mathrm{T}} + \sum_{i=1}^{r} \delta_i A_i Q A_i^{\mathrm{T}} + V, \tag{31}$$

which is identical to (27) with S replaced by $\hat{S}$.

Justification and insight into the meaning of the modified Lyapunov equation (27) can be obtained from several diverse, but interrelated, points of view. These are summarized as follows.

Multiplicative White Noise. As discussed in Section II, the linear bound (20) can be viewed as a consequence of a multiplicative white noise model. In fact, the operator $\Omega(Q)$ defined by (21) also arises from a multiplicative noise model so long as stochastic integration is interpreted in the sense of Stratonovich rather than Ito[31,32]. The interesting feature of Stratonovich stochastic integration (besides the fact that its differentials obey the "standard" rules of calculus) is that it

arises naturally from the limiting process in approximating the solutions to stochastic differential equations[33]. This property was used in conjunction with the Maximum Entropy Principle of Jaynes[34] to justify (21) as an uncertainty model in Refs. 2, 5, whence the name "maximum entropy" controller synthesis.

Uniform Right Shift Versus Variable Left Shift. By comparing (27) to (31) it can be seen that they differ only in the shift (S or $\hat{S}$) to the nominal dynamics matrix A. Whereas the shift $\hat{S}$ in (31) corresponds to a uniform right shift to the open-loop dynamics, the shift S in (27) represents a variable (mode-by-mode) *left* shift. In the context of closed-loop feedback control this distinction has the following effect: Rather than imposing a minimal stability margin on each mode as guaranteed by the uniform right shift, the variable left shift introduces a fictitious augmentation of the open-loop damping to induce the controller to "ignore" uncertain modes and thereby not risk destabilizing them.

Energy Flow and Statistical Energy Analysis. To further motivate the form of $\Omega(Q)$ given by (21), consider the differential equation

$$\dot{Q}(t) = \Omega(Q(t)) \tag{32}$$

for the case $r = 1$ and $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, that is, for the case in which $Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$ is a 2×2 matrix. In this case (32) is equivalent to the three scalar differential equations

$$\dot{Q}_{11}(t) = \delta_1(Q_{22}(t) - Q_{11}(t)), \tag{33}$$

$$\dot{Q}_{12}(t) = -\delta_1 Q_{12}(t), \tag{34}$$

$$\dot{Q}_{22}(t) = \delta_1(Q_{11}(t) - Q_{22}(t)). \tag{35}$$

As $t \to \infty$, note that (33) and (35) have the effect of *equilibrating* (equalizing) $Q_{11}(t)$ and $Q_{22}(t)$ since $Q_{22}(t) > Q_{11}(t)$ implies that $Q_{11}(t)$ is increasing and $Q_{22}(t)$ is decreasing, and vice versa for $Q_{11}(t) > Q_{22}(t)$. Although $Q_{11}(t)$ and $Q_{22}(t)$ may have very different values at $t = 0$, they eventually become equal as $t \to \infty$. Physically, this equilibration (which also occurs for multimodal systems with $r > 1$ and more general choices of skew-symmetric A_i) reflects energy flow among states. It is important to point out that energy flow in dynamic systems need not be viewed strictly as a consequence of modeling uncertainty, but rather as a highly probable dynamic event, much like heat flow in thermodynamics. This viewpoint forms the basis for Statistical Energy Analysis[35] and energy flow models[36–38]. Returning to (33)–(35), note that the effect of $\Omega(Q)$ when utilized

in the context of a differential equation of the form

$$\dot{Q}(t) = AQ(t) + Q(t)A^{\mathrm{T}} + \Omega(Q(t)) + V \tag{36}$$

is to induce equilibration of the diagonal elements of $Q(t)$ in accordance with the size of the scaling parameters δ_i. Note also that the effect of (34) is to suppress the off-diagonal element Q_{12}, an effect known as *decorrelation*[35], which can be viewed as a natural consequence of uncertainty. The combination of these effects in the limit of large δ_i is to induce a covariance matrix of the form βI, where $\beta > 0$, a state known as *equipartition of energy*[35].

Positive Real Theory. The uncertainty ΔA defined by (22) is skew symmetric, that is, $\Delta A^{\mathrm{T}} = -\Delta A$. Thus ΔA is a special case of a positive real uncertainty matrix F satisfying the condition $F + F^{\mathrm{T}} \geq 0$. Robust stability conditions for positive real uncertainty have been developed in Ref. 39, where it was shown that the structure of these conditions is closely related to the modified Lyapunov equation (27). Important differences arise, however, due to the fact that the conditions given in Ref. 39 involve a quadratic term within a matrix Riccati equation.

Structured Covariance. A more mathematical and less physical interpretation of the structure of $\Omega(Q)$ given by (21) can be given in terms of its effect on the *structure* of the covariance Q. To illustrate this idea, which was proposed in Ref. 40, let Q be a covariance matrix corresponding to a nominally stable matrix A so that

$$AQ + QA^{\mathrm{T}} < 0. \tag{37}$$

If A is perturbed by ΔA, then we wish to also have

$$(A + \Delta A)Q + Q(A + \Delta A)^{\mathrm{T}} < 0. \tag{38}$$

One way to achieve (38) is to seek Q satisfying (37) so that the term $\Delta AQ + Q\Delta A^{\mathrm{T}}$ is sufficiently small compared to $AQ + QA^{\mathrm{T}}$. For example, let $Q = \beta I$ and suppose that $A + A^{\mathrm{T}} < 0$. Then for all skew-symmetric ΔA it follows that $\Delta A + \Delta A^{\mathrm{T}} = 0$ so that (38) holds. Thus the effect of the perturbation is small due to the *structure* (rather than the size) of Q. Note that this approach is potentially less conservative than using the bounding technique in which $\Omega(\cdot)$ is required to satisfy (12) for *all* nonnegative-definite Q, not just for $Q = \beta I$. This particular structure for Q thus corresponds to energy equipartition as discussed above.

Covariance Averaging. Suppose that in the uncertainty model (22) for ΔA the uncertain parameters $\sigma = (\sigma_1, \ldots, \sigma_r)$ are modeled as constant random variables with a given probability

distribution. Then the covariance $Q(\sigma)$ given by

$$0 = (A + \Delta A)Q(\sigma) + Q(\sigma)(A + \Delta A)^{\mathrm{T}} + V \tag{39}$$

is also a random variable. The expected value of $Q(\sigma)$ defined by $\bar{Q} = \mathbb{E}[Q(\sigma)]$ can thus be viewed as the "average" covariance over the uncertainty model. Although an exact characterization of $\bar{Q}$ is difficult to obtain, approximations to $\bar{Q}$ can be shown[41] to satisfy equations similar to (27).

Given the modified Lyapunov equation (27), the ME/OP design equations can be derived in a straightforward manner following the technique given in Ref. 15. Hence consider the nominal plant

$$\dot{x} = Ax + Bu + D_1 w_1, \quad x \in \mathbb{R}^{n_x}, \quad u \in \mathbb{R}^{n_u}, \quad w_1 \in \mathbb{R}^{n_{w_1}} \tag{40}$$

$$y = Cx + w_2, \quad y \in \mathbb{R}^{n_y}, \tag{41}$$

$$z = E_1 x, \quad z \in \mathbb{R}^{n_z}, \tag{42}$$

where y is the sensor output, z is the performance variable, and w_1 and w_2 are (for convenience only) uncorrelated white noise disturbances with intensities $V_1 \geq 0$ and $V_2 > 0$, respectively. Also, consider the H_2 cost functional

$$J(A_c, B_c, C_c) = \lim_{t \to \infty} \mathbb{E}[x^{\mathrm{T}} R_1 x + u^{\mathrm{T}} R_2 u], \tag{43}$$

where $R_1 = E_1^{\mathrm{T}} E_1$ and $R_2 > 0$. The matrices A_c, B_c and C_c characterize the n_cth-order dynamic compensator $(n_c \leq n)$

$$\dot{x}_c = A_c x_c + B_c y, \quad x_c \in \mathbb{R}^{n_c}, \tag{44}$$

$$u = -C_c x_c. \tag{45}$$

Optimization of the performance functional (43) with the modified covariance model (27) applied to the closed-loop system yields dynamic compensator gains

$$A_c = \Gamma(A_s - Q\bar{\Sigma} - \Sigma P)G^{\mathrm{T}}, \tag{46}$$

$$B_c = \Gamma Q C^{\mathrm{T}} V_2^{-1}, \tag{47}$$

$$C_c = R_2^{-1} B^{\mathrm{T}} P G^{\mathrm{T}}, \tag{48}$$

where the $n \times n$ nonnegative-definite matrices $Q, P, \hat{Q}, \hat{P}$ satisfy

$$0 = A_s Q + Q A_s^{\mathrm{T}} + \bar{V}_1 + \sum_{i=1}^{r} \delta_i A_i (Q + \hat{Q}) A_i^{\mathrm{T}} - Q\bar{\Sigma}Q + \tau_\perp Q \bar{\Sigma} Q \tau_\perp^{\mathrm{T}}, \tag{49}$$

$$0 = A_s^{\mathrm{T}} P + P A_s + R_1 + \sum_{i=1}^{r} \delta_i A_i^{\mathrm{T}} (P + \hat{P}) A_i - P\Sigma P + \tau_\perp^{\mathrm{T}} P \Sigma P \tau_\perp, \tag{50}$$

$$0 = (A_s - \Sigma P)\hat{Q} + \hat{Q}(A_s - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{51}$$

$$0 = (A_s - Q\bar{\Sigma})^{\mathrm{T}} \hat{P} + \hat{P}(A_s - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}} P \Sigma P \tau_\perp, \tag{52}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c, \tag{53}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}, \quad (\cdot)^{\#} \text{denotes the group generalized inverse} \tag{54}$$

and τ has the factorization

$$\tau = G^{\mathrm{T}}\Gamma, \quad G, \Gamma \in \mathbb{R}^{n_c \times n}. \tag{55}$$

Here,

$$A_s \triangleq A + \sum_{i=1}^{r} \frac{1}{2}\delta_i A_i^2, \quad \Sigma \triangleq BR_2^{-1}B^{\mathrm{T}}, \quad \bar{\Sigma} \triangleq C^{\mathrm{T}}V_2^{-1}C, \quad \bar{V}_1 \triangleq D_1V_1D_1^{\mathrm{T}}. \tag{56}$$

The matrix τ defined in (54) is the optimal projection matrix which is responsible for enforcing the reduced-order constraint $n_c \le n$ on the compensator.

The ME/OP design equations (49)-(54) can be solved by using a homotopy algorithm[21,42]. As illustrated by Figure 1, this homotopy algorithm allows the deformation of an LQG controller into a full-order Maximum Entropy controller. The Maximum Entropy controller is then reduced to an appropriate order by using an indirect controller reduction method. It is important that this initial reduced-order controller approximately solve the ME/OP design equations to within a small error, although it is not required to be a stabilizing controller. This can be achieved by beginning with a low authority LQG design and/or incorporating a sufficiently high level of uncertainty in the ME design. In practice a slight modification of the balanced controller reduction algorithm of Yousuff and Skelton[43] is currently used as the indirect controller reduction method. Once this initial reduced-order controller is obtained, the homotopy algorithm is used to deform this controller into a ME/OP controller. Then, if a higher authority controller is desired, the final step of the algorithm is to increase the controller authority to a desirable level.

IV. A Benchmark Problem

We now turn our attention to the benchmark problem of Ref. 19. The dynamical system is described and two associated design problems are presented.

Consider the two-mass/spring system shown in Figure 2, which is a generic model of an uncertain dynamical system with a noncolocated sensor and actuator pair. A control force acts on body 1, and the position of body 2 is measured resulting in a noncolocated control problem. This system can be represented in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w_1 \tag{57}$$

$$y' = z = x_2, \quad y = y' + v \tag{58}$$

where

$x_1 =$ position of body 1

$x_2 =$ position of body 2

$x_3 =$ velocity of body 1

$x_4 =$ velocity of body 2

$u =$ control input

$w_1 =$ plant disturbance

$z =$ performance variable (output to be controlled)

$y' =$ noise-free measurement

$v =$ sensor noise

Design Problems

Design #1. Design a constant gain linear feedback compensator of the form

$$\dot{x}_c = A_c x_c + B_c y \tag{59}$$

$$u = C_c x_c + D_c y \tag{60}$$

(any of these matrices may of course be zero) with the following properties:

i) The closed-loop system is stable for $m_1 = m_2 = 1$ and $0.5 < k < 2.0$.

ii) For $w(t) =$ unit impulse at $t = 0$, the performance variable z has a settling time of about 15 seconds for the nominal system $m_1 = m_2 = k = 1$.

iii) The control system can tolerate reasonable measurement noise signals $v(t)$.

iv) Achieve reasonable performance/stability robustness with reasonable bandwidth.

v) Use reasonable controller effort.

vi) Use reasonable controller complexity.

Design #2. Same as Design #1 except in place of *ii)* insert:

ii') $w(t)$ is a sinusoidal disturbance of frequency 0.5 rad/sec but whose amplitude and phase, although constant, are not available to the designer. Achieve asymptotic rejection of $w(t)$ at the performance variable $z(t)$ (i.e., minimize $\limsup_{t\to\infty} z(t)$ with a 20 second settling time) for $m_1 = m_2 = 1$, $0.5 < k < 2.0$.

V. ME/OP Control Design for the Benchmark Problem

This section considers design problems #1 and #2 of the benchmark problem described in the previous section. In particular, the development of robust controllers using the ME/OP approach is described. This approach was first applied to the benchmark problem in Refs. 44 and 45.

We begin by introducing some notation. Consider the plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w_1(t), \tag{61}$$

$$z(t) = E_1 x(t), \tag{62}$$

$$y'(t) = Cx(t). \tag{63}$$

Then $G(s)$ is said to be the ***transfer matrix representation*** of (57)–(59) if

$$\begin{bmatrix} z(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} G(s) \end{bmatrix} \begin{bmatrix} w_1(s) \\ u(s) \end{bmatrix}. \tag{64}$$

Likewise (A, B, C, D_1, E_1) is said to be the ***state representation*** of (60) if (57)–(59) hold.

Now, for a nominal value k_{nom} of the spring stiffness let the corresponding state representation of the benchmark system shown in Figure 2 be given by

$$\dot{x}(t) = A_0(k_{\mathrm{nom}})x(t) + B_0 u(t) + D_{1,0} w_1(t), \tag{65}$$

$$z(t) = E_{1,0} x(t), \tag{66}$$

$$y' = z(t). \tag{67}$$

Also, let $G_0(s)$ be the transfer matrix representation of (65)–(67).

A precompensation strategy was used for control law design. This precompensation strategy is illustrated by Figures 3 and 4. As shown in Figure 3 we simply embed the precompensation filters $C_u(s)$, $C_{y'}(s)$, $C_{w_1}(s)$ and $C_z(s)$ in the plant a priori and design the ME/OP controller $\hat{H}(s)$ for this modified design plant. Then, as illustrated in Figure 4, the precompensation dynamics $C_u(s)$ and $C_{y'}(s)$ are included in the implemented compensator $H(s)$. It is not difficult to show that for both Figures 3 and 4 the closed-loop transfer function $G_{c\ell}(s)$ satisfying

$$\begin{bmatrix} z(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} G_{c\ell}(s) \end{bmatrix} \begin{bmatrix} w_1(s) \\ u(s) \end{bmatrix} \tag{68}$$

is identical in Figures 3 and 4. Hence this methodology ensures that if $\hat{H}(s)$ is stabilizing in the feedback loop of Figure 3, then $H(s)$ is stabilizing in the feedback loop of Figure 4. In addition, (64) also ensures that the transfer function between $z(s)$ and $w_1(s)$ is preserved, thus ensuring the preservation of the attenuation from w_1 to z. This precompensation methodology was used in Ref. 9 to achieve controller roll-off and to force the design plant to appear to be rate feedback. Its use for the benchmark problem is detailed below.

To describe the control design process for each controller, assume that (A, B, C, D_1, E_1) is the state space representation corresponding to $\hat{G}(s)$ in Figure 3, such that

$$\dot{x}(t) = Ax(t) + B\hat{u}(t) + D_1\hat{w}_1(t), \tag{69}$$

$$\hat{z}(t) = E_1 x(t), \tag{70}$$

$$\hat{y}'(t) = Cx(t), \tag{71}$$

$$\hat{y}(t) = \hat{y}'(t) + \hat{w}_2(t), \tag{72}$$

is the state space representation of the design plant. The synthesis of $\hat{H}(s)$ in Figure 3 was based upon the solution of the design equations (49)–(54).

The state space basis of (69)–(72) was chosen such that A is block-diagonal with a 2×2 diagonal block of the form $\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}$ corresponding to the vibrational mode of the system with nominal natural frequency ω_0. System uncertainty was assumed to be in the frequency of this mode and thus the parameter r in eqns. (49), (50) and (56) is given by $r = 1$. The corresponding uncertainty pattern matrix A_1 was given by

$$A_1 = \text{block-diag}(0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0) \tag{73}$$

where location of the nonzero 2×2 block corresponded to the location of the dynamics of the vibrational mode in A. The design weights, $R_1, R_2, \bar{V}_1$ and V_2 were given by

$$R_1 = E_1^{\mathrm{T}} E_1, \ \bar{V}_1 = D_1 D_1^{\mathrm{T}}, \ R_2 = V_2 = \rho. \tag{74}$$

The design parameters are ρ, which determines the control authority and $\delta_1^{1/2}$, which weights A_1 and reflects the level of modal uncertainty. Note that since the stiffness k was assumed to be in the interval [0.5 N/m, 2.0 N/m], the natural frequency ω of the vibrational mode was in the interval [1.0 rad/s, 2.0 rad/s].

Three controllers are described below. Controllers 1 and 2 were developed to meet the objectives of design problem #1 while Controller 3 was developed for problem #2. Controllers 1 and 3 were

formulated as standard H_2 disturbance rejection problems, i.e., $C_u(s) = C_y(s) = 1$ in Figures 3 and 4. The disturbance weighting matrix $C_{w_1}(s)$ was chosen in the control design process for Controller 3 to reflect knowledge of the sinusoidal nature of the disturbance. For Controller 2, precompensation was added to nullify the effects of the rigid body mode in a frequency band approximately one decade above and one decade below the frequency of the vibrational mode. Basically, this was an attempt to make the problem "easier" for the LQG part of the design. $C_u(s)$, as will be shown, is simply a second-order lead-lag filter. The "lag" poles were included not only to make the precompensation realizable in state-space, but also to prevent the LQG segment of the controller from having to provide additional roll-off. The description of each controller includes the precompensation dynamics used to developed the design model and the stiffness k_{nom} of the design model. The parameter k_{nom} was not chosen to be 1 N/m as might be expected because it was experimentally observed that as δ_1 increased the closed-loop system tended to become robust with respect to positive perturbations in k_{nom} faster than with respect to negative perturbations.

The settling time for each system was chosen to be the time required for the displacement of mass 2 to reach and stay within the interval [-0.1m, 0.1m]. Each of the controllers satisfied the corresponding settling time objectives when connected to the model corresponding to $k = 1$ N/m. Also, each of the controllers stabilizes the plant for $k \in$ [0.5 N/m, 2.0 N/m]. To illustrate the effectiveness of Maximum Entropy design in inducing robustness, each of the three controllers is compared with the LQG design which was used to initialize the design process illustrated by Figure 1. The gain margin (GM) and phase margin (PM) listed for each controller are the margins yielded by implementing each controller with the corresponding design plant. In addition, for Controllers 1 and 2 a simulation is provided which shows the mass 2 displacement response when the sensor measurements are corrupted by a noise process $w_2(t)$. In these simulations the noise was chosen to be white with a uniform distribution in the interval [-.01m, .01m].

Controller 1 (for Design Problem #1)

The parameters for Controller 1 are as follows:

$$C_u(s) = C_{y'}(s) = C_{w_1}(s) = C_z(s) = 1$$

$$\rho = .00001, \quad \delta_1^{1/2} = .2$$

$$k_{\text{nom}} = 0.6 \text{ N/m} \quad \Rightarrow \quad \omega_0 = 1.0954 \text{ rad/s}$$

$$\text{order of } \hat{G}(s) \;=\; \text{order of } \hat{H}(s) \;=\; 4 \quad \Rightarrow \quad \text{full-order design}$$

$$\text{settling time of the mass 2 displacement} = 15\text{s} \quad (\text{for } k = 1)$$

$$\text{peak response of the mass 2 displacement} = .7\text{m} \quad (\text{for } k = 1)$$

$$H(s) = \frac{194390(s + 0.36679)[(s - 0.11735)^2 + 0.90996^2]}{(s + 81.438)(s + 131.04)[(s + 2.9049)^2 + 1.8615^2]}$$

$$\text{Controller 1: stable for } 0.45 \leq k \leq 2.05, \text{GM=3dB, PM} = 10\,\text{deg}$$

$$\text{LQG Controller: stable for } 0.59 \leq k \leq 1.06, \text{GM} = 1\text{dB, PM} < 1\,\text{deg}.$$

The impulse responses of the mass 2 displacement (the output performance variable), the mass 1 displacement, and the control signal are shown respectively in Figures 5–7 for $k = 1.0$ N/m, 0.5 N/m and 2.0 N/m. The impulse response of the displacement of mass 2 for $k = 1.0$ N/m with noise corrupted measurements is shown in Figure 8. The root locus with respect to the controller gain is shown in Figure 9 while the Nyquist plot of the loop transfer function is shown in Figure 11.

Controller 2 (for Design Problem #2)

The parameters for Controller 2 are as follows:

$$C_u(s) = \frac{100[(s + .04)^2 + .0693^2]}{(s + 10)^2 + 17.3205^2}, \quad C_{y'}(s) = C_{w_1}(s) = C_x(s) = 1.$$

$$\rho = .0001 \quad \alpha = .2$$

$$k_{\text{nom}} = 0.6 \text{ N/m} \quad \Rightarrow \quad \omega_0 = 1.0954 \text{ rad/s}$$

$$\text{order of } \hat{G}(s) = 6, \text{ order of } \hat{H}(s) = 4 \quad \Rightarrow \quad \text{reduced-order design}$$

$$\text{settling time of the mass 2 displacement} = 5\text{s} \quad (\text{for } k = 1)$$

$$\text{peak response of the mass 2 displacement} = .2\text{m} \quad (\text{for } k = 1)$$

$$H(s) = \frac{2490300(s + 0.93838)[(s + 0.30989)^2 + 0.40237^2][(s + 0.040000)^2 + 0.069282^2]}{(s + 54.835)(s + 18.831)[(s + 10.000)^2 + 17.321^2][(s + 0.014561)^2 + 0.031241^2]}$$

$$\text{Controller 2: stable for } .12 \leq k \leq 2.03, \text{GM=6dB, PM} = 33\,\text{deg}.$$

$$\text{LQG Controller: stable for } .45 \leq k \leq 1.35, \text{ GM=7dB, PM=35 deg}$$

The impulse responses of the mass 2 displacement (the output performance variable), the mass 1 displacement and the control signal are shown respectively in Figures 5–7 for $k = 1.0$ N/m, 0.5 N/m and 2.0 N/m. The impulse response of the displacement of mass 2 for $k = 1.0$ N/m with noise corrupted measurements is show in Figure 8. The root locus with respect to the controller gain is shown in Figure 10 while the Nyquist plot of the loop transfer function is shown in Figure 12.

Controller 3 (for Design Problem #2)

The parameters for Controller 3 are as follows:

$$C_u(s) = C_{y'}(s) = C_z(s) = 1, \quad C_{w_1}(s) = \frac{1}{(s+0.00050)^2 + 0.500000^2}$$

$$k_{\text{nom}} = 0.75 \text{ N/m} \quad \Rightarrow \quad \omega_0 = 1.2247 \text{ rad/s}$$

$$\text{order of } \hat{G}(s) = \text{ order of } \hat{H}(s) = 6 \quad \Rightarrow \quad \text{full-order design}$$

$$\text{settling time of the mass 2 displacement} = 12\text{s} \quad (\text{for } k=1)$$

$$\text{peak response of the mass 2 displacement} = 4.4\text{m} \quad (\text{for } k=1)$$

$$H(s) = \frac{86816(s+0.12320)[(s-0.21281)^2+0.96330^2][(s+0.023916)^2+0.42427^2]}{(s+253.19)(s+38.684)[(s+2.5063)^2+1.6776^2][(s+0.0011218)^2+0.50138^2]}$$

$$\text{Controller 3: stable for } 0.48 \le k \le 2.50, \text{GM=5dB, PM=22 deg}$$

$$\text{LQG Controller: stable for } 0.43 \le k \le 0.78, \text{GM=4dB, PM} = 22\,\text{deg}.$$

The responses of the mass 2 displacement (the output performance variable), the mass 1 displacement and the control signal to a sinusoidal disturbance of frequency 0.5 rad/s are shown respectively in Figures 13–15 for $k = 1.0$ N/m, 0.5 N/m and 2.0 N/m. The root locus with respect to the controller gain is shown in Figure 16.

VI. Discussion of Results

Controllers 1 and 2 both satisfied the settling time objectives of design problem #1 but differed significantly in their basic structure and overall performance. One of the primary differences between these two controllers is best illustrated by the Nyquist diagrams presented in Figures 11 and 12 of the corresponding loop transfer functions. In these figures it is seen that both controllers placed the loop transfer function in the first quadrant just before the nominal frequency ω_0. This ensured stability when 180° of phase lag was added due to the undamped vibrational mode of the plant. However, despite this similarity, the route that the controllers took to place the loop transfer function in the first quadrant was vastly different. As seen in Figure 11, Controller 1 chose to force the loop transfer function to reach the first quadrant via the second quadrant. This required substantial phase lag from the compensator. However, compensator gain was also needed to achieve performance. Hence, Controller 1 included a complex pair of nonminimum phase zeros to achieve both phase lag and increased gain. On the other hand, the precompensation chosen for Controller 2 was such that the loop transfer function reached the first quadrant via the third

and fourth quadrants. The desired lead and gain increase were accomplished via minimum phase compensator zeros.

Figures 5 compares the impulse response of mass 2 for Controllers 1 and 2 while Figure 6 compares the impulse response of mass 1 for the two controllers. It is easily seen from these figures that Controller 2 reduced the effects of the impulse disturbance on the system much more than Controller 1. In addition, it is seen in the data listed for each controller that Controller 2 has significantly higher gain and phase margins than Controller 1, a feature that might be desirable in some applications. However Figure 7 reveals that the greater performance of Controller 2 was achieved at the expense of much greater control authority and Figure 8 shows that, due to its higher bandwidth, Controller 2 yields a closed-loop system that is much more sensitive to sensor noise. In fact, the control authority required by Controller 2 would likely require an actuator with mass many times the total mass of the original system. Controllers 1 and 3 avoided the use of an impractical amount of actuator authority by the judicious use of nonminimum phase zeros. We conjecture that a low authority controller that meets the performance objectives cannot be designed for the benchmark problem without using nonminimum phase zeros.

It is interesting to note that the data for Controller 2 shows that the initializing LQG controller has larger gain and phase margins than the robust controller, but also has less robustness with respect to the uncertainty in the stiffness k. This provides an illustration of a control problem in which the gain and phase margins do not necessarily imply parametric robustness.

The data for each of the three controllers clearly shows that the initializing LQG designs did not satisfy the robustness requirements with respect to the uncertainty in k. The efficacy of Maximum Entropy design in providing the needed robustness was thus clearly illustrated by the results of this paper.

Acknowledgments

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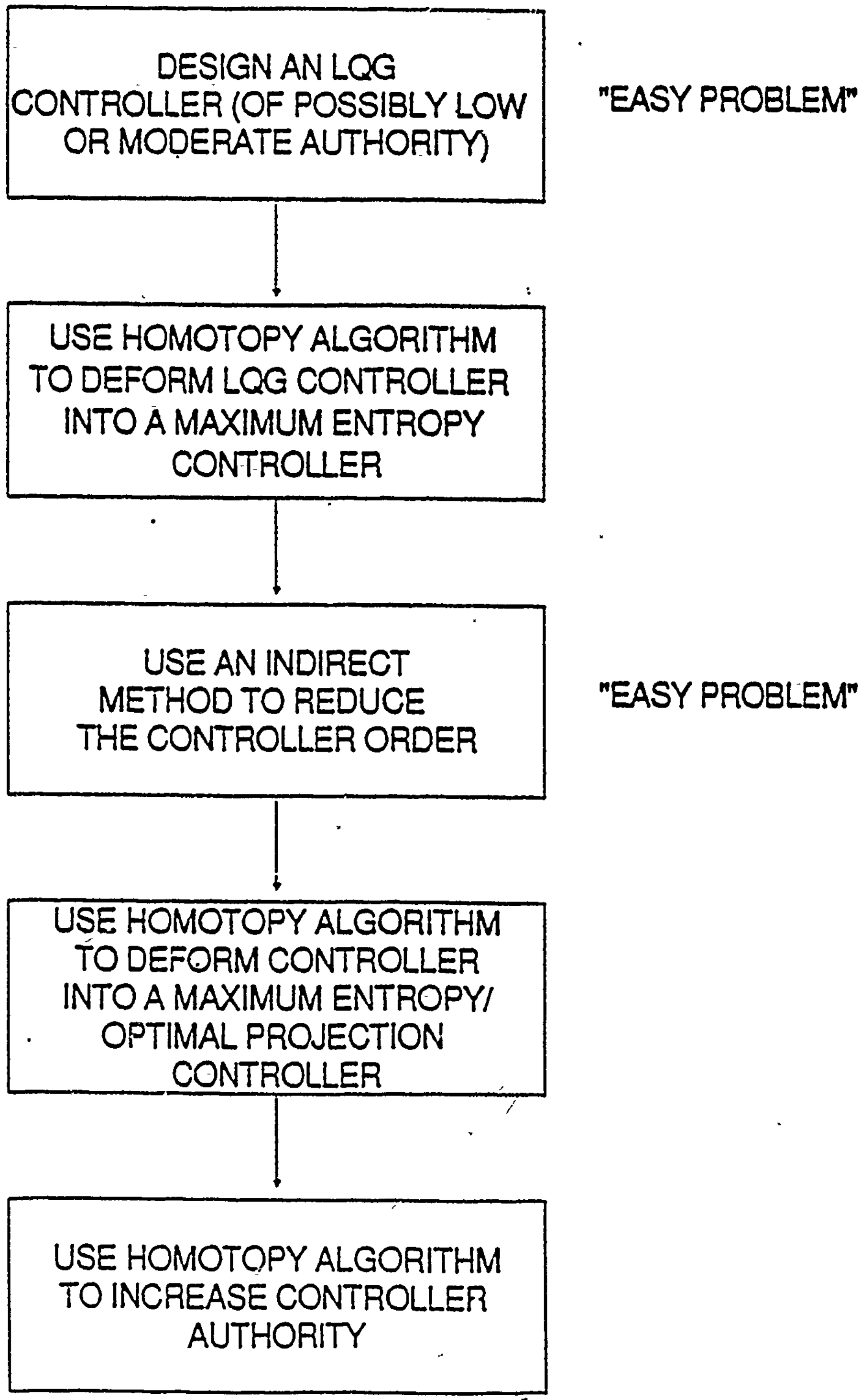


Fig. 1. Practical Application of the Maximum Entropy/Optimal Projection Control-Design Algorithm.

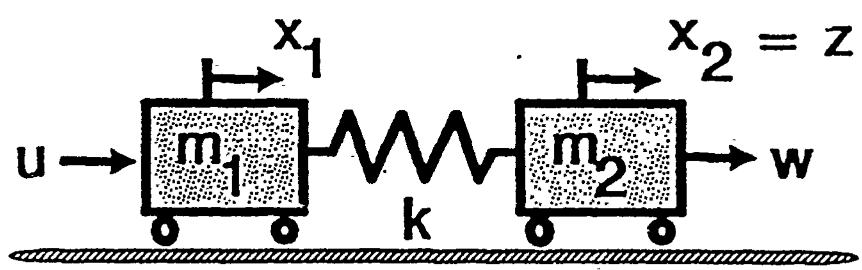


Fig. 2. The Benchmark System

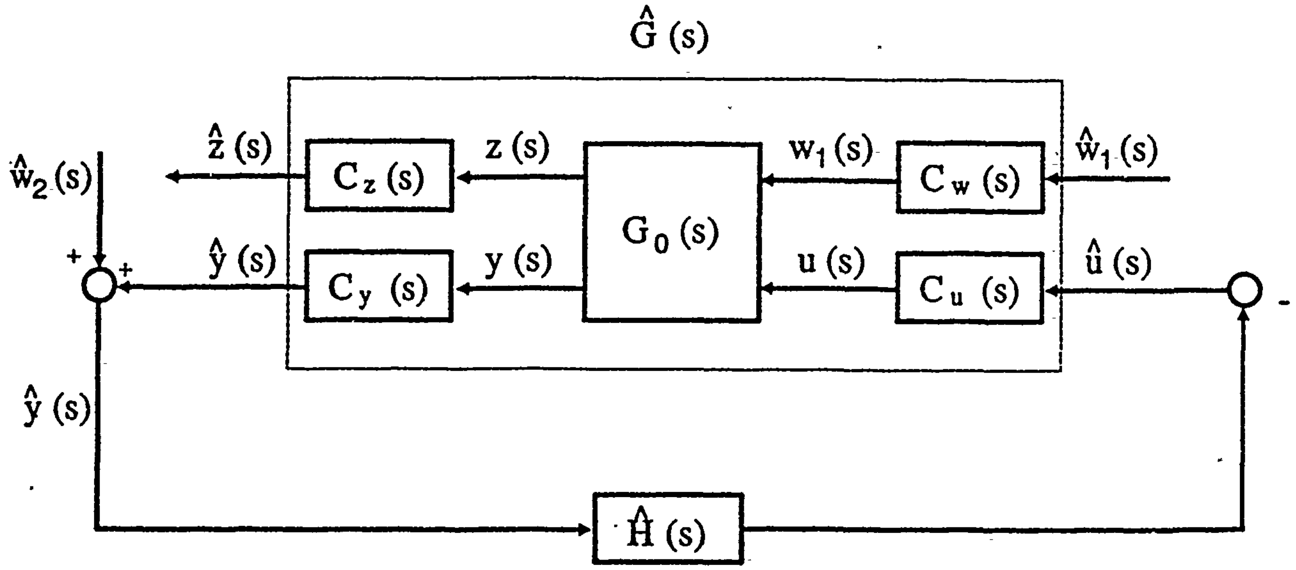


Fig. 3. Design Configuration for the Precompensation Methodology

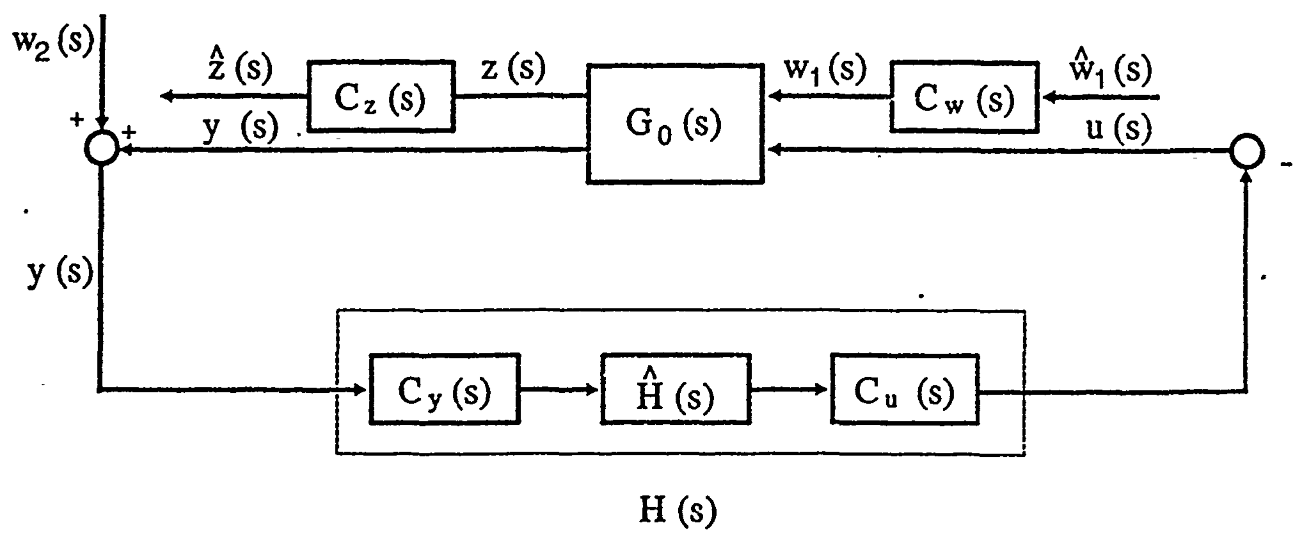


Fig. 4. Implementation Configuration for the Precompensation Methodology

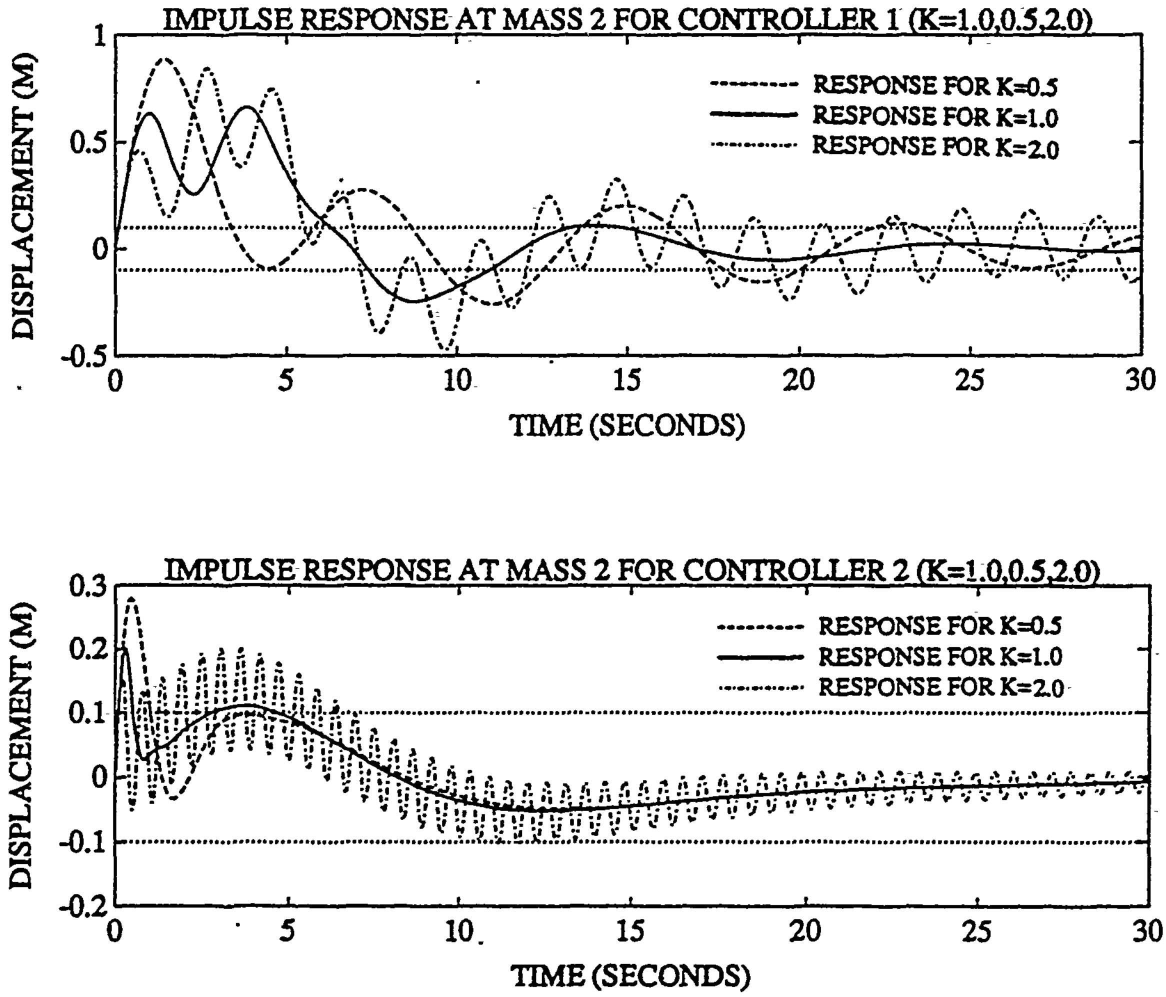
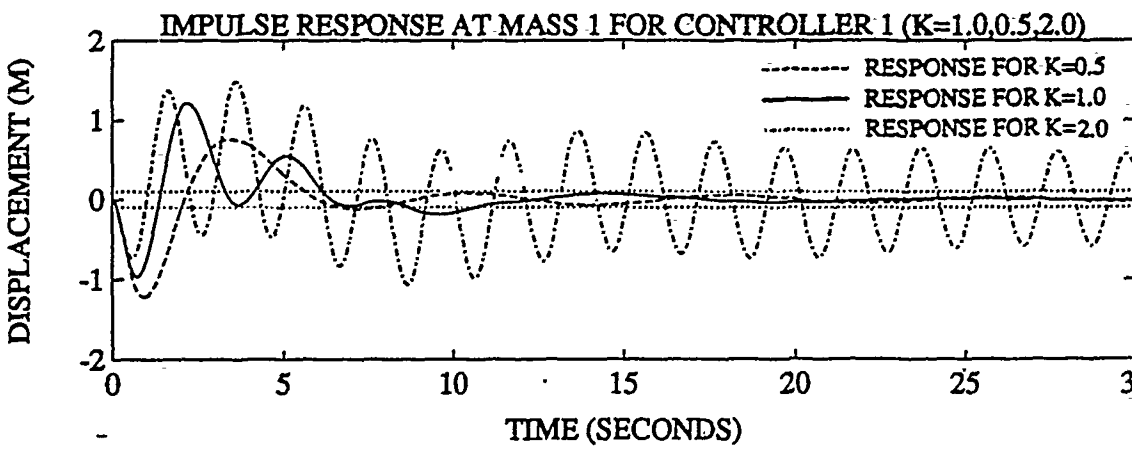


Fig. 5. Mass 2 Displacement Response to an Impulse Disturbance for Controller 1 (top) and Controller 2 (bottom)



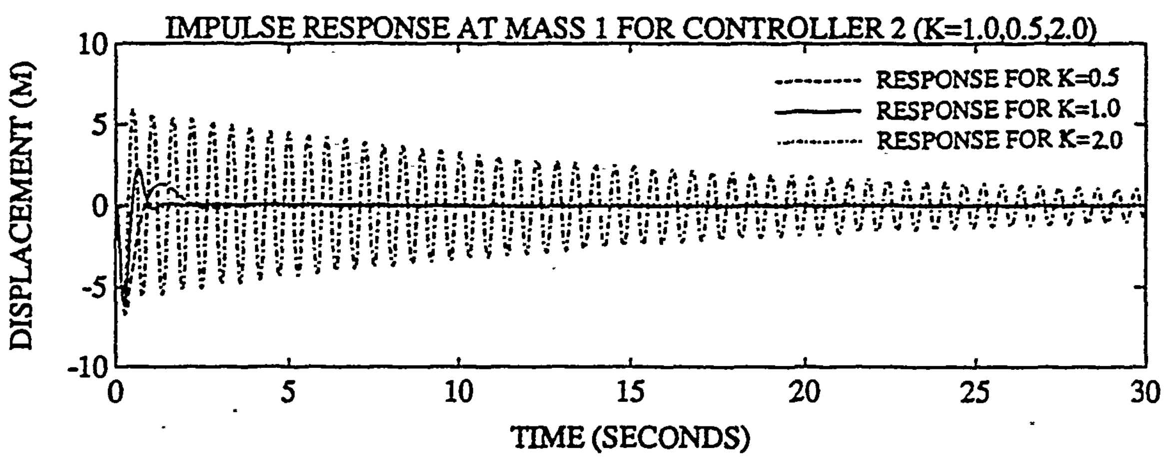


Fig. 6. Mass 1 Displacement Response to an Impulse Disturbance for Controller 1 (top) and Controller 2 (bottom)

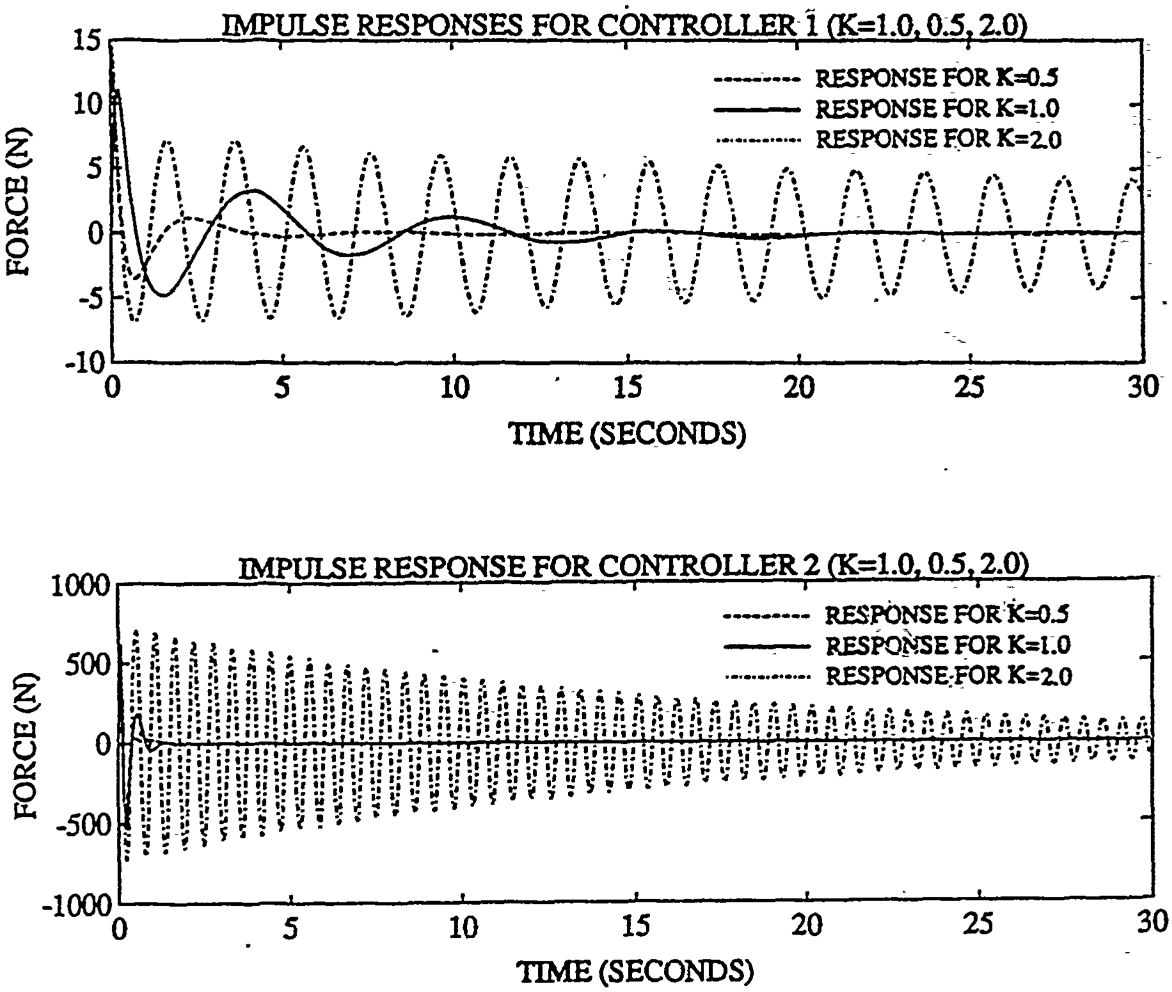


Fig. 7. Control Signal Response to an Impulse Disturbance for Controller 1 (top) and Controller 2 (bottom)

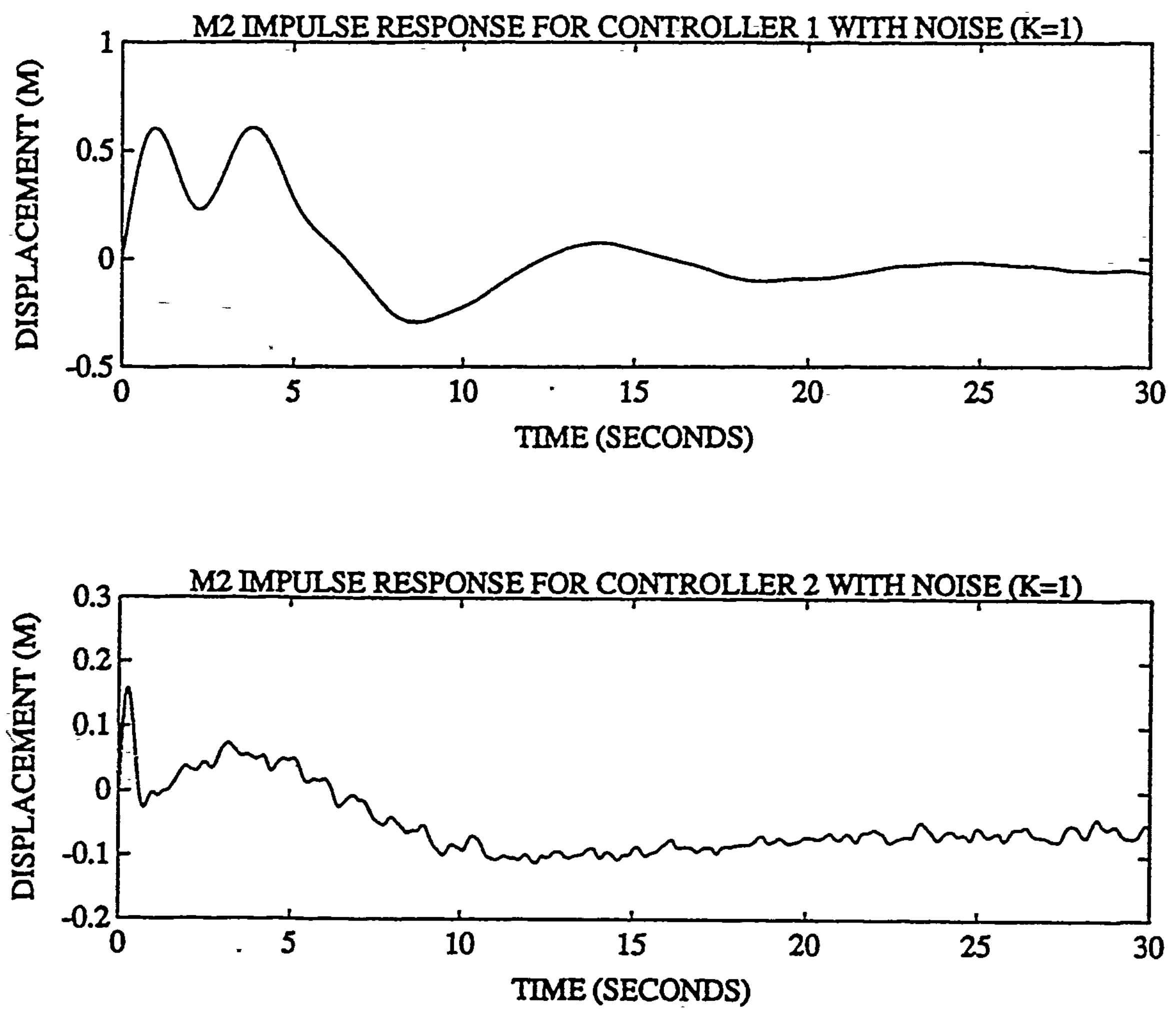


Fig. 8. Mass 2 Displacement Response to an Impulse Disturbance with Sensor Noise for Controller 1 (top) and Controller 2 (bottom)

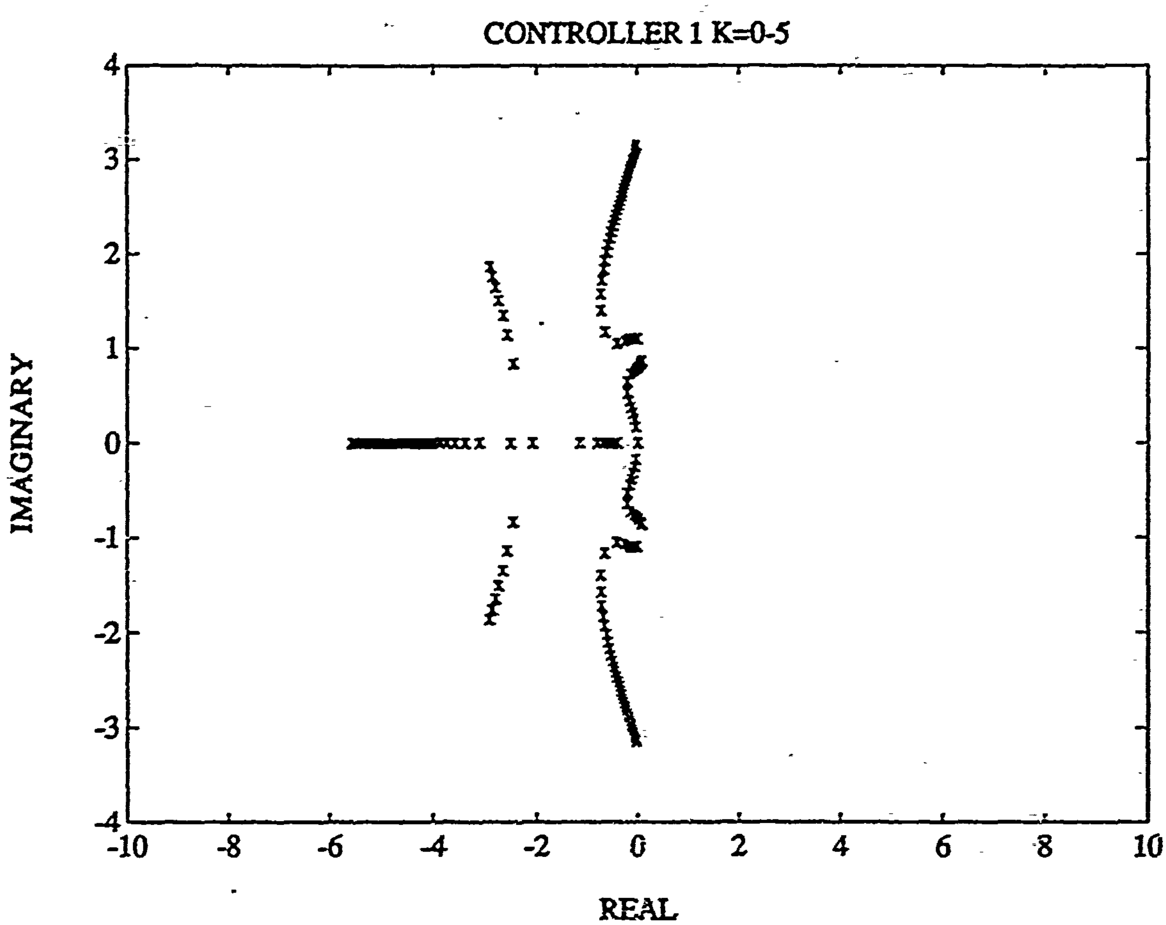


Fig. 9. Root Locus for Gain of Controller 1

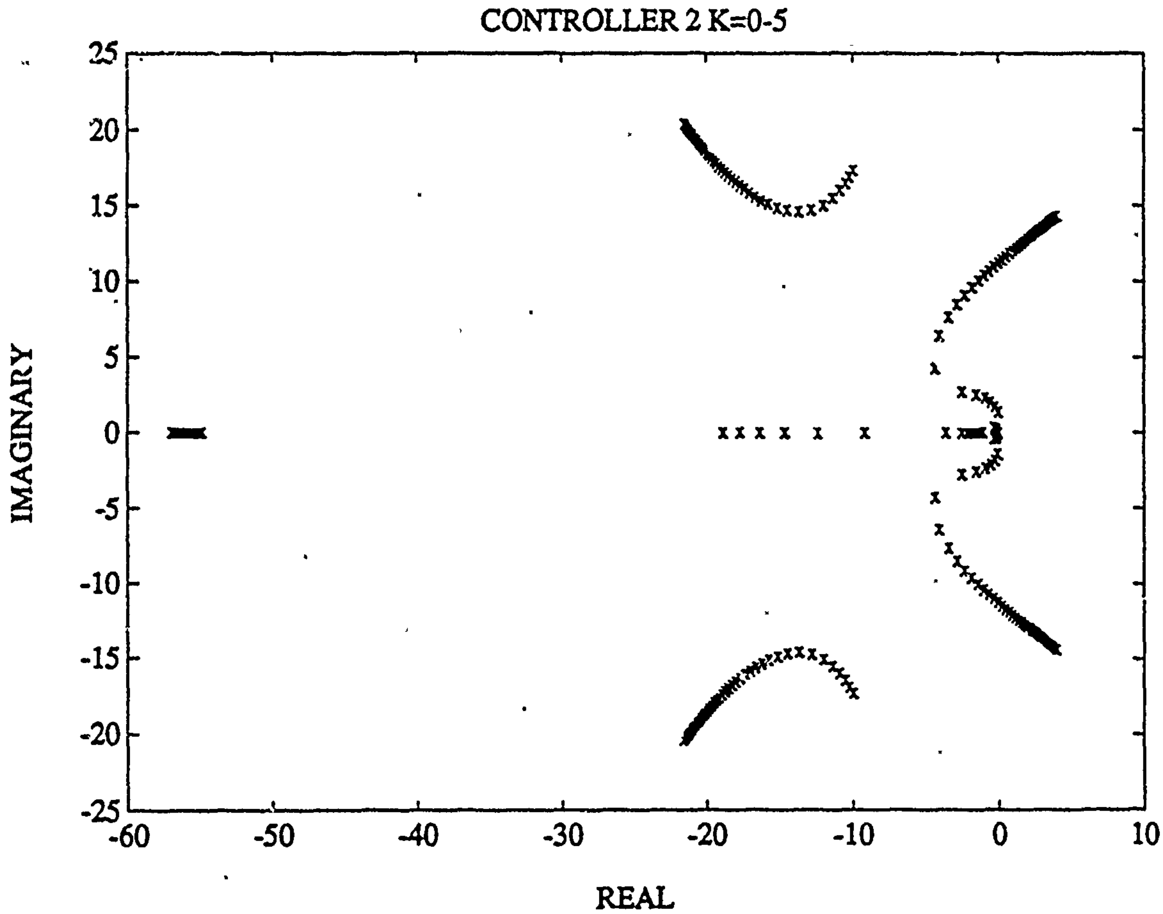


Fig. 10. Root Locus for Gain of Controller 2

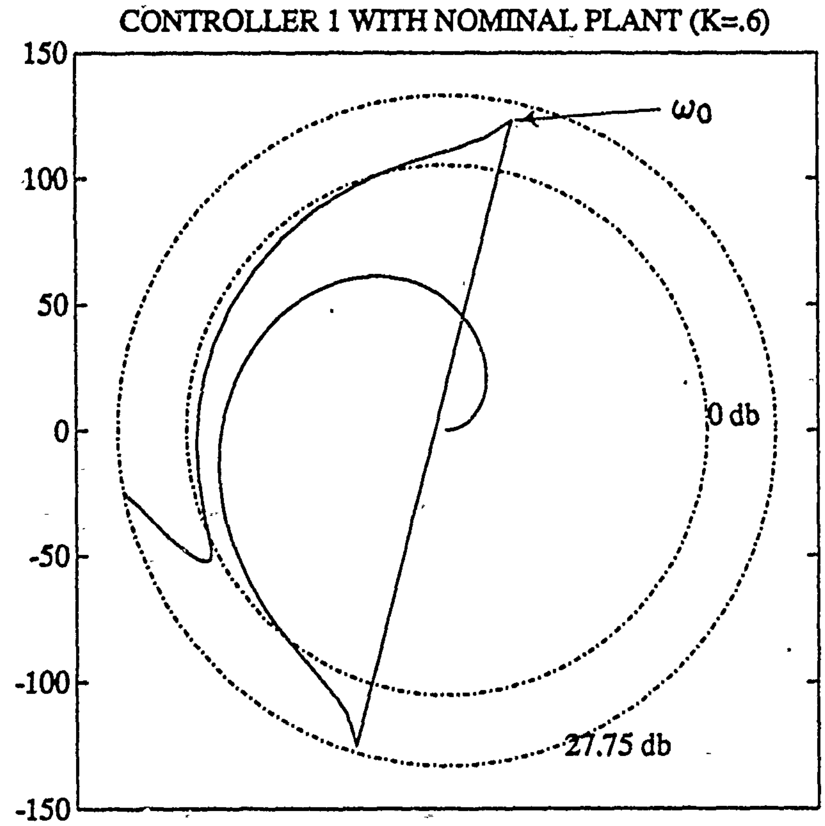


Fig. 11. Nyquist Plot of Loop Transfer Function for Controller 1

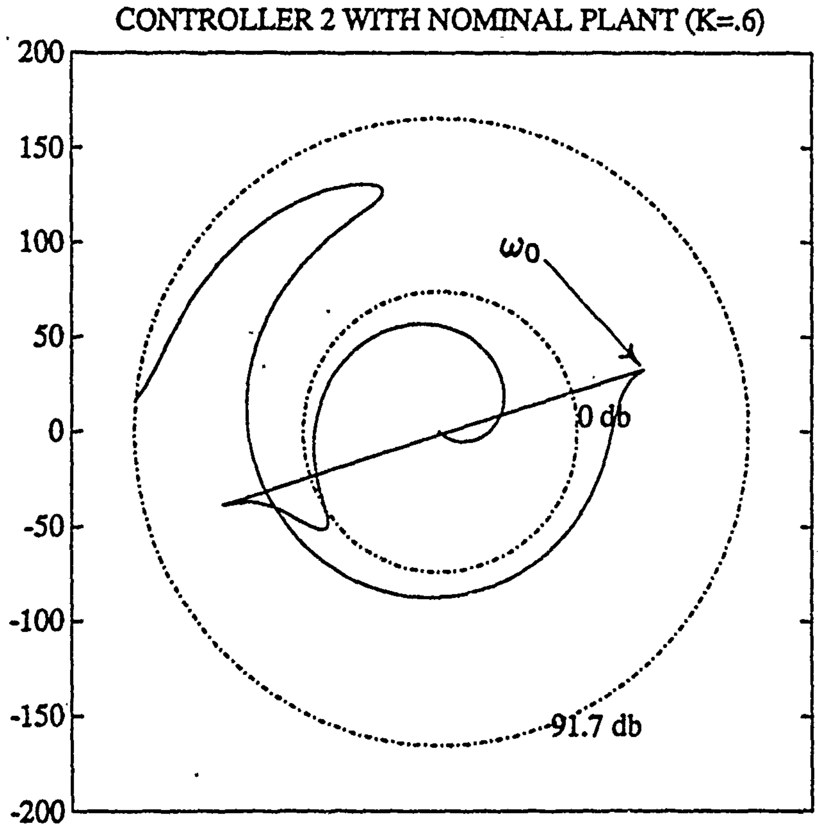


Fig. 12. Nyquist Plot of Loop Transfer Function for Controller 2

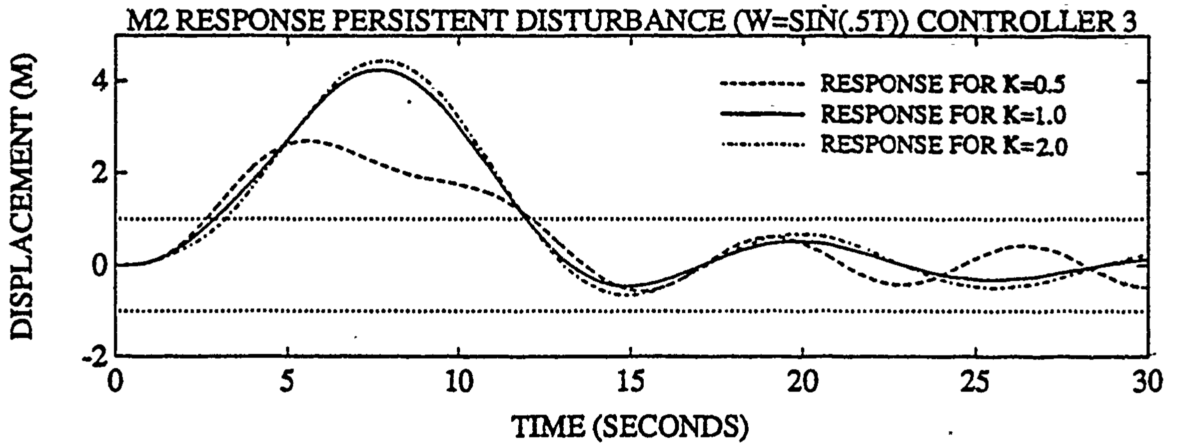


Fig. 13. Mass 2 Displacement Response to a .5 rad/s Sinusoidal Disturbance for Controller 3

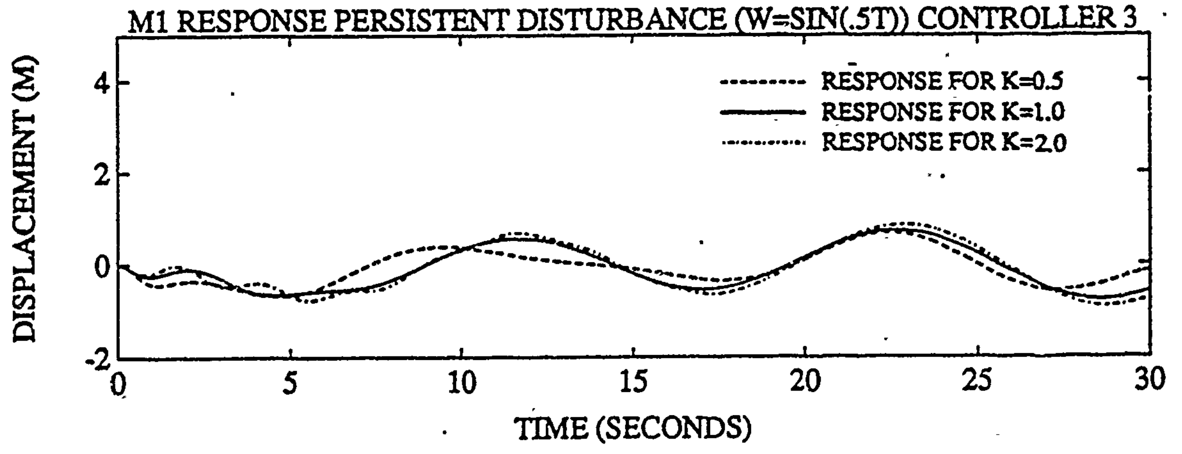


Fig. 14. Mass 1 Displacement Response to a .5 rad/s Sinusoidal Disturbance for Controller 3

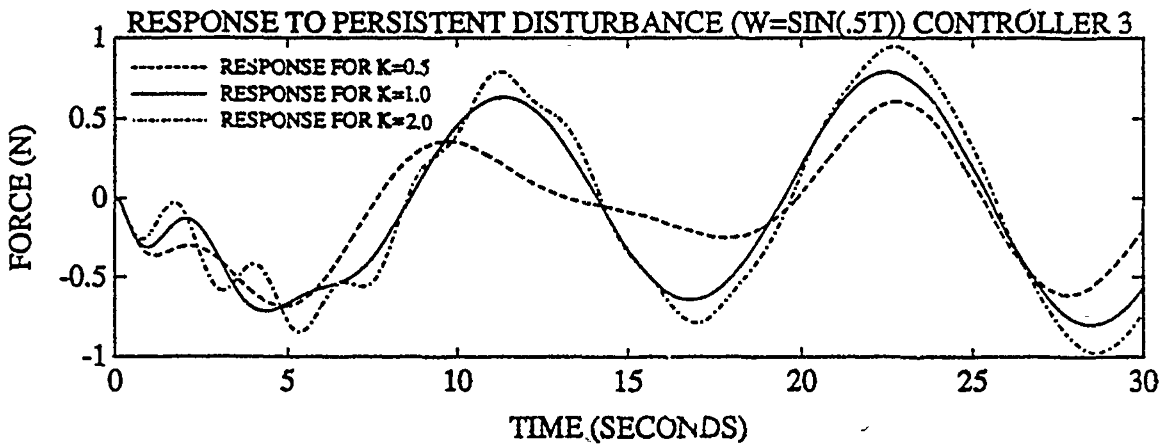
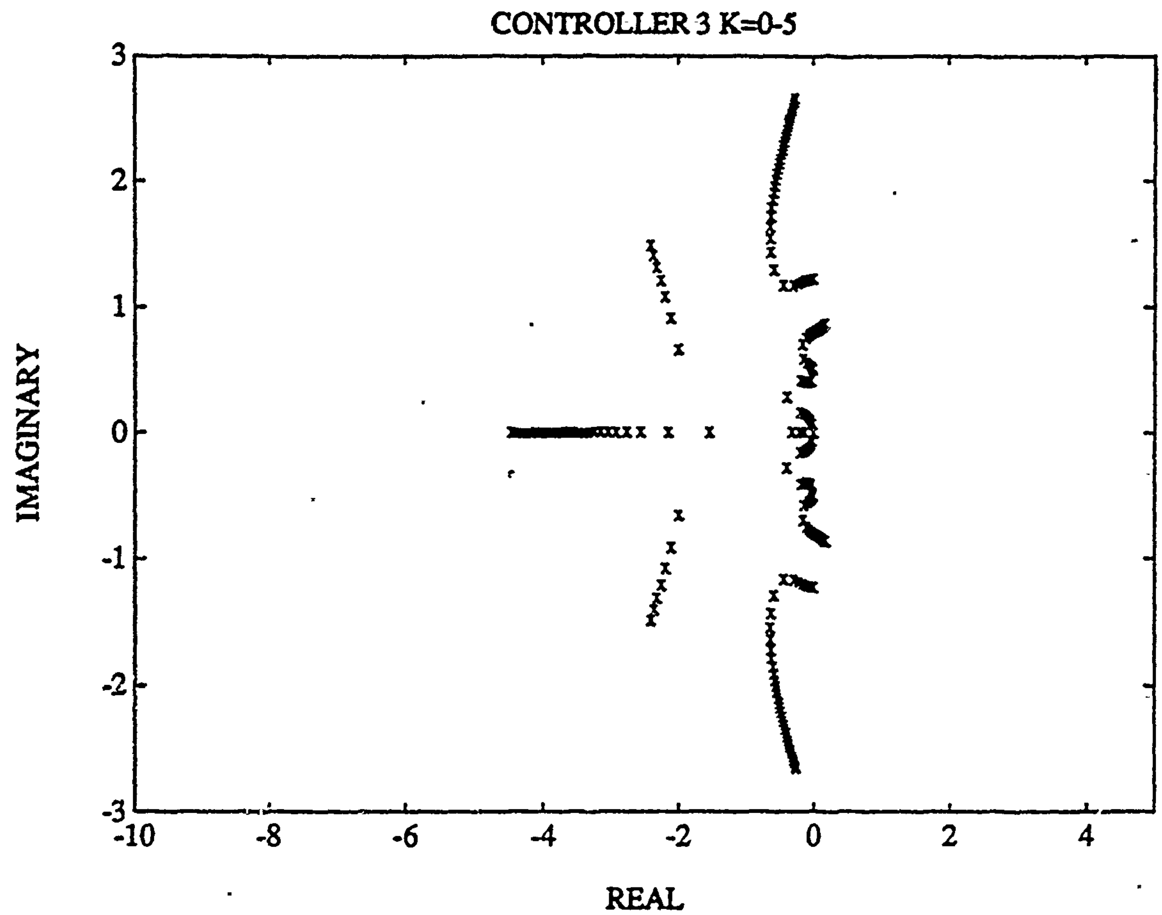


Fig. 15. Control Signal Response to a .5 rad/s Sinusoidal Disturbance for Controller 3



Fig, 16. Root Locus for Gain of Controller 3

Appendix B

INT. J. CONTROL, 1989, VOL. 49, NO. 5, 1523–1535

Combined L_2/H_∞ model reduction

WASSIM M. HADDAD† and DENNIS S. BERNSTEIN‡

A model-reduction problem is considered which involves both L_2 (quadratic) and H_∞ (worst-case frequency-domain) aspects. Specifically, the goal of the problem is to minimize an L_2 model-reduction criterion subject to a prespecified H_∞ constraint on the model-reduction error. The principal result is a sufficient condition for characterizing reduced-order models with bounded L_2 and H_∞ approximation error. The sufficient condition involves a system of modified Riccati equations coupled by an oblique projection, i.e. idempotent matrix. When the H_∞ constraint is absent, the sufficient condition specializes to the L_2 model-reduction result given by Hyland and Bernstein (1985).

Notation and definitions

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
$I_r, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$r\times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
$(\ )^*$	complex conjugate transpose
tr	trace
$\sigma_{\max}(Z)$	largest singular value of matrix Z
$\lambda_{\max}(Z)$	largest eigenvalue of matrix Z with real spectrum
$\Vert Z\Vert_F$	$[\mathrm{tr}\, ZZ^*]^{1/2}$ (Frobenius matrix norm)
$\Vert h(t)\Vert_2$	$[\int_0^\infty \Vert h(t)\Vert_F^2\, dt]^{1/2}$
$\Vert H(s)\Vert_2$	$[(1/2\pi)\int_{-\infty}^{\infty} \Vert H(j\omega)\Vert_F^2\, d\omega]^{1/2}$
$\Vert H(s)\Vert_\infty$	$\sup_{\omega\in\mathbb{R}} \sigma_{\max}[H(j\omega)]$
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$r\times r$ symmetric, non-negative-definite, positive-definite matrices
$Z_1 \leqslant Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r, Z_2 - Z_1 \in \mathbb{P}^r, Z_1, Z_2 \in \mathbb{S}^r$
$n, m, l, n_m, q, p; \tilde{n}$	positive integers; $n + n_m$
$x, y, y_m, x_m, \tilde{y}, \tilde{x}$	$n, l, l, n_m, l, \tilde{n}$-dimensional vectors
$\tilde{y}, \tilde{x}$	$y - y_m, \begin{bmatrix} x \\ x_m \end{bmatrix}$
A, B, C	$n\times n, n\times m, l\times n$ matrices
D, E	$m\times p, q\times l$ matrices
A_m, B_m, C_m	$n_m\times n_m, n_m\times m, l\times n_m$ matrices
$\tilde{A}, \tilde{B}, \tilde{C}$	$\begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}, \begin{bmatrix} B \\ B_m \end{bmatrix}, [C \quad -C_m]$

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† Department of Mechanical Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

‡ Harris Corporation, Government Aerospace Systems Division, MS 22/4848, Melbourne, FL 32902, U.S.A.

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$\tilde{D}, \tilde{E}$ $\tilde{B}D = \begin{bmatrix} BD \\ B_m D \end{bmatrix}$, $E\tilde{C} = [EC \quad -EC_m]$

R $E^{\mathrm{T}}E$, model-reduction error-weighting matrix in $\mathbb{P}^l$

$w(\cdot)$ p-dimensional standard white noise process

V intensity of $Dw(\cdot)$, $V = DD^{\mathrm{T}} \in \mathbb{P}^m$

$\tilde{R}, \tilde{V}$ $\begin{bmatrix} C^{\mathrm{T}}RC & -C^{\mathrm{T}}RC_m \\ -C_m^{\mathrm{T}}RC & C_m^{\mathrm{T}}RC_m \end{bmatrix}, \begin{bmatrix} BVB^{\mathrm{T}} & BVB_m^{\mathrm{T}} \\ B_mVB^{\mathrm{T}} & B_mVB_m^{\mathrm{T}} \end{bmatrix}$

γ positive constant

1. Introduction

One of the most fundamental problems in dynamic systems theory is to approximate a high-order, complex system with a low-order, relatively simpler model. The resulting reduced-order model can then be used to facilitate the analysis of complex systems as well as the design and implementation of feedback controllers and electronic filters. The model-reduction problem thus reflects the fundamental engineering desire for simplicity of implementation and parsimony of hardware.

In view of the practical motivations for the model-reduction problem, it is not surprising that significant effort has been devoted to this problem in recent years. Indeed, there now exists a well-developed theoretical foundation for model reduction under a variety of approximation criteria. Expanding on the original work of Adamjan *et al.* (1971), progress was achieved by Kung and Lin (1981), Lin and Kung (1982), Glover (1984), Latham and Anderson (1985), Hung and Glover (1986), Anderson (1986), Ball and Ran (1987) and Parker and Anderson (1987) for the Hankel-norm approximation criterion. Many of the cited works also present bounds for the closely related H_∞ approximation error, although the optimal H_∞ model-reduction problem remains open. Alternatively, early progress on the model-reduction problem with a quadratic (L_2) criterion was achieved by Wilson (1970) and further explored by Hyland and Bernstein (1985).

Although the Hankel norm, H_∞, and L_2 model-reduction criteria represent distinct approximation objectives, there exist significant connections. For example, it was shown by Wilson (1985), that for systems which are either single input or single output, the input and output space topologies can be redefined so that the induced norm of the Hankel operator coincides with the L_2 system norm. In addition, the optimization technique utilized by Wilson (1970) was re-applied to the Hilbert–Schmidt Hankel operator topology by Wilson (1988). In a recent work, Wilson (1989) has shown that for single-input or single-output systems the quadratic model-reduction criterion is actually an induced norm of the convolution operator itself.

In the present paper we attempt a further unification of the L_2 and H_∞ model-reduction objectives. Specifically, we consider an L_2 model-reduction problem with a constraint on the H_∞ approximation error. The underlying idea involves the suitable application of a frequency-domain inequality due to Willems (1971), which has recently been applied to H_∞ control-design problems by Petersen (1987), Khargonekar *et al.* (1987) and Bernstein and Haddad (1989). The principal result of the present paper is a sufficient condition which characterizes reduced-order models

satisfying an optimized L_2 bound as well as a pre-specified H_∞ bound. The sufficient condition is a direct generalization of the optimal projection approach developed by Hyland and Bernstein (1985) for the unconstrained L_2 problem. While the L_2-optimal reduced-order model was characterized by Hyland and Bernstein (1985) by means of a coupled system of two modified Lyapunov equations, the H_∞-constrained solution in the present paper involves a coupled system consisting of four modified Riccati equations. As in Hyland and Bernstein (1985), the coupling is due to the presence of an oblique projection (idempotent matrix) that determines the constrained reduced-order model. When the H_∞ constraint is sufficiently relaxed, we show that the conditions given herein specialize directly to those given by Hyland and Bernstein (1985). Although our result gives sufficient conditions for H_∞ approximation, we also state hypotheses under which these conditions are also necessary.

Although numerical algorithms were developed by Hyland and Bernstein (1985) for the 'pure' L_2 problem, computational methods for the H_∞-constrained problem are beyond the scope of the present paper. In view of the additional complexity engendered by the H_∞ constraint, more sophisticated algorithms appear necessary. Hence computational methods will focus on the homotopic continuation algorithm developed by Richter (1987) for reduced-order dynamic compensation.

2. Statement of the problem

In this section we introduce the model-reduction problem with constrained H_∞ norm of the model-reduction error. Specifically, we constrain the transfer function of the reduced-order model to lie within a specified H_∞ radius of the original system. In this paper we assume that the full-order model is asymptotically stable, i.e. the matrix A is asymptotically stable.

H_∞-Constrained L_2 model-reduction problem

Given the nth-order controllable and observable model

$$\dot{x}(t) = Ax(t) + BDw(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $t \in [0, \infty)$, determine an n_mth-order model

$$\dot{x}_m(t) = A_m x_m(t) + B_m D w(t) \tag{2.3}$$

$$y_m(t) = C_m x_m(t) \tag{2.4}$$

which satisfies the following criteria:

(i) A_m is asymptotically stable;

(ii) the transfer function of the reduced-order model lies within a radius-γ H_∞ neighbourhood of the full-order model, i.e.

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{2.5}$$

where

$$H(s) \triangleq EC(sI_n - A)^{-1}BD, \quad H_m(s) \triangleq EC_m(sI_{n_m} - A_m)^{-1}B_m D \tag{2.6}$$

and $\gamma > 0$ is a given constant; and

(iii) the L_2 model-reduction criterion

$$J(A_m, B_m, C_m) \triangleq \lim_{t\to\infty} \mathbb{E}\{[y(t) - y_m(t)]^{\mathrm{T}} R[y(t) - y_m(t)]\} \tag{2.7}$$

is minimized.

Note that the full- and reduced-order systems (2.1)–(2.4) can be written as a single augmented system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty) \tag{2.8}$$

so that the $q \times p$ transfer function from $w(t)$ to $E\tilde{y}(t) = \tilde{E}\tilde{x}(t)$ is

$$\tilde{H}(s) = \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{2.9}$$

and (2.7) can be written as

$$J(A_m, B_m, C_m) = \lim_{t\to\infty} \mathbb{E}\{[E\tilde{y}(t)]^{\mathrm{T}}[E\tilde{y}(t)]\} = \lim_{t\to\infty} \mathbb{E}[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)] \tag{2.10}$$

Before continuing it is useful to note that if A_m is asymptotically stable then the L_2 model-reduction criterion (2.7) is given by

$$J(A_m, B_m, C_m) = \operatorname{tr} \tilde{Q}\tilde{R} \tag{2.11}$$

where the steady-state covariance

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{2.12}$$

satisfies the augmented Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{2.13}$$

Using (2.11) and (2.13) it can be shown that the L_2 criterion (2.7) is an approximation measure involving the full- and reduced-order impulse responses with respect to an L_2 norm.

Proposition 2.1

The L_2 model-reduction criterion (2.11) can be written as

$$J(A_m, B_m, C_m) = \int_0^\infty \|EC\exp(At)BD - EC_m\exp(A_m t)B_m D\|_F^2\, dt \tag{2.14 a}$$

or, equivalently

$$J(A_m, B_m, C_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(j\omega) - H_m(j\omega)\|_F^2\, d\omega \tag{2.14 b}$$

Proof

It need only be noted that (2.11) is equivalent to

$$\begin{aligned}\operatorname{tr}\int_0^\infty \exp(\tilde{A}t)\tilde{V}\exp(\tilde{A}^{\mathrm{T}}t)\,dt\tilde{R} &= \operatorname{tr}\int_0^\infty \tilde{E}\exp(\tilde{A}t)\tilde{D}\tilde{D}^{\mathrm{T}}\exp(\tilde{A}^{\mathrm{T}}t)\tilde{E}^{\mathrm{T}}\,dt \\ &= \operatorname{tr}\int_0^\infty (\tilde{E}\exp(\tilde{A}t)\tilde{D})(\tilde{E}\exp(\tilde{A}t)\tilde{D})^{\mathrm{T}}\,dt \\ &= \int_0^\infty \|\tilde{E}\exp(\tilde{A}t)\tilde{D}\|_F^2\,dt\end{aligned}$$

which is equivalent to (2.14 *a*). Finally, (2.14 *b*) follows from Plancherel's Theorem. □

The key step in enforcing (2.5) is to replace the algebraic Lyapunov equation (2.13) by an algebraic Riccati equation. Justification for this technique is provided by the following result.

Lemma 2.1

Let (A_m, B_m, C_m) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.15}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V} \tag{2.16}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.17}$$

if and only if

$$A_m \text{ is asymptotically stable} \tag{2.18}$$

Furthermore, in this case

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{2.19}$$

$$\tilde{Q} \leqslant \mathcal{Q} \tag{2.20}$$

and

$$J(A_m, B_m, C_m) \leqslant \mathcal{J}(A_m, B_m, C_m, \mathcal{Q}) \tag{2.21}$$

where

$$\mathcal{J}(A_m, B_m, C_m, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R} \tag{2.22}$$

Proof

Using the assumed existence of a non-negative-definite solution to (2.16) and the stabilizability condition (2.17), it follows from the dual of Lemma 12.2 of Wonham (1979) that $\tilde{A}$ is asymptotically stable. Since $\tilde{A}$ is block diagonal, A_m is also asymptotically stable. Conversely, since A is assumed to be asymptotically stable, (2.18) implies (2.17). The proof of (2.19) follows from a standard manipulation of (2.16): for details see Lemma 1 of Willems (1971). To prove (2.20), subtract (2.13) from (2.16) to obtain

$$0 = \tilde{A}(\mathcal{Q} - \tilde{Q}) + (\mathcal{Q} - \tilde{Q})\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} \tag{2.23}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q} - \tilde{Q} = \int_0^\infty \exp(\tilde{A}t)[\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}]\exp(\tilde{A}^{\mathrm{T}}t)\,dt \geqslant 0 \tag{2.24}$$

Finally, (2.21) follows immediately from (2.20). □

Lemma 2.1 shows that the H_∞ constraint is automatically enforced when a non-

negative-definite solution to (2.16) is known to exist. Furthermore, the solution $\mathscr{Q}$ provides an upper bound for the actual state covariance $\tilde{Q}$ along with a bound on the L_2 model-reduction criterion. Next, we present a partial converse of Lemma 2.1 which guarantees the existence of a non-negative definite solution to (2.16) when (2.19) is satisfied.

Lemma 2.2

Let (A_m, B_m, C_m) be given, suppose $\tilde{A}$ is asymptotically stable, and assume the H_∞ approximation constraint (2.19) is satisfied. Then there exists a unique non-negative-definite solution $\mathscr{Q}$ satisfying (2.16) and such that $\tilde{A}+\gamma^{-2}\mathscr{Q}\tilde{R}$ is asymptotically stable. Furthermore, this solution is minimal.

Proof

The result is an immediate consequence of Theorems 3 and 2, of Brockett (1970; pp. 150, 167) and the dual of Lemma 12.2 of Wonham (1979). □

Finally, we show that the quadratic term $\gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q}$ in (2.16) also constrains the Hankel norm of the approximation error $E\tilde{y}$ when $\mathscr{Q}$ is positive-definite. To show this, let $\tilde{P}\in\mathbb{N}^{\tilde{n}}$ be the observability Gramian for the augmented system $(\tilde{A}, \tilde{D}, \tilde{E})$ which satisfies

$$0=\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}\tilde{A}+\tilde{R} \tag{2.25}$$

Furthermore, note that $\tilde{Q}$ satisfying (2.13) is the dual controllability Gramian.

Proposition 2.2

Let (A_m, B_m, C_m) be given and assume there exists $\mathscr{Q}\in\mathbb{P}^{\tilde{n}}$ satisfying (2.16) and (2.17) or, equivalently, (2.18). Then

$$\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})\leqslant\gamma \tag{2.26}$$

Proof

Since $\mathscr{Q}$ is invertible, (2.16) implies

$$0=\gamma^2\tilde{A}^{\mathrm{T}}\mathscr{Q}^{-1}+\gamma^2\mathscr{Q}^{-1}\tilde{A}+\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1}+\tilde{R} \tag{2.27}$$

Next, subtract (2.25) from (2.27) to obtain

$$0=\tilde{A}^{\mathrm{T}}(\gamma^2\mathscr{Q}^{-1}-\tilde{P})+(\gamma^2\mathscr{Q}^{-1}-\tilde{P})\tilde{A}+\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1} \tag{2.28}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\gamma^2\mathscr{Q}^{-1}-\tilde{P}=\int_0^\infty \exp(\tilde{A}^{\mathrm{T}}t)[\gamma^2\mathscr{Q}^{-1}\tilde{V}\mathscr{Q}^{-1}]\exp(\tilde{A}t)\,dt\geqslant 0 \tag{2.29}$$

Thus, (2.29) implies $\tilde{P}\leqslant\gamma^2\mathscr{Q}^{-1}$, or, equivalently, $\mathscr{Q}^{1/2}\tilde{P}\mathscr{Q}^{1/2}\leqslant\gamma^2 I_{\tilde{n}}$. Hence, $\lambda_{\max}^{1/2}(\tilde{P}\mathscr{Q})\leqslant\gamma$. Finally, (2.26) follows immediately from (2.20). □

3. Auxiliary minimization problem and necessary conditions for optimality

As discussed in the previous section, the replacement of (2.13) by (2.16) enforces the H_∞ approximation constraint between the full- and reduced-order systems and

results in an upper bound for the L_2 model-reduction criterion. That is, if (2.16) is solvable then the reduced-order model (A_m, B_m, C_m) satisfies the H_∞ approximation constraint (2.5) while the actual L_2 model-reduction criterion is guaranteed to be no worse than the bound given by $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$. Hence, $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ can be interpreted as an auxiliary cost that leads to the following mathematical programming problem.

Auxiliary minimization problem

Determine $(A_m, B_m, C_m, \mathscr{Q})$ that minimizes $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ subject to (2.15) and (2.16).

It follows from Lemma 2.1 that the satisfaction of (2.15)–(2.17) leads to (i) A_m stable; (ii) a bound on the H_∞ distance between the full-order and reduced-order systems; and (iii) an upper bound for the L_2 model-reduction criterion. Hence, it remains to determine (A_m, B_m, C_m) that minimizes $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ and thus provides an optimized bound for the actual L_2 criterion $J(A_m, B_m, C_m)$. Rigorous derivation of the necessary conditions for the auxiliary minimization problem requires additional technical assumptions. Specifically, we restrict $(A_m, B_m, C_m, \mathscr{Q})$ to the open set

$$\mathscr{S} \triangleq \{(A_m, B_m, C_m, \mathscr{Q}) : \mathscr{Q} \in \mathbb{P}^{\tilde{n}}, \tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R} \text{ is asymptotically stable, and } (A_m, B_m, C_m) \text{ is controllable and observable}\} \qquad (3.1)$$

Remark 1

The set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $\mathscr{Q}$ be positive-definite replaces (2.15) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ serves as a normality condition and (A_m, B_m, C_m) minimal is a non-degeneracy condition.

The following lemma is needed for the statement of the main result.

Lemma 3.1

Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose rank $\hat{Q}\hat{P} = n_m$. Then there exist $n_m \times n$ G, Γ and $n_m \times n_m$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_m}$, such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma \qquad (3.2)$$

$$\Gamma G^{\mathrm{T}} = I_{n_m} \qquad (3.3)$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}} \Gamma \qquad (3.4)$$

$$\tau_\perp \triangleq I_n - \tau \qquad (3.5)$$

are idempotent and have rank n_m and $n - n_m$, respectively. If, in addition

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = n_m \qquad (3.6)$$

then

$$\hat{Q} = \tau\hat{Q}, \quad \hat{P} = \hat{P}\tau \qquad (3.7), (3.8)$$

Finally, if $P \in \mathbb{N}^n$ then the inverse

$$S \triangleq (I_n + \gamma^{-2}\hat{Q}P)^{-1} \tag{3.9}$$

exists.

Proof

Conditions (3.2)–(3.8) are a direct consequence of Theorem 6.2.5 of Rao and Mitra (1971). To prove that the inverse in (3.9) exists, note that since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the non-negative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has non-negative eigenvalues. Thus, the eigenvalues of $I_n + \gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists. □

Finally, for convenience define

$$\Sigma \triangleq BVB^{\mathrm{T}}, \quad \tilde{\Sigma} \triangleq C^{\mathrm{T}}RC$$

Theorem 3.1

If $(A_m, B_m, C_m, \mathcal{Q}) \in \mathcal{S}$ solves the auxiliary minimization problem then there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that

$$A_m = \Gamma(A - \gamma^{-4}Q\tilde{\Sigma}QPS)G^{\mathrm{T}} \tag{3.10}$$

$$B_m = \Gamma B \tag{3.11}$$

$$C_m = C(I_n + \gamma^{-2}QPS)G^{\mathrm{T}} \tag{3.12}$$

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix} \tag{3.13}$$

and such that $Q, P, \hat{Q}, \hat{P}$ satisfy

$$0 = AQ + QA^{\mathrm{T}} + \gamma^{-2}Q\tilde{\Sigma}Q + \tau_\perp \Sigma \tau_\perp^{\mathrm{T}} \tag{3.14}$$

$$0 = A^{\mathrm{T}}P + PA - \gamma^{-4}S^{\mathrm{T}}PQ\tilde{\Sigma}QPS + \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\tilde{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp \tag{3.15}$$

$$0 = (A - \gamma^{-4}Q\tilde{\Sigma}QPS)\hat{Q} + \hat{Q}(A - \gamma^{-4}Q\tilde{\Sigma}QPS)^{\mathrm{T}} + \gamma^{-6}\hat{Q}S^{\mathrm{T}}PQ\tilde{\Sigma}QPS\hat{Q} + \Sigma - \tau_\perp \Sigma \tau_\perp^{\mathrm{T}} \tag{3.16}$$

$$0 = (A + \gamma^{-2}Q\tilde{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A + \gamma^{-2}Q\tilde{\Sigma}) + (I_n + \gamma^{-2}QPS)^{\mathrm{T}}\tilde{\Sigma}(I_n + \gamma^{-2}QPS) - \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\tilde{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp \tag{3.17}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_m \tag{3.18}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(A_m, B_m, C_m, \mathcal{Q}) = \operatorname{tr} \tilde{\Sigma}(Q + \gamma^{-4}QPS\hat{Q}S^{\mathrm{T}}PQ) \tag{3.19}$$

Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.18), then $(A_m, B_m, C_m, \mathcal{Q})$ given by (3.10)–(3.13) satisfy (2.15) and (2.16) with the auxiliary cost (2.22) given by (3.19).

Proof

See Appendix A.

Remark 2

Theorem 3.1 presents necessary conditions for the auxiliary minimization problem that explicitly synthesize extremal reduced-order models (A_m, B_m, C_m). As a check of these conditions, consider the extreme case $n_m = n$. Then $G = \Gamma^{-1}$ and thus, without loss of generality, $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$. Furthermore, (3.14) implies that $Q = 0$ and (3.15) implies that $P = 0$. Hence the H_∞-constrained full-order model is given (as expected) by (A, B, C) regardless of γ. Furthermore, note that $\mathcal{Q}$ given by (3.13) becomes

$$\mathcal{Q} = \begin{bmatrix} \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix} \tag{3.20}$$

so that the quadratic term $\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}$ in (2.16) vanishes. Thus, (2.16) reduces to (2.13) so that $\mathcal{Q}$ coincides with the controllability Gramian $\tilde{Q}$. If, alternatively, the reduced-order constraint is retained but the transfer function approximation constraint (2.5) is sufficiently relaxed, i.e. $\gamma \to \infty$, then $S = I_n$ so that the reduced-order model (3.10)–(3.12) is given by $(A_m, B_m, C_m) = (\Gamma A G^{\mathrm{T}}, \Gamma B, C G^{\mathrm{T}})$. In this case, (3.14) and (3.15) are superfluous and (3.16) and (3.17) reduce to the optimal projection equations obtained by Hyland and Bernstein (1985) for the unconstrained L_2 problem.

4. Sufficient conditions for combined L_2/H_∞ approximation

In this section we combine Lemma 2.1 with the converse of Theorem 3.1 to obtain our main result guaranteeing constrained H_∞ approximation along with an optimized L_2 model-reduction bound.

Theorem 4.1

Suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.18) and let $(A_m, B_m, C_m, \mathcal{Q})$ be given by (3.10)–(3.13). Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if A_m is asymptotically stable. In this case, the reduced-order transfer function $H_m(s)$ satisfies the H_∞ approximation constraint

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{4.1}$$

and the L_2 approximation bound

$$\|H(s) - H_m(s)\|_2 \leqslant [\mathrm{tr}\, \tilde{\Sigma}(Q + \gamma^{-4} QPS\hat{Q}S^{\mathrm{T}}PQ)]^{1/2} \tag{4.2}$$

Proof

The converse portion of Theorem 3.1 implies that $\mathcal{Q}$ given by (3.13) satisfies (2.15) and (2.16) with auxiliary cost given by (3.19). It now follows from Lemma 2.1 that the stabilizability condition (2.17) is equivalent to the asymptotic stability of A_m, the H_∞ approximation condition (2.19) holds, and the L_2 model-reduction criterion satisfies the bound (2.21) which is equivalent to (4.2). □

In applying Theorem 4.1, the principal issue concerns conditions on the problem data under which the coupled Riccati equations (3.14)–(3.17) possess non-negative-

definite solutions. Clearly, for γ sufficiently large, (3.14)–(3.17) approximate the 'pure' L_2 solution obtained by Hyland and Bernstein (1985). In practice, we would numerically solve (3.14)–(3.17) for successively smaller values of γ until solutions are no longer obtainable. The important case of interest, however, involves small γ so that accurate H_∞ approximation is enforced. Thus, if (4.1) can be satisfied for a given $\gamma > 0$ by a class of reduced-order models, it is of interest to know whether one such reduced-order model can be obtained by solving (3.14)–(3.17). Lemma 2.2 guarantees that (2.16) possesses a solution for any model satisfying (4.1). Thus our sufficient conditions will also be necessary so long as the auxiliary minimization problem possesses at least one extremal over $\mathscr{S}$. When this is the case we have the following immediate result.

Proposition 4.1

Let γ^* denote the infimum of $\|H(s) - H_m(s)\|_\infty$ over all asymptotically stable reduced-order models and suppose that the auxiliary minimization problem has a solution for all $\gamma > \gamma^*$. Then for all $\gamma > \gamma^*$ there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.17).

Remark 3

As in Hyland and Bernstein (1985), it can be expected that (3.14)–(3.17) possess multiple solutions. Theorem 4.1 guarantees, however, that the bounds (4.1) and (4.2) are enforced for all such extremals obtained by solving (3.14)–(3.17).

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Appendix A

Proof of Theorem 3.1

To optimize (2.22) over the open set $\mathscr{S}$ subject to the constraint (2.16), form the lagrangian

$$\mathscr{L}(A_m, B_m, C_m, \mathscr{Q}, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\{\lambda \mathscr{Q}\tilde{R} + [\tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q} + \tilde{V}]\mathscr{P}\} \qquad (\mathrm{A}\ 1)$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R} \qquad (\mathrm{A}\ 2)$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R} \qquad (\mathrm{A}\ 3)$$

Since $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ is assumed to be stable, $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathscr{P}$ is non-negative-definite.

Now partition $\tilde{n} \times \tilde{n}$, $\mathscr{Q}$, $\mathscr{P}$, into $n \times n$, $n \times n_m$, and $n_m \times n_m$ sub-blocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \quad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}$$

and for notational convenience define

$$\mathscr{P}\mathscr{Q} = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix}$$

where

$$Z_1 \triangleq P_1 Q_1 + P_{12} Q_{12}^{\mathrm{T}}, \quad Z_{12} \triangleq P_1 Q_{12} + P_{12} Q_2$$
$$Z_{21} \triangleq P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}, \quad Z_2 \triangleq P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = (\tilde{A} + \gamma^{-2} \mathscr{Q}\tilde{R})^{\mathrm{T}} \mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2} \mathscr{Q}\tilde{R}) + \tilde{R} = 0 \tag{A 4}$$

$$\frac{\partial \mathscr{L}}{\partial A_m} = Z_2 = 0 \tag{A 5}$$

$$\frac{\partial \mathscr{L}}{\partial B_m} = P_{12}^{\mathrm{T}} B V + P_2 B_m V = 0 \tag{A 6}$$

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial C_m} &= 2RC_m Q_2 + 2\gamma^{-2} RC_m Z_{12}^{\mathrm{T}} Q_{12} - 2RCQ_{12} - \gamma^{-2} RCZ_1^{\mathrm{T}} Q_{12} \\ &\quad - \gamma^{-2} RCQ_1 Z_{12} - \gamma^{-2} RCZ_{21}^{\mathrm{T}} Q_2 = 0\end{aligned} \tag{A 7}$$

Expanding (2.16) and (A 4) yields

$$0 = AQ_1 + Q_1 A^{\mathrm{T}} + \gamma^{-2}(Q_1 C^{\mathrm{T}} - Q_{12} C_m^{\mathrm{T}}) R (Q_1 C^{\mathrm{T}} - Q_{12} C_m^{\mathrm{T}})^{\mathrm{T}} + BVB^{\mathrm{T}} \tag{A 8}$$

$$\begin{aligned}0 &= AQ_{12} + Q_{12} A_m^{\mathrm{T}} + \gamma^{-2} Q_1 C^{\mathrm{T}} RCQ_{12} - \gamma^{-2} Q_{12} C_m^{\mathrm{T}} RCQ_{12} - \gamma^{-2} Q_1 C^{\mathrm{T}} RC_m Q_2 \\ &\quad + \gamma^{-2} Q_{12} C_m^{\mathrm{T}} RC_m Q_2\end{aligned} \tag{A 9}$$

$$0 = A_m Q_2 + Q_2 A_m^{\mathrm{T}} + \gamma^{-2}(Q_{12}^{\mathrm{T}} C^{\mathrm{T}} - Q_2 C_m^{\mathrm{T}}) R (Q_{12}^{\mathrm{T}} C^{\mathrm{T}} - Q_2 C_m^{\mathrm{T}})^{\mathrm{T}} + B_m V B_m^{\mathrm{T}} \tag{A 10}$$

$$\begin{aligned}0 &= A^{\mathrm{T}} P_1 + P_1 A + \gamma^{-2} C^{\mathrm{T}} RCZ_1^{\mathrm{T}} - \gamma^{-2} C^{\mathrm{T}} RC_m Z_{12}^{\mathrm{T}} + \gamma^{-2} Z_1 C^{\mathrm{T}} RC \\ &\quad - \gamma^{-2} Z_{12} C_m^{\mathrm{T}} RC + C^{\mathrm{T}} RC\end{aligned} \tag{A 11}$$

$$\begin{aligned}0 &= A^{\mathrm{T}} P_{12} + P_{12} A_m + \gamma^{-2} C^{\mathrm{T}} RCZ_{21}^{\mathrm{T}} - \gamma^{-2} Z_1 C^{\mathrm{T}} RC_m + \gamma^{-2} Z_{12} C_m^{\mathrm{T}} RC_m \\ &\quad - C^{\mathrm{T}} RC_m\end{aligned} \tag{A 12}$$

$$0 = A_m^{\mathrm{T}} P_2 + P_2 A_m - \gamma^{-2} C_m^{\mathrm{T}} RCZ_{21}^{\mathrm{T}} - \gamma^{-2} Z_{21} C^{\mathrm{T}} RC_m + C_m^{\mathrm{T}} RC_m \tag{A 13}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^{\mathrm{T}}$$
$$\hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^{\mathrm{T}}$$
$$\tau \triangleq -Q_{12} Q_2^{-1} P_2^{-1} P_{12}^{\mathrm{T}}$$

and the $n_m \times n$, $n_m \times n_m$ and $n_m \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1} P_{12}^{\mathrm{T}}$$

The existence of Q_2^{-1} and P_2^{-1} follows from the fact that (A_m, B_m, C_m) is minimal. See Bernstein and Haddad (1989) and Hyland and Bernstein (1985) for details. Note that $\tau = G^{\mathrm{T}} \Gamma$. Clearly, Q, P, $\hat{Q}$ and $\hat{P}$ are symmetric and non-negative-definite.

Next note that with the above definitions, (A 5) implies (3.3) and that (3.2) holds. Hence, $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e. $\tau^2 = \tau$. Sylvester's inequality yields (3.18). Note also that (3.7) and (3.8) hold.

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G and Γ as

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P} \tag{A 14}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \quad P_{12} = -\hat{P}G^{\mathrm{T}} \tag{A 15}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \quad P_2 = G\hat{P}G^{\mathrm{T}} \tag{A 16}$$

Next note that by using (A 14)–(A 16), (A 7) becomes

$$C_m\hat{S} = C[I_n + \gamma^{-2}(Q + \hat{Q})P]G^{\mathrm{T}}$$

where

$$\hat{S} \triangleq I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$$

To prove that $\hat{S}$ is invertible use (3.7) and (3.4) and note that

$$\begin{aligned} I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}} &= I_{n_m} + \gamma^{-2}\Gamma\hat{Q}\tau^{\mathrm{T}}PG^{\mathrm{T}} \\ &= I_{n_m} + \gamma^{-2}(\Gamma\hat{Q}\Gamma^{\mathrm{T}})(GPG^{\mathrm{T}}) \end{aligned}$$

Since $\Gamma\hat{Q}\Gamma^{\mathrm{T}}$ and GPG^{T} are non-negative-definite, their product has non-negative eigenvalues. Thus each eigenvalue of $I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$ is real and is greater than unity. Hence $\hat{S}$ is invertible. Now note that by using (3.3) and (3.4) it can be shown that

$$G^{\mathrm{T}}\hat{S}^{-1}\Gamma = S\tau$$

The expressions (3.11), (3.12) and (3.13) follow from (A 6), (A 7), (3.9) and the definition of $\mathcal{Q}$ by using the above identities. Next, computing either Γ(A 9) $-$ (A 10) or G(A 12) + (A 13) yields (3.10). Substituting this expression for A_m into (A 8)–(A 13) it follows that (A 10) = Γ(A 9) and (A 13) = G(A 12). Thus, (A 10) and (A 13) are superfluous and can be omitted. Next, using (A 8) + $G^{\mathrm{T}}\Gamma$(A 9)G $-$ (A 9)G $-$ [(A 9)G]$^{\mathrm{T}}$ and $G^{\mathrm{T}}\Gamma$(A 9)G $-$ (A 9)G $-$ [(A 9)G]$^{\mathrm{T}}$ yields (3.14) and (3.16). Using (A 11) + $\Gamma^{\mathrm{T}}G$(A 12)Γ $-$ (A 12)Γ $-$ [(A 12)Γ]$^{\mathrm{T}}$ and $\Gamma^{\mathrm{T}}G$(A 12)Γ $-$ (A 12)Γ $-$ [(A 12)Γ]$^{\mathrm{T}}$ yields (3.15) and (3.17).

Finally, to prove the converse we use (3.10)–(3.18) to obtain (2.16) and (A 4)–(A 7). Let A_m, B_m, C_m, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 3.1 and define $Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2$ by (A 14)–(A 16). Using (3.3), (3.11) and (3.12) it is easy to verify (A 6) and (A 7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ and τ into (3.14)–(3.17) along with (3.3), (3.4), (3.7) and (3.8) to obtain (2.16) and (A 4). Finally, note that

$$\mathcal{Q} = \begin{bmatrix} Q & 0_{n \times n_m} \\ 0_{n_m \times n} & 0_{n_m} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q} [I_n \quad \Gamma^{\mathrm{T}}]$$

which shows that $\mathcal{Q} \geqslant 0$.

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Steady-state Kalman filtering with an H_∞ error bound *

Dennis S. BERNSTEIN

Harris Corporation, Government Aerospace Systems Division, MS 22/4848, Melbourne, FL 32902, U.S.A.

Wassim M. HADDAD

Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

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Abstract: An estimator design problem is considered which involves both L_2 (least squares) and H_∞ (worst-case frequency-domain) aspects. Specifically, the goal of the problem is to minimize an L_2 state-estimation error criterion subject to a prespecified H_∞ constraint on the state-estimation error. The H_∞ estimation-error constraint is embedded within the optimization process by replacing the covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the L_2 state-estimation error. The principal result is a sufficient condition for characterizing fixed-order (i.e., full- and reduced-order) estimators with bounded L_2 and H_∞ estimation error. The sufficient condition involves a system of modified Riccati equations coupled by an oblique projection, i.e., idempotent matrix. When the H_∞ constraint is absent, the sufficient condition specializes to the L_2 state-estimation result given in [2].

Keywords: Kalman filter; H_∞ norm; reduced-order state estimation; optimal projection equations; Hankel norm.

1. Introduction

One of the fundamental problems in dynamic systems theory is the observation of state variables. Although an extensive theoretical foundation has been developed for the quadratic (least squares) error criterion, state estimation with a worst-case frequency-domain design objective has apparently not been considered. In the present paper we thus extend the least squares formulation to include a frequency-domain bound on the state-estimation error. The underlying idea involves the application of state-space techniques which have recently been developed for H_∞ control-design in [1,4-6]. The results of the present paper are thus complementary to the results obtained in [1].

The principal result of the present paper is a sufficient condition which yields full- and reduced-order estimators satisfying an optimized L_2 error bound as well as a prespecified H_∞ error bound. In the full-order case, the H_∞-constrained estimator involves a modified Riccati equation which specializes to the standard steady-state Kalman filter when the H_∞ constraint is absent. In the reduced-order case the H_∞-constrained result leads to a direct generalization of the optimal projection approach developed in [2] for the unconstrained L_2 state-estimation problem. While the L_2-optimal reduced-order state estimator was characterized in [2] by means of a coupled system of one modified Riccati equation and two modified Lyapunov equations, the H_∞-constrained solution involves a coupled system consisting of three modified Riccati equations and one modified Lyapunov equation. As in [2], the coupling is due to the presence of an oblique projection (idempotent matrix) with additional coupling now arising from the H_∞ constraint. When the H_∞ constraint is sufficiently relaxed, these conditions again specialize directly to those given in [2].

We note that the development in the present paper is limited to the case in which the plant is asymptotically stable. These results can also be extended to the unstable plant case, although with additional complexity. This case will thus be treated in a future paper.

The contents of the paper are as follows. After collecting notation in Section 2, the statement of the H_∞-Constrained State-Estimation Problem is given in Section 3. The principal result of this section (Lemma 3.1) shows that if the algebraic

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Lyapunov equation for the covariance is replaced by a modified Riccati equation possessing a non-negative-definite solution, then the H_∞ estimation-error constraint is enforced and the L_2 state-estimation error criterion is bounded above by an auxiliary cost function. The problem of determining a reduced-order estimator which minimizes this upper bound subject to the Riccati equation constraint is considered in Section 4 as the Auxiliary Minimization Problem. Necessary conditions for the Auxiliary Minimization Problem (Theorem 4.1) are given in the form of a coupled system of modified algebraic Riccati equations. To develop connections with standard Kalman filter theory the full-order estimator result is also given. In Section 5 the necessary conditions of Theorem 4.1 are combined with Lemma 3.1 to yield sufficient conditions for bounded H_∞ and L_2 estimation error. Although our result gives sufficient conditions for H_∞ estimation error, we also state hypotheses under which these conditions are also necessary (Proposition 5.1).

2. Notation and definitions

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$: real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.

I_r, $()^{\mathrm{T}}$, $0_{r\times s}$, 0_r: $r\times r$ identity matrix, transpose, $r\times s$ zero matrix, $0_{r\times r}$.

tr: trace.

$\sigma_{\max}(Z)$: largest singular value of matrix Z.

$\lambda_{\max}(Z)$: largest eigenvalue of matrix Z with a real spectrum.

$\|Z\|_F$: $[\operatorname{tr} ZZ^{\mathrm{T}}]^{1/2}$ (Frobenius matrix norm).

$\|H(s)\|_\infty$: $\sup_{\omega\in\mathbb{R}}\sigma_{\max}[H(j\omega)]$.

$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$: $r\times r$ symmetric, nonnegative-definite, positive-definite matrices.

$Z_1\le Z_2$, $Z_1<Z_2$: $Z_2-Z_1\in\mathbb{N}^r$, $Z_2-Z_1\in\mathbb{P}^r$, $Z_1, Z_2\in\mathbb{S}^r$.

n, l, n_e, p, q, r; $\tilde n$ positive integers: $n+n_e$; $n_e\le n$.

x, y, y_e, x_e, $\tilde x$: n, l, q, n_e, $\tilde n$-dimensional vectors.

$$\tilde x \triangleq \begin{bmatrix} x \\ x_e \end{bmatrix}.$$

A, C: $n\times n$, $l\times n$ matrices.

D_1, D_2, E: $n\times p$, $l\times p$, $r\times q$ matrices.

L: $q\times n$ matrix.

A_e, B_e, C_e: $n_e\times n_e$, $n_e\times l$, $q\times n_e$ matrices.

$$\tilde A \triangleq \begin{bmatrix} A & 0_{n\times n_e} \\ B_eC & A_e \end{bmatrix},$$

$$\tilde D \triangleq \begin{bmatrix} D_1 \\ B_eD_2 \end{bmatrix}, \qquad \tilde E \triangleq [EL \quad -EC_e].$$

R: $E^{\mathrm{T}}E$, estimation error weighting in $\mathbb{P}^q$.

$w(\cdot)$: p-dimensional standard white noise process.

V_1, V_2: intensity of $D_1w(\cdot)$, $D_2w(\cdot)$; $V_1\triangleq D_1D_1^{\mathrm{T}}\in\mathbb{N}^n$, $V_2\triangleq D_2D_2^{\mathrm{T}}\in\mathbb{P}^l$.

V_{12}: cross intensity of $D_1w(\cdot)$, $D_2w(\cdot)$; $V_{12}\triangleq D_1D_2^{\mathrm{T}}\in\mathbb{R}^{n\times l}$.

$$\tilde R \triangleq \begin{bmatrix} L^{\mathrm{T}}RL & -L^{\mathrm{T}}RC_e \\ -C_e^{\mathrm{T}}RL & C_e^{\mathrm{T}}RC_e \end{bmatrix},$$

$$\tilde V \triangleq \begin{bmatrix} V_1 & V_{12}B_e^{\mathrm{T}} \\ B_eV_{12}^{\mathrm{T}} & B_eV_2B_e^{\mathrm{T}} \end{bmatrix}.$$

γ: positive constant.

3. Statement of the problem

In this section we introduce the reduced-order state-estimation problem with a constraint on the H_∞ norm of the state-estimation error. Specifically, the transfer function between disturbances and error states is constrained to have H_∞ norm less than γ. In this paper it is assumed that the plant is asymptotically stable, i.e., the eigenvalues of A are in the open left half plane.

H_∞-Constrained State-Estimation Problem. Given the n-th-order observed system

$$\dot x(t) = Ax(t) + D_1w(t), \tag{3.1}$$

$$y(t) = Cx(t) + D_2w(t), \tag{3.2}$$

where $t\in[0,\infty)$, determine an n_e-th-order state estimator

$$\dot x_e(t) = A_ex_e(t) + B_ey(t), \tag{3.3}$$

$$y_e(t) = C_ex_e(t), \tag{3.4}$$

where $n_e\le n$, which satisfies the following design criteria:

(i) A_e is asymptotically stable;
(ii) the $r \times p$ transfer function

$$H(s) \triangleq \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{3.5}$$

from disturbances $w(t)$ to error states $E[Lx(t) - y_e(t)] = \tilde{E}\tilde{x}(t)$ satisfies the constraint

$$\| H(s) \|_\infty \le \gamma, \tag{3.6}$$

where $\gamma > 0$ is a given constant; and
(iii) the L_2 state-estimation error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{t\to\infty} \mathbb{E}\{[Lx(t) - y_e(t)]^{\mathrm{T}} R[Lx(t) - y_e(t)]\} \tag{3.7}$$

is minimized.

It is useful to note that the augmented system (3.1)–(3.4) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty), \tag{3.8}$$

and that (3.7) is equivalent to

$$J(A_e, B_e, C_e) = \lim_{t\to\infty} \mathbb{E}\{[\tilde{E}\tilde{x}(t)]^{\mathrm{T}}[\tilde{E}\tilde{x}(t)]\} = \lim_{t\to\infty} \mathbb{E}[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)]. \tag{3.9}$$

Furthermore, if A_e is asymptotically stable for a given estimator (A_e, B_e, C_e) then the L_2 state-estimation error criterion is given by

$$J(A_e, B_e, C_e) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{3.10}$$

where the steady-state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{3.11}$$

satisfies the $\tilde{n} \times \tilde{n}$ Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{3.12}$$

Using (3.10) and (3.12) we now show that the criterion (3.7) is an error measure involving the impulse response of (3.8) with respect to an L_2 norm.

Proposition 3.1. *If A_e is asymptotically stable then the L_2 state-estimation criterion* (3.7) *can be written as*

$$J(A_e, B_e, C_e) = \int_0^\infty \| \tilde{E}e^{\tilde{A}t}\tilde{D} \|_F^2 \, dt. \tag{3.13}$$

Proof. It need only be noted that (3.10) is equivalent to

$$\operatorname{tr} \int_0^\infty e^{\tilde{A}t}\tilde{V}e^{\tilde{A}^{\mathrm{T}}t} \, dt\tilde{R} = \operatorname{tr} \int_0^\infty (\tilde{E}e^{\tilde{A}t}\tilde{D})(\tilde{E}e^{\tilde{A}t}\tilde{D})^{\mathrm{T}} \, dt,$$

which is equivalent to (3.13). □

The key step in enforcing (3.6) is to replace the algebraic Lyapunov equation (3.12) by an algebraic Riccati equation. Justification for this technique is provided by the following result.

Lemma 3.1. *Let (A_e, B_e, C_e) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying*

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{3.14}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}. \tag{3.15}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{3.16}$$

if and only if

$$A_e \text{ is asymptotically stable.} \tag{3.17}$$

Furthermore, in this case

$$\| H(s) \|_\infty \le \gamma, \tag{3.18}$$

$$\tilde{Q} \le \mathcal{Q}, \tag{3.19}$$

and

$$J(A_e, B_e, C_e) \le \mathcal{J}(A_e, B_e, C_e, \mathcal{Q}), \tag{3.20}$$

where

$$\mathcal{J}(A_e, B_e, C_e, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{3.21}$$

Proof. Theorem 3.6 of [9] and (3.16) imply that $(\tilde{A}, [\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V}]^{1/2})$ is also stabilizable. Using Lemma 12.2 of [9] and the assumed existence of a nonnegative-definite solution to (3.15), it follows that $\tilde{A}$ is asymptotically stable. Since $\tilde{A}$ is lower block triangular, $\tilde{A}$ asymptotically stable implies A_e is asymptotically stable. Conversely, since A is assumed to be asymptotically stable, (3.17) implies $\tilde{A}$ is asymptotically stable and thus (3.16) holds. The proof of (3.18) follows from a standard

manipulation of (3.15); for details see Lemma 1 of [8]. To prove (3.19) subtract (3.12) from (3.15) to obtain

$$0=\tilde{A}(\mathcal{Q}-\tilde{Q})+(\mathcal{Q}-\tilde{Q})\tilde{A}^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}, \quad (3.22)$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q}-\tilde{Q}=\int_0^{\infty}\mathrm{e}^{\tilde{A}t}\left[\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}\right]\mathrm{e}^{\tilde{A}^{\mathrm{T}}t}\,\mathrm{d}t\geq 0. \quad (3.23)$$

Finally, (3.20) follows immediately from (3.19). □

Lemma 3.1 shows that the H_∞ constraint is automatically enforced when a nonnegative-definite solution to (3.15) can be shown to exist. Furthermore, the solution $\mathcal{Q}$ provides an upper bound for the steady-state covariance $\tilde{Q}$ along with a bound on the L_2 state-estimation error criterion. Next, we present a partial converse of Lemma 3.1 which guarantees the existence of a nonnegative-definite solution to (3.15) when (3.18) is satisfied.

Lemma 3.2. *Let (A_e, B_e, C_e) be given, suppose A_e is asymptotically stable, and assume the H_∞ state-estimation error constraint* (3.18) *is satisfied. Then there exists a unique nonnegative-definite solution $\mathcal{Q}$ satisfying* (3.15) *and such that $\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}$ is asymptotically stable. Furthermore, $\mathcal{Q}$ is the minimal solution to* (3.15).

Proof. The result is an immediate consequence of Theorems 3 and 2 of [3], pp. 150 and 167, along with the dual of Lemma 12.2 of [9]. □

Finally, we show that the quadratic term $\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}$ in (3.15) also constrains the Hankel norm of the estimation error $E[Lx(t)-y_e(t)]$ when $\mathcal{Q}$ is positive definite. To show this let $\tilde{P}\in\mathbb{N}^{\tilde{n}}$ be the observability Gramian for the augmented system $(\tilde{A}, \tilde{D}, \tilde{E})$ which satisfies

$$0=\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}\tilde{A}+\tilde{R}. \quad (3.24)$$

Proposition 3.2. *Let (A_e, B_e, C_e) be given and assume there exists $\mathcal{Q}\in\mathbb{P}^{\tilde{n}}$ satisfying* (3.15) *and* (3.16) *or, equivalently,* (3.17). *Then*

$$\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})\leq\gamma. \quad (3.25)$$

Proof. Since $\mathcal{Q}$ is invertible, (3.15) implies

$$0=\gamma^2\tilde{A}^{\mathrm{T}}\mathcal{Q}^{-1}+\gamma^2\mathcal{Q}^{-1}\tilde{A}+\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1}+\tilde{R}. \quad (3.26)$$

Next, subtract (3.24) from (3.26) to obtain

$$0=\tilde{A}^{\mathrm{T}}(\gamma^2\mathcal{Q}^{-1}-\tilde{P})+(\gamma^2\mathcal{Q}^{-1}-\tilde{P})\tilde{A}+\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1}, \quad (3.27)$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\gamma^2\mathcal{Q}^{-1}-\tilde{P}=\int_0^{\infty}\mathrm{e}^{\tilde{A}^{\mathrm{T}}t}\left[\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1}\right]\mathrm{e}^{\tilde{A}t}\,\mathrm{d}t\geq 0. \quad (3.28)$$

Thus (3.28) implies $\tilde{P}\leq\gamma^2\mathcal{Q}^{-1}$, or equivalently, $\mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}\leq\gamma^2 I_{\tilde{n}}$. Hence,

$$\gamma^2\geq\lambda_{\max}(\mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2})=\lambda_{\max}(\tilde{P}^{1/2}\mathcal{Q}\tilde{P}^{1/2})\geq\lambda_{\max}(\tilde{P}^{1/2}\tilde{Q}\tilde{P}^{1/2})=\lambda_{\max}(\tilde{P}\tilde{Q}). \quad □$$

4. The auxiliary minimization problem and necessary conditions for optimality

As discussed in the previous section, the replacement of (3.12) by (3.15) enforces the H_∞ state-estimation error constraint and results in an upper bound for the L_2 state-estimation error criterion. That is, given an estimator (A_e, B_e, C_e) satisfying the H_∞ estimation constraint, the *actual* L_2 state-estimation error criterion is guaranteed to be no worse than the bound $\mathcal{J}(A_e, B_e, C_e, \mathcal{Q})$ if (3.15) is solvable. Hence, $\mathcal{J}(A_e, B_e, C_e, \mathcal{Q})$ can be interpreted as an *auxiliary cost* which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_e, B_e, C_e, \mathcal{Q})$ which minimizes $\mathcal{J}(A_e, B_e, C_e, \mathcal{Q})$ subject to (3.14) and (3.15).

It follows from Lemma 3.1 that the satisfaction of (3.14)–(3.16) leads to (1) A_e stable; (2) H_∞ estimation error bound γ; and (3) an upper bound (3.21) for the L_2 state-estimation error criterion. Hence it remains to determine (A_e, B_e, C_e) which minimizes $\mathcal{J}(A_e, B_e, C_e, \mathcal{Q})$ and thus provides an optimized bound for the *actual* L_2 criterion $J(A_e, B_e, C_e)$. Rigorous derivation of the necessary conditions for the Auxiliary Minimization Problem requires additional technical assump-

tions. Specifically, we restrict $(A_e, B_e, C_e, \mathscr{Q})$ to the open set

$$\mathscr{S} \triangleq \{(A_e, B_e, C_e, \mathscr{Q}): \mathscr{Q} \in \mathbb{P}^{\tilde{n}},\ \tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R} \text{ is asymptotically stable, and } (A_e, B_e, C_e) \text{ is controllable and observable.}\}$$

Remark 4.1. The set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the requirement that $\mathscr{Q}$ be positive definite replaces (3.14) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ serves as a normality condition, and (A_e, B_e, C_e) minimal is a nondegeneracy condition.

The following lemma is needed for the statement of the main result.

Lemma 4.1. *Let* $\hat{Q}, \hat{P} \in \mathbb{N}^n$ *and suppose* rank $\hat{Q}\hat{P} = n_e$. *Then there exist* $n_e \times n$ G, Γ *and* $n_e \times n_e$ *invertible* M, *unique except for a change of basis in* $\mathbb{R}^{n_e}$, *such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \tag{4.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_e}. \tag{4.2}$$

Furthermore, the $n \times n$ *matrices*

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \tag{4.3}$$

$$\tau_\perp \triangleq I_n - \tau, \tag{4.4}$$

are idempotent and have rank n_e *and* $n - n_e$, *respectively. If, in addition,*

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = n_e, \tag{4.5}$$

then

$$\hat{Q} = \tau\hat{Q}, \qquad \hat{P} = \hat{P}\tau. \tag{4.6), (4.7}$$

Finally, if $P \in \mathbb{N}^n$ *then the inverse*

$$S \triangleq (I_n + \gamma^{-2}\hat{Q}P)^{-1} \tag{4.8}$$

exists.

Proof. Conditions (4.1)–(4.7) are a direct consequence of Theorem 6.2.5 of [7]. To prove that the inverse in (4.8) exists, note that since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the nonnegative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has nonnegative eigenvalues. Thus, the eigenvalues of $I_n + \gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists. □

Finally, for arbitrary $Q \in \mathbb{R}^{n \times n}$ define

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12}, \qquad \Sigma \triangleq L^{\mathrm{T}}RL. \tag{4.9}$$

Theorem 4.1. *If* $(A_e, B_e, C_e, \mathscr{Q}) \in \mathscr{S}$ *solves the Auxiliary Minimization Problem then there exist* $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *such that*

$$A_e = \Gamma(A - Q_a V_2^{-1}C - \gamma^{-4}Q\Sigma QPS)G^{\mathrm{T}}, \tag{4.10}$$

$$B_e = \Gamma Q_a V_2^{-1}, \tag{4.11}$$

$$C_e = L(I_n + \gamma^{-2}QPS)G^{\mathrm{T}}, \tag{4.12}$$

$$\mathscr{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}, \tag{4.13}$$

and such that $Q, P, \hat{Q}, \hat{P}$ *satisfy*

$$0 = AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2}Q\Sigma Q - Q_a V_2^{-1} Q_a^{\mathrm{T}} + \tau_\perp Q_a V_2^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \tag{4.14}$$

$$0 = A^{\mathrm{T}}P + PA - \gamma^{-4}S^{\mathrm{T}}PQ\Sigma QPS + \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\Sigma(I_n + \gamma^{-2}QPS)\tau_\perp, \tag{4.15}$$

$$0 = (A - \gamma^{-4}Q\Sigma QPS)\hat{Q} + \hat{Q}(A - \gamma^{-4}Q\Sigma QPS)^{\mathrm{T}} + \gamma^{-6}\hat{Q}S^{\mathrm{T}}PQ\Sigma QPS\hat{Q} + Q_a V_2^{-1}Q_a^{\mathrm{T}} - \tau_\perp Q_a V_2^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \tag{4.16}$$

$$0 = (A - Q_a V_2^{-1}C + \gamma^{-2}Q\Sigma)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q_a V_2^{-1}C + \gamma^{-2}Q\Sigma) + (I_n + \gamma^{-2}QPS)^{\mathrm{T}}\Sigma(I_n + \gamma^{-2}QPS) - \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\Sigma(I_n + \gamma^{-2}QPS)\tau_\perp, \tag{4.17}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_e. \tag{4.18}$$

Furthermore, the auxiliary cost is given by

$$\mathscr{J}(A_e, B_e, C_e, \mathscr{Q}) = \operatorname{tr} L^{\mathrm{T}}RL(Q + \gamma^{-4}QPS\hat{Q}S^{\mathrm{T}}PQ). \tag{4.19}$$

Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *satisfying* (4.14)–(4.18), *then* $(A_e, B_e, C_e, \mathscr{Q})$ *given by*

(4.10)–(4.13) *satisfies* (3.14) *and* (3.15) *with the auxiliary cost* (3.21) *given by* (4.19).

Proof. See Appendix A. □

Remark 4.2. Theorem 4.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly synthesize extremal full- and reduced-order estimators (A_e, B_e, C_e). If the H_∞ estimation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then $S = I_n$. In this case equations (4.16) and (4.17) become decoupled from (4.15) and thus (4.15) becomes superfluous. Furthermore, (4.14), (4.16) and (4.17) specialize to the optimal projection equations obtained in [2].

As discussed in [2], in the full-order (Kalman filter) case $n_e = n$, $G = \Gamma^{-1}$ and thus $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. To develop further connections with the standard Kalman filter theory assume

$$V_{12} = 0. \tag{4.20}$$

In this case (4.15) implies that $P = 0$ so that the gain expressions (4.10)–(4.12) become

$$A_e = A - QC^{\mathrm{T}}V_2^{-1}C, \tag{4.21}$$

$$B_e = QC^{\mathrm{T}}V_2^{-1}, \tag{4.22}$$

$$C_e = L, \tag{4.23}$$

while equations (4.14)–(4.16) and auxiliary cost (4.19) specialize to

$$0 = AQ + QA^{\mathrm{T}} + V_1 + \gamma^{-2}QL^{\mathrm{T}}RLQ - QC^{\mathrm{T}}V_2^{-1}CQ, \tag{4.24}$$

$$\mathscr{J}(A_e, B_e, C_e, \mathscr{Q}) = \operatorname{tr} L^{\mathrm{T}}RLQ. \tag{4.25}$$

Remark 4.3. Note that the necessary conditions for the full-order problem involve one modified Riccati equation. This equation is similar to the observer Riccati equation with the additional quadratic term $\gamma^{-2}QL^{\mathrm{T}}RLQ$. Finally, note that when the H_∞ estimation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, (4.24) reduces to the standard observer Riccati equation of steady-state Kalman filter theory.

5. Sufficient conditions for combined L_2/H_∞ estimation

In this section we combine Lemma 3.1 with the converse of Theorem 4.1 to obtain our main result guaranteeing constrained H_∞ state-estimation error and an optimized L_2 state-estimation error bound.

Theorem 5.1. *Suppose there exist* $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ *satisfying* (4.14)–(4.18) *and let* $(A_e, B_e, C_e, \mathscr{Q})$ *be given by* (4.10)–(4.13). *Then* $(\tilde{A}, \tilde{D})$ *is stabilizable if and only if* A_e *is asymptotically stable. In this case, the transfer function* $H(s)$ *defined by* (3.5) *satisfies the* H_∞ *state-estimation error constraint*

$$\| H(s) \|_\infty \le \gamma, \tag{5.1}$$

and the L_2 *state-estimation error criterion* (3.7) *satisfies the bound*

$$J(A_e, B_e, C_e) \le \operatorname{tr} L^{\mathrm{T}}RL(Q + \gamma^{-2}QPS\hat{Q}S^{\mathrm{T}}PQ). \tag{5.2}$$

Proof. The converse portion of Theorem 4.1 implies that $\mathscr{Q}$ given by (4.13) satisfies (3.14) and (3.15). It now follows from Lemma 3.1 that the stabilizability condition (3.16) is equivalent to the asymptotic stability of A_e, the H_∞ state-estimation error constraint (3.18) holds, and the L_2 state-estimation error criterion (3.7) satisfies the bound (3.20) which, by (4.19), is equivalent to (5.2). □

In applying Theorem 5.1 the principal issue concerns conditions on the problem data under which the coupled Riccati equations (4.14)–(4.17) possess nonnegative-definite solutions. Clearly, for γ sufficiently large, (4.14)–(4.17) approximate the pure least squares problem considered in [2]. The important case of interest, however, involves small γ so that significant H_∞ estimation is enforced. Thus, if (5.1) can be satisfied for a given $\gamma > 0$, it is of interest to know whether one such fixed-order estimator can be obtained by solving (4.14)–(4.17). Lemma 3.2 guarantees that (3.15) possesses a solution for any fixed-order estimator satisfying (5.1). Thus our sufficient conditions will also be *necessary* so long as the Auxiliary Minimization Problem possesses at least one extremal over $\mathscr{S}$. When this is the case we have the following result.

Proposition 5.1. *Let* γ^* *denote the infimum of* $\| H(s) \|_\infty$ *over all asymptotically stable fixed-order estimators and suppose that the Auxiliary Minimization Problem has an extremal for all* $\gamma > \gamma^*$. *Then*

for all $\gamma > \gamma^*$ *there exist* Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ *satisfying* (4.14)–(4.17).

Appendix A: Proof of Theorem 4.1

To optimize (3.21) over the open set $\mathscr{S}$ subject to the constraint (3.15), form the Lagrangian

$$\mathscr{L}(A_e, B_e, C_e, \mathscr{Q}, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\{\lambda\mathscr{Q}\tilde{R} + [\tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q} + \tilde{V}]\mathscr{P}\}, \tag{A.1}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial\mathscr{L}}{\partial\mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R}. \tag{A.2}$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R}. \tag{A.3}$$

Since $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ is assumed to be stable, $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathscr{P}$ is nonnegative definite.

Now partition $\tilde{n}\times\tilde{n}$, Q, P into $n\times n$, $n\times n_e$, and $n_e\times n_e$ subblocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \qquad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix},$$

and for notational convenience define

$$\mathscr{P}\mathscr{Q} = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix},$$

where

$$Z_1 \triangleq P_1Q_1 + P_{12}Q_{12}^{\mathrm{T}}, \qquad Z_{12} \triangleq P_1Q_{12} + P_{12}Q_2,$$
$$Z_{21} \triangleq P_{12}^{\mathrm{T}}Q_1 + P_2Q_{12}^{\mathrm{T}}, \qquad Z_2 \triangleq P_{12}^{\mathrm{T}}Q_{12} + P_2Q_2.$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial\mathscr{L}}{\partial\mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \tilde{R} = 0, \tag{A.4}$$

$$\frac{\partial\mathscr{L}}{\partial A_e} = Z_2 = 0, \tag{A.5}$$

$$\frac{\partial\mathscr{L}}{\partial B_e} = Z_{21}C^{\mathrm{T}} + P_{12}^{\mathrm{T}}V_{12} + P_2B_eV_2 = 0, \tag{A.6}$$

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial C_e} &= 2RC_eQ_2 + 2\gamma^{-2}RC_eZ_{12}^{\mathrm{T}}Q_{12} \\ &\quad - 2RLQ_{12} - \gamma^{-2}RLZ_1^{\mathrm{T}}Q_{12} \\ &\quad - \gamma^{-2}RLQ_1Z_{12} - \gamma^{-2}RLZ_{21}^{\mathrm{T}}Q_2 \\ &= 0.\end{aligned} \tag{A.7}$$

Expanding (3.15) and (A.4) yields

$$\begin{aligned}0 &= AQ_1 + Q_1A^{\mathrm{T}} + V_1 \\ &\quad + \gamma^{-2}(Q_1L^{\mathrm{T}} - Q_{12}C_e^{\mathrm{T}})R(Q_1L^{\mathrm{T}} - Q_{12}C_e^{\mathrm{T}})^{\mathrm{T}},\end{aligned} \tag{A.8}$$

$$\begin{aligned}0 &= AQ_{12} + Q_{12}A_e^{\mathrm{T}} + Q_1C^{\mathrm{T}}B_e^{\mathrm{T}} + V_{12}B_e^{\mathrm{T}} \\ &\quad + \gamma^{-2}Q_1L^{\mathrm{T}}RLQ_{12} - \gamma^{-2}Q_{12}C_e^{\mathrm{T}}RLQ_{12} \\ &\quad - \gamma^{-2}Q_1L^{\mathrm{T}}RC_eQ_2 + \gamma^{-2}Q_{12}C_e^{\mathrm{T}}RC_eQ_2,\end{aligned} \tag{A.9}$$

$$\begin{aligned}0 &= A_eQ_2 + Q_2A_e^{\mathrm{T}} + B_eCQ_{12} + Q_{12}^{\mathrm{T}}C^{\mathrm{T}}B_e^{\mathrm{T}} + B_eV_2B_e^{\mathrm{T}} \\ &\quad + \gamma^{-2}(Q_{12}^{\mathrm{T}}L^{\mathrm{T}} - Q_2C_e^{\mathrm{T}})R(Q_{12}^{\mathrm{T}}L^{\mathrm{T}} - Q_2C_e^{\mathrm{T}})^{\mathrm{T}},\end{aligned} \tag{A.10}$$

$$\begin{aligned}0 &= A^{\mathrm{T}}P_1 + P_1A + C^{\mathrm{T}}B_e^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}B_eC \\ &\quad + \gamma^{-2}L^{\mathrm{T}}RLZ_1^{\mathrm{T}} + \gamma^{-2}Z_1L^{\mathrm{T}}RL \\ &\quad - \gamma^{-2}L^{\mathrm{T}}RC_eZ_{12}^{\mathrm{T}} - \gamma^{-2}Z_{12}C_e^{\mathrm{T}}RL + L^{\mathrm{T}}RL,\end{aligned} \tag{A.11}$$

$$\begin{aligned}0 &= A^{\mathrm{T}}P_{12} + P_{12}A_e + C^{\mathrm{T}}B_e^{\mathrm{T}}P_2 \\ &\quad + \gamma^{-2}L^{\mathrm{T}}RLZ_{21}^{\mathrm{T}} - \gamma^{-2}Z_1L^{\mathrm{T}}RC_e \\ &\quad + \gamma^{-2}Z_{12}C_e^{\mathrm{T}}RC_e - L^{\mathrm{T}}RC_e,\end{aligned} \tag{A.12}$$

$$\begin{aligned}0 &= A_e^{\mathrm{T}}P_2 + P_2A_e - \gamma^{-2}C_e^{\mathrm{T}}RLZ_{21}^{\mathrm{T}} \\ &\quad - \gamma^{-2}Z_{21}L^{\mathrm{T}}RC_e + C_e^{\mathrm{T}}RC_e.\end{aligned} \tag{A.13}$$

Now define the $n\times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$
$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$
$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^{\mathrm{T}},$$

and the $n_e\times n$, $n_e\times n_e$, and $n_e\times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^{\mathrm{T}}.$$

The existence of Q_2^{-1} and P_2^{-1} follows from the

fact that (A_e, B_e, C_e) is minimal. See [1,2] for details. Note that $\tau = G^{\mathrm{T}}\Gamma$. Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and nonnegative definite.

Next note that with the above definitions, (A.5) implies (4.2) and that (4.1) holds. Hence $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Sylvester's inequality yields (4.18). Note also that (4.6) and (4.7) hold.

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \tag{A.14}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \qquad P_{12} = -\hat{P}G^{\mathrm{T}}, \tag{A.15}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \qquad P_2 = G\hat{P}G^{\mathrm{T}}. \tag{A.16}$$

Next note that by using (A.14)–(A.16), (A.7) becomes

$$C_e\hat{S} = L\left[I_n + \gamma^{-2}(Q + \hat{Q})P\right]G^{\mathrm{T}},$$

where

$$\hat{S} \triangleq I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}.$$

To prove that $\hat{S}$ is invertible use (4.6) and (4.3) and note that

$$I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}} = I_{n_e} + \gamma^{-2}\Gamma\hat{Q}\tau^{\mathrm{T}}PG^{\mathrm{T}}$$
$$= I_{n_e} + \gamma^{-2}(\Gamma\hat{Q}\Gamma^{\mathrm{T}})(GPG^{\mathrm{T}}).$$

Since $\Gamma\hat{Q}\Gamma^{\mathrm{T}}$ and GPG^{T} are nonnegative definite, their product has nonnegative eigenvalues. Thus each eigenvalue of $I_{n_e} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$ is real and is greater than unity. Hence $\hat{S}$ is invertible. Now note that by using (4.2) and (4.3) it can be shown that

$$G^{\mathrm{T}}\hat{S}^{-1} = SG^{\mathrm{T}}.$$

The expressions (4.11), (4.12) and (4.13) follow from (A.6), (A.7), (4.8) and the definition of $\mathcal{Q}$ by using the above identities. Next, computing either Γ(A.9) − (A.10) or G(A.12) + (A.13) yields (4.10). Substituting this expression for A_e into (A.8) − (A.13) it follows that (A.10) = Γ(A.9) and (A.13) = G(A.12). Thus, (A.10) and (A.13) are superfluous and can be omitted. Next, using

$$(\text{A.8}) + G^{\mathrm{T}}\Gamma(\text{A.9})G - (\text{A.9})G - \left[(\text{A.9})G\right]^{\mathrm{T}}$$

and

$$G^{\mathrm{T}}\Gamma(\text{A.9})G - (\text{A.9})G - \left[(\text{A.9})G\right]^{\mathrm{T}}$$

yields (4.14) and (4.16). Using

$$(\text{A.11}) + \Gamma^{\mathrm{T}}G(\text{A.12})\Gamma - (\text{A.12})\Gamma - \left[(\text{A.12})\Gamma\right]^{\mathrm{T}}$$

and

$$\Gamma^{\mathrm{T}}G(\text{A.12})\Gamma - (\text{A.12})\Gamma - \left[(\text{A.12})\Gamma\right]^{\mathrm{T}}$$

yields (4.15) and (4.17).

Finally, to prove the converse we use (4.10)–(4.18) to obtain (3.15) and (A.4)–(A.7). Let A_e, B_e, C_e, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 4.1 and define Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2 by (A.14)–(A.16). Using (4.4), (4.11) and (4.12) it is easy to verify (A.6) and (A.7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ and τ into (4.14)–(4.17) along with (4.2), (4.3), (4.6) and (4.7) to obtain (3.15) and (A.4). Finally, note that

$$\mathcal{Q} = \begin{bmatrix} Q & 0_{n\times n_e} \\ 0_{n_e\times n} & 0_{n_e} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix}\hat{Q}\left[I_n \;\; \Gamma^{\mathrm{T}}\right],$$

which shows that $\mathcal{Q} \geq 0$. □

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LQG Control with an H_∞ Performance Bound: A Riccati Equation Approach

DENNIS S. BERNSTEIN, MEMBER, IEEE, AND WASSIM M. HADDAD, MEMBER, IEEE

Abstract—An LQG control-design problem involving a constraint on H_∞ disturbance attenuation is considered. The H_∞ performance constraint is embedded within the optimization process by replacing the covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on L_2 performance. In contrast to the pair of separated Riccati equations of standard LQG theory, the H_∞-constrained gains are given by a coupled system of three modified Riccati equations. The coupling illustrates the breakdown of the separation principle for the H_∞-constrained problem. Both full- and reduced-order design problems are considered with an H_∞ attenuation constraint involving both state and control variables. An algorithm is developed for the full-order design problem and illustrative numerical results are given.

I. INTRODUCTION

THE fundamental differences between Wiener-Hopf-Kalman (WHK) control design (for example, LQG theory [1]) and H_∞ control theory [2]-[4] can be traced to the modeling and treatment of uncertain exogenous disturbances. As explained by Zames in the classic paper [2], LQG design is based upon a stochastic noise disturbance model possessing a fixed covariance (power spectral density), while H_∞ theory is predicated on a deterministic disturbance model consisting of bounded power (square-integrable) signals. Since LQG design utilizes a quadratic cost criterion, it follows from Plancherel's theorem that WHK theory strives to minimize the L_2 norm of the closed-loop frequency response, while H_∞ theory seeks to minimize the worst-case attenuation. For systems with poorly modeled disturbances which may possess significant power within arbitrarily small bandwidths, H_∞ is clearly appropriate, while for systems with well-known disturbance power spectral densities, WHK design may be less conservative.

In addition to the fact that H_∞ design embodies many classical design objectives [5], it also presents a natural tool for modeling plant uncertainty in terms of normed H_∞ plant neighborhoods. In contrast, the H_2 topology has been shown in [6] to be too weak for a practical robustness theory, while the H_∞ norm is not only suitable for robust stabilization but is also conveniently submultiplicative. Within the WHK state-space theory, however, the appropriate robustness model appears not to be a nonparametric normed plant neighborhood as in H_∞ theory, but rather a parametric uncertainty model. The principal technique for bounding the effects of real parameters within state-space models is Lyapunov function theory (see, e.g., [7]-[16] and the references therein). Such structured uncertainties are difficult to capture nonconservatively within H_∞ theory except with specialized refinements [17].

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D. S. Bernstein is with the Government Aerospace Systems Division, Harris Corporation, Melbourne, FL 32902.

W. M. Haddad is with the Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901.

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In spite of the fundamental differences between WHK design and H_∞ theory, a significant connection was discovered in [18]. Specifically, Petersen observed that a modified algebraic Riccati equation developed for parameter-robust full-state-feedback control can be reinterpreted to yield controllers satisfying H_∞ disturbance attenuation bounds. This relationship was further explored in [19] where it was shown that the H_∞-optimal static full-state-feedback controller is also optimal over the class of dynamic full-state-feedback controllers. The results of [18]-[20] thus solve the standard problem considered in [3] and [4] for the full-state-feedback case.

The extension of these results to the standard problem for dynamic output-feedback compensation, however, was not given in [18]-[20]. Within the realm of quadratic robust stabilization, the dynamic output-feedback problem was addressed in [7]. The results of [7] involve a pair of decoupled modified Riccati equations along with an auxiliary inequality. Using different techniques, a more complete solution was obtained in [13] and [14] involving a coupled system of three modified Riccati equations for full-order dynamic compensation and a coupled system of four modified Riccati and Lyapunov equations in the fixed-order (i.e., reduced-order) case as in [21]. The results of [13] and [14] thus raise the following question: What is the relevance of this system of coupled design equations to the problem of H_∞ disturbance attenuation via fixed-order compensation?

To begin to address this question, the goal of the present paper is to develop a design methodology for fixed-order, i.e., full- and reduced-order, L_2 optimal control which includes as a design constraint a bound on H_∞ disturbance attenuation. There are three principal motivations for developing such a theory. First, the results of [18]-[20] present full-state-feedback controllers whose form is directly analogous to the standard LQR solution. However, no L_2 interpretation was provided in [18]-[20] to explain this similarity. The present paper thus provides an L_2 interpretation within the context of an H_∞ design constraint. A novel feature of this mathematical formulation is the dual interpretation of the disturbances. That is, within the context of L_2 optimality the disturbances are interpreted as white noise signals while, simultaneously, for the purpose of H_∞ attenuation the very same disturbance signals have the alternative interpretation of deterministic L_2 functions. This dual interpretation is unique to the present paper since stochastic modeling plays no role in [18]-[20]. We also note recent results obtained in [22] which essentially show that the H_2 plant topology can be induced by imposing L_2 and L_∞ topologies on the disturbance and output spaces, respectively. For further investigation into the relationships between L_2 and H_∞ control, see [22a].

The second motivation for our approach is the simultaneous treatment of both L_2 and H_∞ performance criteria which quantitatively demonstrates design tradeoffs. Specifically, in order to enforce the H_∞ constraint we derive an upper bound for the L_2 criterion. Minimization of this upper bound shows that the enforcement of an H_∞ disturbance attenuation constraint leads directly to an increase in the L_2 performance criterion.

The third motivation for our approach is to provide a characterization of fixed-order dynamic output-feedback control-

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lers yielding specified disturbance attenuation. Existing optimal H_∞ design methods tend to yield high-order controllers. Intuitively, solving the fixed-order design equations for progressively smaller H_∞ disturbance attenuation constraints should, in the limit, yield an H_∞-optimal controller over the class of fixed-order stabilizing controllers. Although our main result gives sufficient conditions, we also state hypotheses under which these conditions are also necessary (Proposition 4.[illegible]). It should also be noted that the inherent coupling among the modified Riccati equations shows that the classical separation principle of LQG theory is not valid for the H_∞-constrained full- and reduced-order design problems.

In the full-order case involving equalized L_2 and H_∞ performance weights, we also show that the H_∞-constrained gains are given by two rather than three equations (Section V). These two equations are precisely those given in [26] for the pure H_∞ problem without an L_2 interpretation. Since the results of [26] are necessary as well as sufficient, these connections show that our sufficient conditions (at least in this special case) are also necessary. The authors are indebted to Prof. J. C. Doyle for pointing out these relationships and to D. Mustafa for providing a preprint of [45] which further clarifies these connections.

Besides establishing connections with robust stabilizability in state-space systems, an immediate benefit of the modified Riccati equation characterization of H_∞-constrained controllers is the opportunity to develop novel computational algorithms for controller synthesis. To this end a continuation algorithm has been developed for solving the coupled system of three modified Riccati equations. In a numerical study (see Section VIII) we have demonstrated convergence of the algorithm and reasonable computational efficiency. Homotopy methods were suggested for the coupled Riccati equations because of their demonstrated effectiveness in related problems which also involve coupled modified Riccati equations [23]–[25]. Since H_∞ control problems are solvable by established numerical methods [4], it should be stressed that the objective of these numerical studies is not to make direct comparisons with existing H_∞ synthesis algorithms, but rather to demonstrate solvability of the coupled modified Riccati equations.

The contents of the paper are as follows. After presenting notation at the end of this section, the statement of the H_∞-constrained LQG control problem is given in Section II. The principal result of Section II (Lemma 2.1) shows that if the algebraic Lyapunov equation for the closed-loop covariance is replaced by a modified Riccati equation possessing a nonnegative-definite solution, then the closed-loop system is asymptotically stable, the H_∞ disturbance attenuation constraint is satisfied, and the L_2 performance is bounded above by an auxiliary cost function. The problem of determining compensator gains which minimize this upper bound subject to the Riccati equation constraint is considered in Section III as the auxiliary minimization problem. Necessary conditions for the auxiliary minimization problem (Theorem 3.1) are given in the form of a coupled system of three modified Riccati equations. In Section IV the necessary conditions of Theorem 3.1 are combined with Lemma 2.1 to yield sufficient conditions for closed-loop stability, H_∞ disturbance attenuation, and bounded L_2 performance. In Section V we derive alternative forms of the design equations and specialize the results to the simpler case in which the LQG weights are equal to the H_∞ weights To achieve further design flexibility, the reduced-order control-design problem is considered in Section VI. A simplified qualitative analysis of the full-order design equations is given in Section VII to highlight important features with regard to existence and multiplicity of solutions. Finally, a numerical algorithm is presented in Section VIII along with illustrative numerical results. A series of designs is obtained to illustrate tradeoffs between the L_2 and H_∞ aspects and the conservatism of the L_2 performance bound. Although in the present paper the numerical results are limited to the case of full-order dynamic compensation, reduced-order designs have been obtained in [27] using Theorem 6.1.

Notation

Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
I_r, $(\ )^T$, $0_{r\times s}$, 0_r	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
tr, $\rho(\ )$	Trace, spectral radius
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \leq Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
$n, m, l, n_c, p, q, q_\infty; \tilde{n}$	Positive integers; $n + n_c (n_c \leq n)$
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors
$\tilde{x}$	$\begin{bmatrix} x \\ x_c \end{bmatrix}$
A, B, C	$n \times n$, $n \times m$, $l \times n$ matrices
A_c, B_c, C_c	$n_c \times n_c$, $n_c \times l$, $m \times n_c$ matrices
$\tilde{A}$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$
$w(\cdot)$	p-dimensional standard white noise
D_1, D_2	$n \times p$, $l \times p$ matrices; $D_1D_2^T = 0$
V_1, V_2	$D_1D_1^T$, $D_2D_2^T$; $V_2 \in \mathbb{P}^l$
$\tilde{D}$	$\begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$
$\tilde{V}$	$\begin{bmatrix} V_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & B_cV_2B_c^T \end{bmatrix}$
E_1, E_2	$q \times n$, $q \times m$ matrices; $E_1^TE_2 = 0$
$\tilde{E}$	$[E_1\ E_2C_c]$
R_1, R_2	$E_1^TE_1$, $E_2^TE_2$; $R_2 \in \mathbb{P}^m$
$\tilde{R}$	$\begin{bmatrix} R_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_2C_c \end{bmatrix} = \tilde{E}^T\tilde{E}$
$E_{1\infty}, E_{2\infty}$	$q_\infty \times n$, $q_\infty \times m$ matrices; $E_{1\infty}^TE_{2\infty} = 0$
$\tilde{E}_\infty$	$[E_{1\infty}\ E_{2\infty}C_c]$
$R_{1\infty}, R_{2\infty}$	$E_{1\infty}^TE_{1\infty}$, $E_{2\infty}^TE_{2\infty}$
$\tilde{R}_\infty$	$\begin{bmatrix} R_{1\infty} & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_{2\infty}C_c \end{bmatrix} = \tilde{E}_\infty^T\tilde{E}_\infty$
$\Sigma, \hat{\Sigma}$	$BR_2^{-1}B^T$, $C^TV_2^{-1}C$
β, γ	Nonnegative constant, positive constant

II. Statement of the Problem

In this section we introduce the LQG dynamic output-feedback control problem with constrained H_∞ disturbance attenuation between the plant and sensor disturbances and the state and control variables. Without the L_2 performance criterion, the problem considered here essentially corresponds to the standard problem of [3] and [4]. For simplicity we restrict our attention to controllers of order $n_c = n$ only, i.e., controllers whose order is equal to the dimension of the plant. This constraint is removed in Section VI where controllers of reduced order are considered. Hence, throughout Sections II–V the controller dimension n_c and closed-loop plant dimension $\tilde{n} \triangleq n + n_c$ should be interpreted as n and $2n$, respectively. Controllers of order greater than n are generally of less interest in practice and thus are not considered in this paper.

H_∞-Constrained LQG Control Problem: Given the nth-order stabilizable and detectable plant

$$\dot{x}(t)=Ax(t)+Bu(t)+D_1w(t), \tag{2.1}$$

$$y(t)=Cx(t)+D_2w(t) \tag{2.2}$$

determine an nth-order dynamic compensator

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t), \tag{2.3}$$

$$u(t)=C_cx_c(t) \tag{2.4}$$

which satisfies the following design criteria:

i) the closed-loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;

ii) the $q_\infty \times p$ closed-loop transfer function

$$H(s) \triangleq \tilde{E}_\infty(sI_{\tilde{n}}-\tilde{A})^{-1}\tilde{D} \tag{2.5}$$

from $w(t)$ to $E_{1\infty}x(t) + E_{2\infty}u(t)$ satisfies the constraint

$$\|H(s)\|_\infty \le \gamma \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[x^T(t)R_1x(t)+u^T(t)R_2u(t)] \tag{2.7}$$

is minimized.

Note that the closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)+\tilde{D}w(t) \tag{2.8}$$

and that (2.7) becomes

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \mathbb{E}[(\tilde{E}\tilde{x}(t))^T(\tilde{E}\tilde{x}(t))]$$

$$=\lim_{t\to\infty} \mathbb{E}[\tilde{x}^T(t)\tilde{R}\tilde{x}(t)]. \tag{2.9}$$

Remark 2.1: Since (A, B, C) is assumed to be stabilizable and detectable the set of nth-order stabilizing compensators is nonempty.

Remark 2.2: It is easy to show that the performance functional (2.7) is equivalent to the more familiar expression involving an averaged integral, i.e.,

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left\{\int_0^t [x^T(s)R_1x(s)+u^T(s)R_2u(s)]\,ds\right\}.$$

Remark 2.3: For convenience we assume $D_1D_2^T = 0$, which effectively implies that the plant disturbance and sensor noise are uncorrelated.

Remark 2.4: One may also consider a general L_2 optimization problem of the form min $\|T - UQV\|_2$, where Q is a parameterization of stabilizing controllers. In this case, without a constraint on the MacMillan degree of Q, it may be possible to satisfy (2.6) with smaller values of γ.

Note that the problem statement involves both L_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the L_2 weights for the state and control variables. By introducing L_2-weighted variables

$$z(t)=E_1x(t), \quad v(t)=E_2u(t)$$

the cost (2.7) can be written as

$$J(A_c, B_c, C_c)=\lim_{t\to\infty} \mathbb{E}[z^T(t)z(t)+v^T(t)v(t)].$$

For convenience we thus define $R_1 \triangleq E_1^TE_1$ and $R_2 \triangleq E_2^TE_2$ which appear in subsequent expressions. Although an L_2 cross-weighting term of the form $2x^T(t)R_{12}u(t)$ can also be included, we shall not do so here to facilitate the presentation.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$ and $E_{2\infty}$ for the state and control variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^TE_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^TE_{2\infty}$ are thus the H_∞ counterparts of the L_2 weights R_1 and R_2. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2, we shall require in the next section that $R_{2\infty} = \beta^2R_2$, where the nonnegative scalar β is a design variable. Finally, the condition $E_{1\infty}^TE_{2\infty} = 0$ precludes an H_∞ cross-weighting term which again facilitates the presentation.

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given compensator (A_c, B_c, C_c), then the performance (2.7) is given by

$$J(A_c, B_c, C_c)=\operatorname{tr} \tilde{Q}\tilde{R} \tag{2.10}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)] \tag{2.11}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0=\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V}. \tag{2.12}$$

Remark 2.5: Using (2.10) and (2.12) it can be shown that the L_2 cost criterion (2.7) can be written in terms of the L_2 norm of the impulse response of the closed-loop system. Specifically, by writing $\tilde{Q}$ satisfying (2.12) as

$$\tilde{Q}=\int_0^\infty e^{\tilde{A}t}\tilde{V}e^{\tilde{A}^Tt}\,dt$$

(2.10) becomes

$$J(A_c, B_c, C_c)=\int_0^\infty \|\tilde{E}e^{\tilde{A}t}\tilde{D}\|_F^2\,dt$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.12) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1: Let (A_c, B_c, C_c) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.13}$$

and

$$0=\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^T+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}+\tilde{V}. \tag{2.14}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.15}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.16}$$

In this case,

$$\|H(s)\|_\infty \le \gamma \tag{2.17}$$

and

$$\tilde{Q} \le \mathcal{Q}. \tag{2.18}$$

Consequently,

$$J(A_c, B_c, C_c) \le \mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) \tag{2.19}$$

where

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{2.20}$$

Proof: It follows from [28, Theorem 3.6] that (2.15) implies that $(\tilde{A}, [\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V}]^{1/2})$ is also stabilizable. Using the assumed existence of a nonnegative-definite solution to (2.14) and [28, Lemma 12.2], it now follows that $\tilde{A}$ is asymptotically stable. The converse is immediate. The proof of (2.17) follows from a standard manipulation of (2.14); for details see [29, Lemma 1]. To prove (2.18), subtract (2.12) from (2.14) to obtain

$$0 = \tilde{A}(\mathcal{Q} - \tilde{Q}) + (\mathcal{Q} - \tilde{Q})\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} \tag{2.21}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q} - \tilde{Q} = \int_0^\infty e^{\tilde{A}t}[\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}]e^{\tilde{A}^Tt}\, dt \ge 0. \tag{2.22}$$

Finally, (2.19) follows immediately from (2.18). □

Remark 2.6: Note that (2.15) is actually a closed-loop disturbability condition which is not concerned with control as such. This condition guarantees that the system does not possess undisturbed unstable modes. Of course, if $\tilde{V}$ is positive definite or $(\tilde{A}, \tilde{D})$ is controllable, then (2.15) is satisfied.

Lemma 2.1 shows that the H_∞ disturbance attenuation constraint is automatically enforced when a nonnegative-definite solution to (2.14) is known to exist and $\tilde{A}$ is asymptotically stable. Furthermore, all such solutions provide upper bounds for the actual closed-loop state covariance $\tilde{Q}$ along with a bound on the L_2 performance criterion. Next, we present a partial converse of Lemma 2.1 which guarantees the existence of a unique minimal nonnegative-definite solution to (2.14) when (2.17) is satisfied. The minimal solution is desirable since it yields the least performance bound in (2.19). This was first pointed out in [45].

Lemma 2.2: Let (A_c, B_c, C_c) be given, suppose $\tilde{A}$ is asymptotically stable, and assume the disturbance attenuation constraint (2.17) is satisfied. Then there exists a unique nonnegative-definite solution $\mathcal{Q}$ satisfying (2.14) and such that $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ is asymptotically stable. Furthermore, this solution is also minimal.

Proof: The result is an immediate consequence of [30, pp. 150 and 167], using Theorems 3 and 2, along with the dual version of [28, Lemma 12.2]. The proof of minimality is given in [29]. □

Remark 2.7: To further clarify the relationships between the L_2 and H_∞ aspects of the problem, we note that the closed-loop system can be represented by two possibly different transfer functions. Specifically, with respect to the L_2 cost criterion, the closed-loop transfer function between disturbances and controlled variables is given by the triple $(\tilde{A}, \tilde{D}, \tilde{E})$ while for the H_∞ constraint the closed-loop transfer function (2.5) corresponds to the triple $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$.

Finally, it can be shown that the closed-loop Riccati equation (2.14) also enforces a constraint on the norm of the Hankel operator corresponding to the closed-loop system $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$ when $\mathcal{Q}$ is positive definite. Thus, let $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ denote the solution to

$$0 = \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}_\infty \tag{2.23}$$

and note that $\tilde{P}$ and $\tilde{Q}$ [satisfying (2.12)] are the observability and controllability Gramians, respectively, of the system $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$. As shown in [31], the norm of the Hankel operator corresponding to $(\tilde{A}, \tilde{D}, \tilde{E}_\infty)$ is given by $\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})$.

Proposition 2.1: Suppose there exists $\mathcal{Q} \in \mathbb{P}^{\tilde{n}}$ satisfying (2.14) and such that (2.15) or, equivalently, (2.16) holds. Then

$$\lambda_{\max}^{1/2}(\tilde{P}\mathcal{Q}) \le \gamma. \tag{2.24}$$

Proof: Since $\mathcal{Q}$ is assumed to be invertible, (2.14) is equivalent to

$$0 = \gamma^2\tilde{A}^T\mathcal{Q}^{-1} + \gamma^2\mathcal{Q}^{-1}\tilde{A} + \gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1} + \tilde{R}_\infty. \tag{2.25}$$

Subtracting (2.23) from (2.25) shows that $\gamma^2\mathcal{Q}^{-1} - \tilde{P} \ge 0$, or, equivalently, $\gamma^2 I_{\tilde{n}} \ge \mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}$. Thus,

$$\gamma^2 \ge \lambda_{\max}(\mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}) = \lambda_{\max}(\tilde{P}^{1/2}\mathcal{Q}\tilde{P}^{1/2}) \ge \lambda_{\max}(\tilde{P}^{1/2}\tilde{Q}\tilde{P}^{1/2})$$
$$= \lambda_{\max}(\tilde{P}\tilde{Q})$$

which yields (2.24). □

III. The Auxiliary Minimization Problem and Necessary Conditions for Optimality

As discussed in the previous section, the replacement of (2.12) by (2.14) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the L_2 performance criterion. That is, given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.14), the actual L_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$. Hence, $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ can be interpreted as an auxiliary cost which leads to the following mathematical programming problem.

Auxiliary Minimization Problem: Determine $(A_c, B_c, C_c, \mathcal{Q})$ which minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ subject to (2.13) and (2.14).

It follows from Lemma 2.1 that the satisfaction of (2.13) and (2.14) along with the generic condition (2.15) leads to: 1) closed-loop stability; 2) prespecified H_∞ performance attenuation; and 3) an upper bound for the L_2 performance criterion. Hence, it remains to determine (A_c, B_c, C_c) which minimizes $J(A_c, B_c, C_c, \mathcal{Q})$, and thus provides an optimized bound for the actual L_2 performance $J(A_c, B_c, C_c)$. Rigorous derivation of the necessary conditions for the auxiliary minimization problem requires additional technical assumptions. Specifically, we restrict $(A_c, B_c, C_c, \mathcal{Q})$ to the open set

$$\mathfrak{X} \triangleq \{(A_c, B_c, C_c, \mathcal{Q}) : \mathcal{Q} \in \mathbb{P}^{\tilde{n}}, \tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty \text{ is asymptotically stable, and } (A_c, B_c, C_c) \text{ is controllable and observable}\}. \tag{3.1}$$

Remark 3.1: The set $\mathfrak{X}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $\mathcal{Q}$ be positive definite replaces (2.13) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ serves as a normality condition, and (A_c, B_c, C_c) minimal is a nondegeneracy condition. Note that the stabilizability condition (2.15) and stability condition (2.16) play no role in determining solutions of the auxiliary minimization problem.

The following result presents the necessary conditions for optimality in the auxiliary minimization problem. The proof of this result is given in the Appendix as a special case of the corresponding result for reduced-order dynamic compensation considered in Section VI. As mentioned previously, we assume that $R_{2\infty} = \beta^2 R_2$, where $\beta \ge 0$. Furthermore, for arbitrary $\hat{Q}, P \in \mathbb{N}^n$ define

$$S \triangleq (I_n + \beta^2\gamma^{-2}\hat{Q}P)^{-1}. \tag{3.2}$$

Since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the nonnegative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has nonnegative eigenvalues. Thus, the eigenvalues of $I_n + \beta^2\gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists.

Theorem 3.1: If $(A_c, B_c, C_c, \mathfrak{Q}) \in \mathfrak{X}$ solves the auxiliary minimization problem then there exist $Q, P, \hat{Q} \in \mathfrak{N}^n$ such that

$$A_c = A - Q\bar{\Sigma} - \Sigma PS + \gamma^{-2}QR_{1\infty}, \tag{3.3}$$

$$B_c = QC^T V_2^{-1}, \tag{3.4}$$

$$C_c = -R_2^{-1}B^T PS, \tag{3.5}$$

$$\mathfrak{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix} \tag{3.6}$$

and such that $Q, P, \hat{Q}$ satisfy

$$0 = AQ + QA^T + V_1 + \gamma^{-2}QR_{1\infty}Q - Q\bar{\Sigma}Q, \tag{3.7}$$

$$0 = (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty}) + R_1 - S^T P\Sigma PS, \tag{3.8}$$

$$0 = (A - \Sigma PS + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma PS + \gamma^{-2}QR_{1\infty})^T + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2 S^T P\Sigma PS)\hat{Q} + Q\bar{\Sigma}Q. \tag{3.9}$$

Furthermore, the auxiliary cost is given by

$$\mathfrak{J}(A_c, B_c, C_c, \mathfrak{Q}) = \operatorname{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}S^T P\Sigma PS]. \tag{3.10}$$

Conversely, if there exist $Q, P, \hat{Q} \in \mathfrak{N}^n$ satisfying (3.7)–(3.9), then $(A_c, B_c, C_c, \mathfrak{Q})$ given by (3.3)–(3.6) satisfies (2.13) and (2.14) with auxiliary cost (2.20) given by (3.10).

Remark 3.2: If Q and $\hat{Q}$ are nonnegative definite, then the fact that the definiteness condition (2.13) is satisfied can easily be seen by writing $\mathfrak{Q}$ as

$$\mathfrak{Q} = \begin{bmatrix} Q & 0_n \\ 0_n & 0_n \end{bmatrix} + \begin{bmatrix} \hat{Q}^{1/2} \\ \hat{Q}^{1/2} \end{bmatrix} \begin{bmatrix} \hat{Q}^{1/2} \\ \hat{Q}^{1/2} \end{bmatrix}^T.$$

As mentioned in Section II, it is desirable to determine solutions Q and $\hat{Q}$ which yield the minimal solution to (2.14).

Remark 3.3: Setting $\beta = 0$, or equivalently, $E_{2\infty} = 0$, specializes Theorem 3.1 to the cheap H_∞ control case in which H_∞ attenuation between disturbances and controls is not constrained. In this case $S = I_n$, $\mathfrak{Q}$ is given by (3.6), and (3.3)–(3.5) become

$$A_c = A - Q\bar{\Sigma} - \Sigma P + \gamma^{-2}QR_{1\infty}, \tag{3.11}$$

$$B_c = QC^T V_2^{-1}, \tag{3.12}$$

$$C_c = -R_2^{-1}B^T P \tag{3.13}$$

where Q satisfies (3.7), and (3.8) and (3.9) become

$$0 = (A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q+\hat{Q}]R_{1\infty}) + R_1 - P\Sigma P, \tag{3.14}$$

$$0 = (A - \Sigma P + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \gamma^{-2}QR_{1\infty})^T + \gamma^{-2}\hat{Q}R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q. \tag{3.15}$$

Finally, the auxiliary cost reduces to

$$\mathfrak{J}(A_c, B_c, C_c, \mathfrak{Q}) = \operatorname{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}P\Sigma P]. \tag{3.16}$$

Numerical solution of (3.7), (3.14), and (3.15) is discussed in Section VIII.

Remark 3.4: Note that if both $\beta = 0$ and $R_{1\infty} = 0$, then Theorem 3.1 specializes to the standard LQG result.

Theorem 3.1 presents necessary conditions for the auxiliary minimization problem which explicitly synthesize extremal controllers (A_c, B_c, C_c). These necessary conditions comprise a system of three modified algebraic Riccati equations in variables Q, P, and $\hat{Q}$. The Q and P equations are similar to the estimator and regulator Riccati equations of LQG theory, while the $\hat{Q}$ equation has no counterpart in the standard theory. Note that the Q equation is decoupled from the P and $\hat{Q}$ equations and thus can be solved independently. The P equation, however, depends on Q. Thus, regulator/estimator separation holds in only one direction which clearly shows that the certainty equivalence principle is no longer valid for the L_2/H_∞ design problem. Furthermore, since the P and $\hat{Q}$ equations are coupled, they must be solved simultaneously. Finally, note that if the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the P equation becomes decoupled from the $\hat{Q}$ equation and thus the $\hat{Q}$ equation becomes superfluous. Furthermore, the remaining Q and P equations separate and coincide with the standard LQG result.

IV. Sufficient Conditions for H_∞ Disturbance Attenuation

In this section we combine Lemma 2.1 with the converse of Theorem 3.1 to obtain our main result guaranteeing closed-loop stability, H_∞ disturbance attenuation, and an optimized L_2 performance bound.

Theorem 4.1: Suppose there exist $Q, P, \hat{Q} \in \mathfrak{N}^n$ satisfying (3.7)–(3.9) and let $(A_c, B_c, C_c, \mathfrak{Q})$ be given by (3.3)–(3.6). Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ attenuation constraint

$$\|H(s)\|_\infty \le \gamma \tag{4.1}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \le \operatorname{tr}\,[(Q+\hat{Q})R_1 + \hat{Q}S^T P\Sigma PS]. \tag{4.2}$$

Proof: The converse portion of Theorem 3.1 implies that $\mathfrak{Q}$ given by (3.6) satisfies (2.13) and (2.14) with auxiliary cost given by (3.10). It now follows from Lemma 2.1 that the stabilizability condition (2.15) is equivalent to the asymptotic stability of $\tilde{A}$, the H_∞ disturbance attenuation constraint (2.17) holds, and the performance bound (2.19), which is equivalent to (4.2), holds. □

Remark 4.1: In applying Theorem 4.1 it is not actually necessary to check (2.15) which holds generically. Rather, it suffices to check the stability of $\tilde{A}$ directly which is guaranteed to be equivalent to (2.15).

In applying Theorem 4.1 the principal issue concerns conditions on the problem data under which the coupled Riccati equations (3.7)–(3.9) possess nonnegative-definite solutions. Clearly, for γ sufficiently large, (3.7)–(3.9) approximate the standard LQG ... so that existence is assured. The important case of interest, however, involves small γ so that significant H_∞ disturbance attenuation is enforced. Thus, if (4.1) can be satisfied for a given $\gamma > 0$, it is of interest to know whether one such controller can be obtained by solving (3.7)–(3.9). Lemma 2.2 guarantees that (2.14) possesses a solution for any controller satisfying (2.17). Thus, our sufficient condition will also be necessary as long as the auxiliary minimization problem possesses at least one extremal over $\mathfrak{X}$. When this is the case we have the following immediate result.

Proposition 4.1: Let γ^* denote the infimum of $\|H(s)\|_\infty$ over all stabilizing nth-order dynamic compensators and suppose that the auxiliary minimization problem has a solution for all $\gamma > \gamma^*$. Then for all $\gamma > \gamma^*$ there exist $Q, P, \hat{Q} \in \mathfrak{N}^n$ satisfying (3.7)–(3.9).

Unlike the standard LQG result involving a pair of separated Riccati equations, the new result enforcing H_∞ disturbance attenuation involves a nonstandard coupled system of three modified Riccati equations. The asymmetry of these equations suggests the possibility of a dual result in which the modifications

to the standard P and Q Riccati equations are reversed. Such a dual result will generally be different from Theorem 4.1 and thus will yield an improved bound for particular problems. This point was demonstrated in [16] for the problem of robust performance analysis. Due to space limitations, however, we give only a brief outline of the dual H_∞ results. Note that $J(A_c, B_c, C_c)$ given by (2.10) is also given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{P}\tilde{V} \tag{4.3}$$

where $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ is the unique solution to (2.23) with $\tilde{R}_\infty$ replaced by $\tilde{R}$. Next, utilizing (4.3) in place of (2.10), the H_∞ disturbance attenuation constraint (2.6) can now be enforced by replacing (2.23) by the Riccati equation

$$0 = \tilde{A}^T\mathcal{P} + \mathcal{P}\tilde{A} + \gamma^{-2}\mathcal{P}\tilde{V}_\infty\mathcal{P} + \tilde{R} \tag{4.4}$$

where $\tilde{V}_\infty$ has the same form as $\tilde{V}$ but may involve weights $V_{1\infty}$ and $V_{2\infty}$. Note that (4.4) is merely the dual of (2.14). We also require the condition dual to (2.15) given by

$$(\tilde{E}, \tilde{A}) \text{ is detectable} \tag{4.5}$$

and that $\tilde{A} + \gamma^{-2}\tilde{V}_\infty\mathcal{P}$ be asymptotically stable. Once again, the sufficient conditions for H_∞ disturbance attenuation involve a coupled system of three modified Riccati equations dual to (3.7)–(3.9). Similar remarks apply to the reduced-order case given by Theorem 6.1 below. Finally, if $\tilde{R}_\infty = \tilde{R}$ and $\tilde{V}_\infty = \tilde{V}$, then it can be shown that $\operatorname{tr} \mathcal{Q}\tilde{R} = \operatorname{tr} \mathcal{P}\tilde{V}$ and thus the solutions to the primal and dual problems coincide.

V. Alternative Forms of the Riccati Equations

In this section we develop alternative forms of the Riccati equations (3.7)–(3.9). These alternative forms provide further insight into the structure of (3.7)–(3.9) and, in certain cases, are simpler and thus are easier to solve computationally. This section also provides connections between our approach and [26].

First we note that the gains (3.3), (3.5), and (3.10) do not depend upon P and $\hat{Q}$ individually, but rather only upon the term $Z \triangleq PS$. Thus, it is of interest to know whether (3.8) and (3.9) can be transformed to yield an equation which characterizes Z directly. The following result summarizes useful properties of Z.

Lemma 5.1: Let $P, \hat{Q} \in \mathbb{N}^n$ and define $Z \triangleq PS$. Then $Z = Z^T = S^TP$, where $S^T = (I_n + \beta^2\gamma^{-2}P\hat{Q})^{-1}$, and Z is nonnegative definite. If, in addition, P is positive definite, then Z is positive definite and

$$Z = (P^{-1} + \beta^2\gamma^{-2}\hat{Q})^{-1}. \tag{5.1}$$

Proof: The result (5.1) is immediate. The remaining results can be obtained by replacing P by $P + \epsilon I_n$, where $\epsilon > 0$, and taking the limit as $\epsilon \to 0$. □

Proposition 5.1: Let $Q \in \mathbb{N}^n$ and suppose there exist $P \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (3.8) and (3.9). Then $Z \triangleq PS$ satisfies

$$\begin{aligned} 0 = {} & (A + \gamma^{-2}QR_{1\infty} + \gamma^{-2}\hat{Q}[R_{1\infty} - \beta^2R_1])^TZ \\ & + Z(A + \gamma^{-2}QR_{1\infty} + \gamma^{-2}\hat{Q}[R_{1\infty} - \beta^2R_1]) \\ & + R_1 - Z(\Sigma + \beta^2\gamma^{-4}\hat{Q}[R_{1\infty} - \beta^2R_1]\hat{Q})Z \\ & + \beta^2\gamma^{-2}ZQ\hat{\Sigma}QZ \end{aligned} \tag{5.2}$$

and (3.9) is equivalent to

$$\begin{aligned} 0 = {} & (A - \Sigma Z + \gamma^{-2}QR_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma Z + \gamma^{-2}QR_{1\infty})^T \\ & + \gamma^{-2}\hat{Q}(R_{1\infty} + \beta^2Z\Sigma Z)\hat{Q} + Q\hat{\Sigma}Q. \end{aligned} \tag{5.3}$$

Furthermore, (3.3), (3.5), and (3.10) become

$$A_c = A - Q\hat{\Sigma} - \Sigma Z + \gamma^{-2}QR_{1\infty}, \tag{5.4}$$

$$C_c = -R_2^{-1}B^TZ, \tag{5.5}$$

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) = \operatorname{tr}[(Q + \hat{Q})R_1 + \hat{Q}Z\Sigma Z]. \tag{5.6}$$

Proof: Using the identities

$$P = (I_n - \beta^2\gamma^{-2}Z\hat{Q})^{-1}Z = Z(I_n - \beta^2\gamma^{-2}\hat{Q}Z)^{-1}$$

which follow from (5.1), equation (5.2) can be obtained by forming the new equation

$$(I_n - \beta^2\gamma^{-2}Z\hat{Q})(3.8)(I_n - \beta^2\gamma^{-2}\hat{Q}Z) + \beta^2\gamma^{-2}Z(3.9)Z. \tag{5.7}$$

Finally, (5.3)–(5.5) are restatements of (3.9), (3.3), and (3.5) with $Z = PS$. □

Having obtained a single equation (5.2) for $Z = PS$ by combining (3.8) and (3.9) for P and $\hat{Q}$, it is of interest to know whether (3.8) for P can be recovered from (5.2) and (5.3).

Proposition 5.2: Let $Q \in \mathbb{N}^n$, $\beta > 0$, suppose there exist $Z \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (5.2) and (5.3), and assume that

$$\rho(Z\hat{Q}) < \beta^{-2}\gamma^2. \tag{5.8}$$

Then $P \triangleq (Z^{-1} - \beta^2\gamma^{-2}\hat{Q})^{-1}$ is positive definite and satisfies (3.8). Furthermore, P satisfies $Z = PS$.

Proof: If (5.8) holds, then it can be shown that P as defined above is positive definite. Reversing the proof of Proposition 5.1, (3.8) can be recovered by forming

$$(I_n - \beta^2\gamma^{-2}Z\hat{Q})^{-1}[(5.2) - \beta^2\gamma^{-2}Z(5.3)Z](I_n - \beta^2\gamma^{-2}\hat{Q}Z)^{-1}. \quad □$$

Although Proposition 5.2 allows us to reconstruct (3.8) for P, it can only be utilized when (5.8) holds. This fact raises a question as to the sufficiency of (3.7), (5.2), and (5.3) in the absence of (3.8). It turns out that the matrices P and Z need not actually satisfy (3.8) and (5.2) to enforce the H_∞ performance constraint (2.17) since only the Q and $\hat{Q}$ equations are required. Rather, P can be viewed as a parameterization of Z which, in turn, is a parameterization of the gains A_c and C_c given by (5.4) and (5.5) which yield a controller satisfying the desired H_∞ performance. These observations are summarized by the following result which does not require that Z be obtained by solving (5.2).

Proposition 5.3: Let $Z \in \mathbb{N}^n$ and suppose there exist $Q, \hat{Q} \in \mathbb{N}^n$ satisfying (3.7) and (5.3). Then $(A_c, B_c, C_c, \mathcal{Q})$ given by (5.4), (3.4), (5.5), and (3.6) satisfy (2.13) and (2.14). Thus, (2.15) and (2.16) are equivalent, and, in this case, (2.17) and (2.19) hold.

Proof: The result follows by direct verification of (2.14). □

Proposition 5.3 shows that the H_∞ constraint (2.17) is enforced for arbitrary $Z \in \mathbb{N}^n$ as long as (3.7) and (5.3) can be solved for Q and $\hat{Q}$. The price we pay for using arbitrary Z is that we no longer are assured that Z is obtained from (5.2) or from $Z = PS$ where P satisfies (3.8). Since P arises from the Lagrange multiplier for the constraint (2.14) [see (A.3)], it follows that an arbitrary choice of P (or Z) may fail to minimize the L_2 auxiliary cost (2.20). Thus, regarding P and Z as free parameters effectively ignores the L_2 aspect of Theorem 4.1.

It is also of interest to introduce yet another transformation of (3.7)–(3.9) by defining

$$Y \triangleq (Z^{-1} + \beta^2\gamma^{-2}Q)^{-1} = (P^{-1} + \beta^2\gamma^{-2}[Q + \hat{Q}])^{-1} \tag{5.9}$$

when P is positive definite. As in Lemma 5.1, Y is also positive definite.

Proposition 5.4: Let $Q \in \mathbb{N}^n$ and suppose there exist $P \in \mathbb{P}^n$ and $\hat{Q} \in \mathbb{N}^n$ satisfying (3.8) and (3.9). Then Y defined by (5.9) satisfies

$$\begin{aligned} 0 = {} & (A + \gamma^{-2}[Q + \hat{Q}][R_{1\infty} - \beta^2R_1])^TY \\ & + Y(A + \gamma^{-2}[Q + \hat{Q}][R_{1\infty} - \beta^2R_1]) \\ & + R_1 + \beta^2\gamma^{-2}YV_1Y - Y\Sigma Y \\ & - \beta^2\gamma^{-4}Y(Q + \hat{Q})(R_{1\infty} - \beta^2R_1)(Q + \hat{Q})Y \end{aligned} \tag{5.10}$$

and (3.9) is equivalent to

$$0=(A-\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}+\gamma^{-2}QR_{1\infty})\hat{Q}$$
$$+\hat{Q}(A-\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}+\gamma^{-2}QR_{1\infty})^T$$
$$+\gamma^{-2}\hat{Q}(R_{1\infty}+\beta^2[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1}$$
$$\cdot\Sigma[Y^{-1}-\beta^2\gamma^{-2}Q]^{-1})\hat{Q}+Q\hat{\Sigma}Q. \tag{5.11}$$

Furthermore, (3.3), (3.5), and (3.10) become

$$A_c=A-Q\hat{\Sigma}-\Sigma(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}+\gamma^{-2}QR_{1\infty}, \tag{5.12}$$

$$C_c=-R_2^{-1}B^T(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}, \tag{5.13}$$

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})=\mathrm{tr}\,[(Q+\hat{Q})R_1$$
$$+\hat{Q}(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}\Sigma(Y^{-1}-\beta^2\gamma^{-2}Q)^{-1}]. \tag{5.14}$$

Proof: To obtain (5.10), form

$$Y[Z^{-1}(5.2)Z^{-1}+\beta^2\gamma^{-2}(3.7)]Y. \qquad \square$$

The following result allows us to recover (3.8) for P from (5.10) and (5.11).

Proposition 5.5: Let $Q\in\mathbb{N}^n$, $\beta>0$, suppose there exist $Y\in\mathbb{P}^n$ and $\hat{Q}\in\mathbb{N}^n$ satisfying (5.10), (5.11), and assume that

$$\rho(Y[Q+\hat{Q}])<\beta^{-2}\gamma^2. \tag{5.15}$$

Then $P\triangleq(Y^{-1}-\beta^2\gamma^{-2}[Q+\hat{Q}])^{-1}$ is positive definite and satisfies (3.8).

Proof: The result follows by reversing the proof of Proposition 5.4. $\square$

By specializing further, it is possible to achieve even greater simplification. Specifically, we consider the case in which the L_2 and H_∞ weights are equalized, i.e.,

$$R_{1\infty}=R_1,\ \beta=1. \tag{5.16}$$

In this case it is always possible to eliminate (5.3) and (5.11) by noting that they are satisfied by $\hat{Q}=\gamma^2Z^{-1}$ and $\hat{Q}=\gamma^2Y^{-1}-Q$, respectively. However, although this solution enforces the H_∞ constraint, it can be seen from the resulting form of $\mathcal{J}$ that this solution does not correspond to the minimal solution $\mathcal{Q}$ of (2.14). Hence, we impose additional assumptions which allow us to directly characterize the solution which yields the minimal performance bound. We are indebted to D. Mustafa for clarifying this point in [45] where it is also shown that the auxiliary cost (2.20) is equivalent to an entropy integral.

Proposition 5.6: Assume (5.16) is satisfied, suppose there exist $Q\in\mathbb{N}^n$ and $Z_\infty\in\mathbb{P}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\hat{\Sigma}Q, \tag{5.17}$$

$$0=(A+\gamma^{-2}QR_{1\infty})^TZ_\infty+Z_\infty(A+\gamma^{-2}QR_{1\infty})$$
$$+R_{1\infty}-Z_\infty\Sigma Z_\infty+\gamma^{-2}Z_\infty Q\hat{\Sigma}QZ_\infty \tag{5.18}$$

and such that

$$A+\gamma^{-2}QR_{1\infty}+(\gamma^{-2}Q\hat{\Sigma}Q-\Sigma)Z_\infty \text{ is asymptotically stable} \tag{5.19}$$

and

$$(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty},\ \gamma^{-1}[R_{1\infty}+Z_\infty\Sigma Z_\infty]^{1/2}) \text{ is observable.} \tag{5.20}$$

Furthermore, let (A_c, B_c, C_c) be given by

$$A_c=A-Q\hat{\Sigma}-\Sigma Z_\infty+\gamma^{-2}QR_{1\infty}, \tag{5.21}$$

$$B_c=QC^TV_2^{-1}, \tag{5.22}$$

$$C_c=-R_{2\infty}^{-1}B^TZ_\infty. \tag{5.23}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\le\gamma \tag{5.24}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\le\mathrm{tr}[QR_{1\infty}+Q\hat{\Sigma}QZ_\infty]. \tag{5.25}$$

Proof: First note that it follows from (5.18) that

$$-(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty})=Z_\infty[A+\gamma^{-2}QR_{1\infty}$$
$$+(\gamma^{-2}Q\hat{\Sigma}Q-\Sigma)Z_\infty]Z_\infty^{-1} \tag{5.26}$$

and thus (5.19) implies that $-(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty})$ is asymptotically stable. It now follows from (5.20) that there exists $N\in\mathbb{P}^n$ satisfying

$$0=-(A+\gamma^{-2}QR_{1\infty}+Z_\infty^{-1}R_{1\infty})^TN-N(A+\gamma^{-2}QR_{1\infty}$$
$$+Z_\infty^{-1}R_{1\infty})+\gamma^{-2}(R_{1\infty}+Z_\infty\Sigma Z_\infty). \tag{5.27}$$

It can now be shown that $\hat{Q}=\gamma^2Z_\infty^{-1}-N^{-1}$ satisfies (5.3) with $\beta=1$ and $Z=Z_\infty$. Furthermore, (5.8) is satisfied so that the hypotheses of Theorem 4.1 are verified. The expression (5.25) now follows by direct substitution. $\square$

Finally, we consider a simplified version of Proposition 5.4.

Proposition 5.7: Assume (5.16) is satisfied and suppose there exist $Q\in\mathbb{N}^n$ and $Y_\infty\in\mathbb{P}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\hat{\Sigma}Q, \tag{5.28}$$

$$0=A^TY_\infty+Y_\infty A+R_{1\infty}+\gamma^{-2}Y_\infty V_1Y_\infty-Y_\infty\Sigma Y_\infty, \tag{5.29}$$

$$\rho(QY_\infty)<\gamma^2 \tag{5.30}$$

and such that

$$A+(\gamma^{-2}V_1-\Sigma)Y_\infty \text{ is asymptotically stable} \tag{5.31}$$

and

$$(A+Y_\infty^{-1}R_{1\infty},\ \gamma^{-1}[R_{1\infty}+(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}$$
$$\cdot\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)]^{1/2}) \text{ is observable.} \tag{5.32}$$

Furthermore, let (A_c, B_c, C_c) be given by

$$A_c=A-Q\hat{\Sigma}-\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}+\gamma^{-2}QR_{1\infty}, \tag{5.33}$$

$$B_c=QC^TV_2^{-1}, \tag{5.34}$$

$$C_c=-R_{2\infty}^{-1}B^T(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}. \tag{5.35}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\le\gamma \tag{5.36}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\le\mathrm{tr}[QR_{1\infty}+Q\hat{\Sigma}Q(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}]. \tag{5.37}$$

Proof: The proof is similar to the proof of Proposition 5.6 with $\hat{Q}=\gamma^2Y_\infty-Q-\hat{N}^{-1}$, where $\hat{N}$ satisfies

$$0=-(A+Y_\infty^{-1}R_{1\infty})^T\hat{N}-\hat{N}(A+Y_\infty^{-1}R_{1\infty})$$
$$+\gamma^{-2}[R_{1\infty}+(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}\Sigma(Y_\infty^{-1}-\gamma^{-2}Q)^{-1}]. \qquad \square$$

Remark 5.1: The solutions Q and Y_∞ of (5.28) and (5.29) are analogous to the matrices Y_∞ and X_∞ of [26], while (5.30) corresponds to condition 5.2(iii) of [26]. Note that by letting $\gamma \to \infty$, (5.25) and (5.37) coincide with 5-77a of [1] and the LQG result is recovered.

Remark 5.2: It is interesting to note that (5.17) and (5.18) with controller gains (5.21)-(5.23) are already known since they are identical to the optimality conditions for the linear-exponential-of-quadratic-Gaussian problem treated in [33] (see also [34] and [35]). Specifically, see (3.1) and (4.1) on pp. 603 and 609, respectively. As shown in [33], minimizing the criterion

$$J=\lim_{t\to\infty} \mathcal{E}\mu e^{\mu/2(x^TR_1x+u^TR_2u)}$$

leads to the pair of modified Riccati equations (5.17) and (5.18) with γ^{-2} replaced by μ. This implies that the exponential-of-quadratic design problem effectively enforces a bound of $\mu^{-1/2}$ on the H_∞ norm of the closed-loop transfer function. There also exist fundamental connections with the problem of entropy maximization [43]-[45].

VI. Extensions to Reduced-Order Dynamic Compensation

In this section we extend Theorem 4.1 by expanding the formulation of Section III to allow the compensator to be of fixed dimension n_c which may be less than the plant order n. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \leq n$. As in [21] this constraint leads to an oblique projection which introduces additional coupling in the design equations along with an additional equation. The following lemma is required.

Lemma 6.1: Let $\hat{Q}, \hat{P} \in \mathfrak{N}^n$ and suppose rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ G, Γ, and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathfrak{R}^{n_c}$, such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \tag{6.1}$$

$$\Gamma G^T=I_{n_c}. \tag{6.2}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^T\Gamma, \tag{6.3}$$

$$\tau_\perp \triangleq I_n-\tau \tag{6.4}$$

are idempotent and have rank n_c and $n - n_c$, respectively.

Proof: Conditions (6.1)-(6.4) are a direct consequence of [36, Theorem 6.2.5]. □

Theorem 6.1: Let $n_c \leq n$, suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathfrak{N}^n$ satisfying

$$0=AQ+QA^T+V_1+\gamma^{-2}QR_{1\infty}Q-Q\bar{\Sigma}Q+\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T, \tag{6.5}$$

$$0=(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})^TP+P(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})+R_1 -S^TP\Sigma PS+\tau_\perp^TS^TP\Sigma PS\tau_\perp, \tag{6.6}$$

$$0=(A-\Sigma PS+\gamma^{-2}QR_{1\infty})\hat{Q}+\hat{Q}(A-\Sigma PS+\gamma^{-2}QR_{1\infty})^T +\gamma^{-2}\hat{Q}(R_{1\infty}+\beta^2S^TP\Sigma PS)\hat{Q}+Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T, \tag{6.7}$$

$$0=(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty}) +S^TP\Sigma PS-\tau_\perp^TS^TP\Sigma PS\tau_\perp, \tag{6.8}$$

$$\text{rank } \hat{Q}=\text{rank } \hat{P}=\text{rank } \hat{Q}\hat{P}=n_c \tag{6.9}$$

and let $(A_c, B_c, C_c, \mathfrak{Q})$ be given by

$$A_c=\Gamma(A-Q\bar{\Sigma}-\Sigma PS+\gamma^{-2}QR_{1\infty})G^T, \tag{6.10}$$

$$B_c=\Gamma QC^TV_2^{-1}, \tag{6.11}$$

$$C_c=-R_2^{-1}B^TPSG^T, \tag{6.12}$$

$$\mathfrak{Q}=\begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}. \tag{6.13}$$

Then, $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|H(s)\|_\infty\leq\gamma \tag{6.14}$$

and the L_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c)\leq\text{tr}[(Q+\hat{Q})R_1+\hat{Q}S^TP\Sigma PS]. \tag{6.15}$$

Remark 6.1: It is easy to see that Theorem 6.1 is a direct generalization of Theorem 4.1. To recover Theorem 4.1, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.5)-(6.8) can be deleted and (6.8) becomes superfluous. Furthermore, (6.5)-(6.7) now reduce to (3.7)-(3.9), as expected. If, furthermore, $\beta = 0$ then $S = I_n$ so that (6.5)-(6.7) now reduce to the cheap H_∞ control case given by (3.7), (3.14), and (3.15). Alternatively, setting $\gamma = \infty$ and retaining the reduced-order constraint $n_c < n$ yields the result of [21].

Remark 6.2: By introducing a new variable $Z = PS = (P^{-1} + \beta^2\gamma^{-2}\hat{Q})^{-1}$ as in Section V, (6.6) becomes

$$\begin{aligned}0=&(A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}\hat{Q}[R_{1\infty}-\beta^2R_1])^TZ\\&+Z(A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}\hat{Q}[R_{1\infty}-\beta^2R_1])\\&+R_1-Z(\Sigma+\beta^2\gamma^{-4}\hat{Q}[R_{1\infty}-\beta^2R_1]\hat{Q})Z\\&+\tau_\perp^TZ\Sigma Z\tau_\perp+\beta^2\gamma^{-2}Z(Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)Z\end{aligned} \tag{6.16}$$

which specializes to (5.2) when $n_c = n$, i.e., $\tau_\perp = 0$. When (5.16) holds, (6.16) becomes

$$\begin{aligned}0=&(A+\gamma^{-2}QR_{1\infty})^TZ_\infty+Z_\infty(A+\gamma^{-2}QR_{1\infty})\\&+R_{1\infty}-Z_\infty\Sigma Z_\infty+\tau_\perp^TZ_\infty\Sigma Z_\infty\tau_\perp\\&+\gamma^{-2}Z_\infty(Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^T)Z_\infty.\end{aligned} \tag{6.17}$$

Analogous equations for Y defined by (5.9) can also be developed.

VII. Analysis of the Design Equations

Before developing numerical algorithms, it is instructive to analyze the design equations to determine existence and multiplicity of nonnegative-definite solutions. Although a detailed theoretical analysis remains an area for future research, in this section we present a simplified treatment which highlights important asymptotic properties of the equations. it turns out that several key properties are discernible by considering the scalar case $n = 1$. Although many of these properties can be developed for general n, the simplified scalar case suffices for obtaining a useful qualitative analysis. Here we consider only (3.7), (3.14), and (3.15).

Since the Q equation (3.7) is decoupled from (3.14) and (3.15), it can be analyzed separately. It is easy to see that there exists a unique nonnegative solution for $\gamma > (R_1/\bar{\Sigma})^{1/2}$ as in the case of a standard Riccati equation with stabilizability and detectability hypotheses. Furthermore, it can be seen that for

$$(R_1/[\bar{\Sigma}+(A^2/V_1)])^{1/2}<\gamma\leq(R_1/\bar{\Sigma})^{1/2}$$

there exist two nonnegative solutions when A is stable and zero nonnegative solutions when A is unstable. Below this lower bound for γ, nonnegative solutions Q do not exist. This result thus indicates (as in LQG theory [42]) a lower bound to the achievable H_∞ disturbance attenuation as determined by the sensor noise intensity V_2 appearing in $\bar{\Sigma}$.

Since the P and $\hat{Q}$ equations (3.14) and (3.15) are coupled, they

must be analyzed jointly. Since (3.15) is a standard Riccati equation it follows under generic hypotheses that it possesses exactly one nonnegative-definite solution for all values of Q and $\hat{Q}$. The analysis of the $\hat{Q}$ equation is, however, more involved. It can be shown that the existence of real solutions is a complicated function of γ, Q, and P. When real solutions do exist, it follows that there exist either zero or two nonnegative-definite solutions. To obtain further qualitative insight into the solutions P and $\hat{Q}$, we fix γ and allow $R_2 \to 0$, that is, the cheap L_2 control case. It thus follows that $P \sim (R_1\Sigma)^{1/2}$ and that either $\hat{Q} \sim 2\gamma^2(\Sigma/R_1)^{1/2}$ or $\hat{Q} \sim 1/2\hat{\Sigma}Q^2(\Sigma R_1)^{-1/2}$, which correspond to the previously mentioned pair of solutions satisfying (3.15). This result thus indicates that an arbitrarily small H_∞ disturbance attenuation constraint γ can be achieved [subject to the solvability of (3.7)] by sufficiently increasing the L_2 controller authority. That is, since solutions exist in the cheap L_2 control case, the H_∞ disturbance attenuation constraint is achievable. The ability to achieve small γ is also attributable to the fact that since $\beta = 0$, H_∞ disturbance attenuation to the control variables is not limited in (3.7), (3.14), and (3.15) as in Theorems 3.1 and 6.1. Of course, as is well known, it is not possible to make $\gamma \to 0$ by letting $\Sigma \to \infty$ and $\hat{\Sigma} \to \infty$ when the system possesses nonminimum phase zeros. Also, note that both of the asymptotic solutions to (3.15) are guaranteed to yield the bound (4.1). The solution of interest, however, is $\hat{Q} = O(\Sigma^{-1/2})$ since it clearly yields a lower value of $\mathcal{J}(A_c, B_c, C_c, Q)$ than $\hat{Q} = O(\Sigma^{1/2})$.

VIII. Numerical Algorithm and Illustrative Results

In this section we describe a numerical algorithm which has been developed and implemented for solving the coupled Riccati equations (3.7), (3.14), and (3.15). We also present numerical results for an illustrative example.

Coupled modified Riccati equations arise in a variety of problems and homotopic continuation methods have been shown to be particularly successful [23]–[25]. To solve (3.7), (3.14), and (3.15) we have implemented a simplified continuation method involving the constraint constant γ. The idea is to exploit the fact that for large γ the problem is approximated by LQG which provides a reliable starting solution. The continuation parameter γ is then successively decreased until either a desired value of γ is achieved or no further decrease is possible. This algorithm is now summarized. Let $\epsilon > 0$ denote a convergence criterion.

Algorithm 8.1: To solve (3.7), (3.14), and (3.15), perform the following steps:

Step 1: Initialize $\gamma > 0$.

Step 2: Solve (3.7) for Q.

Step 3: Let $k = 0$, $\hat{Q}_0 = 0$.

Step 4: Solve (3.14) for $P_{k+1} = P$ with $\hat{Q} = \hat{Q}_k$.

Step 5: Solve (3.15) for $\hat{Q}_{k+1} = \hat{Q}$ with $P = P_{k+1}$.

Step 6: If $k \geq 1$ check for $\|P_{k+1} - P_k\| < \epsilon$ and $\|\hat{Q}_{k+1} - \hat{Q}_k\| < \epsilon$.

Step 7: If convergence is not achieved in Step 6 (or $k = 0$) let $k \leftarrow k + 1$ and go to Step 4; otherwise decrease γ and go to Step 2.

Steps 2, 4, and 5 were carried out using a standard Riccati solver [37] which proved to be reliable even when the quadratic term was indefinite or nonnegative definite. For instance, for the example considered below, the term $\gamma^{-2}R_1 - \hat{\Sigma}$ was indefinite for all finite γ. The crucial step in the algorithm is the decreasing of γ in Step 7. Significant effort was devoted to providing a smooth transition to smaller values of γ without sacrificing computational efficiency. The development of more sophisticated continuation algorithms remains an area for future research.

The example considered was formulated in [38] and was considered extensively in [24], [25], and [39] to compare reduced-order design methods. The example is interesting since it possesses a complex pair of nonminimum phase zeros due to the fact that the physical system (coupled rotating disks) has noncollocated sensors and actuators. The plant is of eighth order and has two neutrally stable poles. The problem data are as follows:

$$n = n_c = 8, \quad m = l = 1, \quad q = p = 2,$$

$$A = \begin{bmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix} \qquad C = [1 \;\; 0_{1\times 7}]$$

$$E_1 = E_{1\infty} = 10^{-3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_{2\infty} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta = 0,$$

$$D_1 = [B \;\; 0_{8\times 1}], \quad D_2 = [0 \;\; 1].$$

With the problem data as given, the LQG controller was found to yield a closed-loop H_∞ performance of 1.39 (i.e., 2.87 dB above unity gain). Using Algorithm 8.1 we obtained a solution for $\gamma = 0.52$ for a net H_∞ performance improvement of 8.7 dB (see Fig. 1). Note that this result is consistent with [3, Proposition 8.1] which implies that the maximum ratio of the H_∞ performance of the optimal L_2 controller to the H_∞ performance of the optimal H_∞ controller can be no more than twice the number of right-half-plane zeros, which for the present problem with two nonminimum phase zeros corresponds to a factor of 4 (i.e., 12 dB).

Our numerical experience revealed two interesting features. First, the loop between Steps 4 and 6 converged reliably. However, a critical value $\gamma_{\min}$ of γ was invariably found below which solutions could not be computed. This value $\gamma_{\min}$ appears to represent the best achievable H_∞ performance for the given L_2 weights. Second, for each value of $\gamma \geq \gamma_{\min}$ for which a solution was computed, the actual H_∞ performance was close to this value revealing that the H_∞ bound is tight. For example, the actual worst-case attenuation of the $\gamma = 0.52$ design shown in Fig. 1 is 0.511. Controller characteristics are given in Table I and are plotted in Fig. 2 for several values of γ. Note that in each case the L_2 performance bound is within 30 percent of the actual L_2 performance.

IX. Further Extensions

The results obtained herein can readily be extended in several directions. These include the treatment of parameter uncertainties [13]–[15], [46], extensions to controllers with static feedthrough [32], and the inclusion of cross-weighting terms $(x^T(t)R_{12}u(t))$ and noise correlation $(D_1D_2^T \neq 0)$. Finally, as mentioned in Remark 5.2, connections with the exponential-of-quadratic cost criterion [33]–[35] and entropy maximization [43]–[45] are of interest.

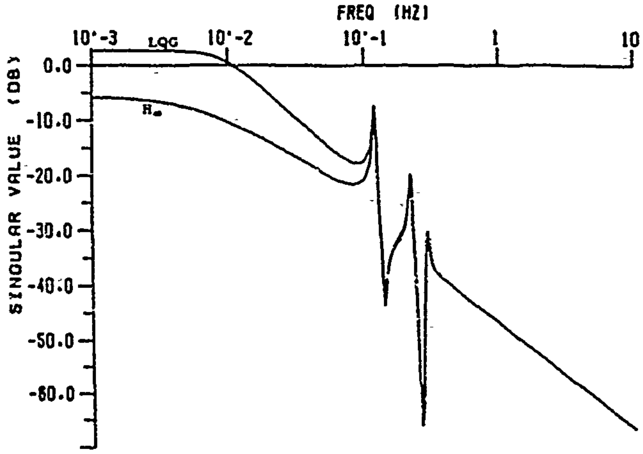

Fig. 1.

TABLE I

H_∞ Attenuation Constraint γ	Actual H_∞ Attenuation $\|\|H(s)\|\|_\infty$	L_2 Performance Bound $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$	Actual L_2 Performance $J(A_c, B_c, C_c)$
∞ (LQG)	1.39	—	.143
2	1.18	.159	.146
1.5	1.06	.171	.151
1.0	.855	.204	.168
.9	.797	.217	.176
.8	.732	.236	.187
.7	.661	.262	.203
.52	.511	.299	.262

Appendix
Proof of Theorem 6.1

To optimize (2.20) over the open set $\mathfrak{X}$ subject to the constraint (2.14), form the Lagrangian

$$\mathcal{L}(A_c, B_c, C_c, \mathcal{Q}, \mathcal{P}, \lambda) \triangleq \operatorname{tr}\{\lambda \mathcal{Q}\tilde{R} + [\tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty \mathcal{Q} + \tilde{V}]\mathcal{P}\} \quad (A.1)$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathcal{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \lambda\tilde{R}. \quad (A.2)$$

Setting $\partial\mathcal{L}/\partial\mathcal{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \lambda\tilde{R}. \quad (A.3)$$

Since $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ is assumed to be stable, $\lambda = 0$ implies $\mathcal{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathcal{P}$ is nonnegative definite.

Now partition $\tilde{n} \times \tilde{n}$ $\mathcal{Q}\mathcal{P}$ into $n \times n$, $n \times n_c$, and $n_c \times n_c$ subblocks as

$$\mathcal{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial \mathcal{L}}{\partial \mathcal{Q}} = (\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty)^T\mathcal{P} + \mathcal{P}(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) + \tilde{R} = 0, \quad (A.4)$$

$$\frac{\partial \mathcal{L}}{\partial A_c} = P_{12}^TQ_{12} + P_2Q_2 = 0, \quad (A.5)$$

$$\frac{\partial \mathcal{L}}{\partial B_c} = P_2B_cV_2 + (P_{12}^TQ_1 + P_2Q_{12}^T)C^T = 0, \quad (A.6)$$

$$\frac{\partial \mathcal{L}}{\partial C_c} = R_2C_cQ_2 + \beta^2\gamma^{-2}R_2C_c(P_1Q_{12} + P_{12}Q_2)^TQ_{12} + B^T(P_1Q_{12} + P_{12}Q_2) = 0. \quad (A.7)$$

Expanding (2.14) and (A.4) yields

$$0 = AQ_1 + Q_1A^T + BC_cQ_{12}^T + Q_{12}C_c^TB^T + \gamma^{-2}Q_1R_{1\infty}Q_1 + \beta^2\gamma^{-2}Q_{12}C_c^TR_2C_cQ_{12}^T + V_1, \quad (A.8)$$

$$0 = AQ_{12} + Q_{12}A_c^T + BC_cQ_2 + Q_1C^TB_c^T + \gamma^{-2}Q_1R_{1\infty}Q_{12} + \beta^2\gamma^{-2}Q_{12}C_c^TR_2C_cQ_2, \quad (A.9)$$

$$0 = A_cQ_2 + Q_2A_c^T + B_cCQ_{12} + Q_{12}^TC^TB_c^T + \gamma^{-2}Q_{12}^TR_{1\infty}Q_{12} + \beta^2\gamma^{-2}Q_2C_c^TR_2C_cQ_2 + B_cV_2B_c^T, \quad (A.10)$$

$$0 = A^TP_1 + P_1A + C^TB_c^TP_{12}^T + P_{12}B_cC + \gamma^{-2}R_{1\infty}(P_1Q_1 + P_{12}Q_{12}^T)^T + \gamma^{-2}(P_1Q_1 + P_{12}Q_{12}^T)R_{1\infty} + R_1, \quad (A.11)$$

$$0 = A^TP_{12} + P_{12}A_c + C^TB_c^TP_2 + P_1BC_c + \gamma^{-2}R_{1\infty}(P_{12}^TQ_1 + P_2Q_{12}^T)^T + \beta^2\gamma^{-2}(P_1Q_{12} + P_{12}Q_2)C_c^TR_2C_c, \quad (A.12)$$

$$0 = A_c^TP_2 + P_2A_c + P_{12}^TBC_c + C_c^TB^TP_{12} + C_c^TR_2C_c. \quad (A.13)$$

Lemma A.1: Q_2 and P_2 are positive definite.

Proof: By a minor extension of results from [40], (A.10) can be rewritten as

$$0 = (A_c + B_cCQ_{12}Q_2^+)Q_2 + Q_2(A_c + B_cCQ_{12}Q_2^+)^T + \Psi$$

where

$$\Psi \triangleq \gamma^{-2}Q_{12}^TR_{1\infty}Q_{12} + \beta^2\gamma^{-2}Q_2C_c^TR_2C_cQ_2 + B_cV_2B_c^T$$

and Q_2^+ is the Moore-Penrose or Drazin generalized inverse of Q_2. Next note that since (A_c, B_c) is controllable it follows from [28, Lemma 2.1 and Theorem 3.6] that $(A_c + B_cCQ_{12}Q_2^+, \Psi^{1/2})$ is also controllable. Now, since Q_2 and Ψ are nonnegative definite, [28, Lemma 12.2] implies that Q_2 is positive definite. Using (A.13), similar arguments show that P_2 is positive definite. □

Since R_2, V_2, Q_2, P_2 are invertible, (A.5)-(A.7) can be written as

$$-P_2^{-1}P_{12}^TQ_{12}Q_2^{-1} = I_{n_c}, \quad (A.14)$$

$$B_c = -P_2^{-1}(P_{12}^TQ_1 + P_2Q_{12}^T)C^TV_2^{-1}, \quad (A.15)$$

$$C_c[I_{n_c} + \beta^2\gamma^{-2}(Q_{12}^TP_1 + Q_2P_{12}^T)Q_{12}Q_2^{-1}] = -R_2^{-1}B^T(P_1Q_{12} + P_{12}Q_2)Q_2^{-1}. \quad (A.16)$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^T, \quad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^T,$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^T, \quad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^T,$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^T$$

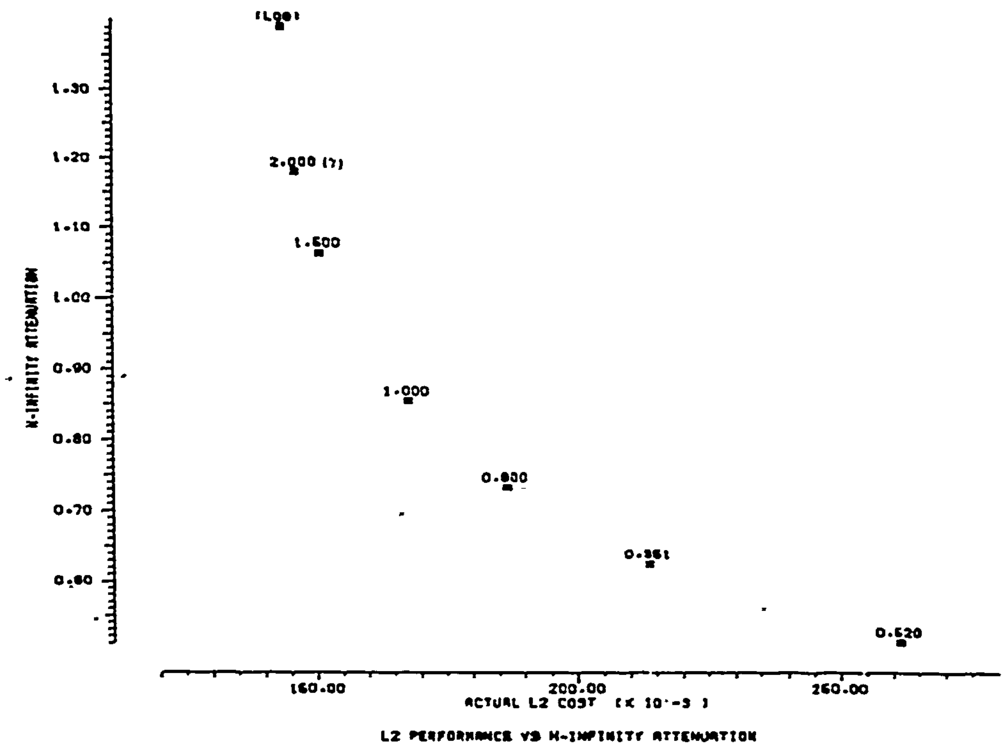

Fig. 2.

and the $n_c \times n$, $n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^T,\quad M \triangleq Q_2P_2,\quad \Gamma \triangleq -P_2^{-1}P_{12}^T.$$

Note that $\tau = G^T\Gamma$.

Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and $\hat{Q}$ and $\hat{P}$ are nonnegative definite. To show that Q and P are also nonnegative definite, note that Q is the upper left-hand block of the nonnegative definite matrix $\tilde{\mathcal{Q}}\mathcal{Q}\tilde{\mathcal{Q}}^T$, where

$$\tilde{\mathcal{Q}} \triangleq \begin{bmatrix} I_n & -Q_{12}Q_2^{-1} \\ 0_{n_c\times n} & I_{n_c} \end{bmatrix}.$$

Similarly, P is nonnegative definite.

Next note that with the above definitions (A.14) is equivalent to (6.2) and that (6.1) holds. Hence, $\tau = G^T\Gamma$ is idempotent, i.e., $\tau^2 = \tau$.

It is helpful to note the identities

$$\hat{Q} = Q_{12}G = G^TQ_{12}^T = G^TQ_2G,\quad \hat{P} = -P_{12}\Gamma = -\Gamma^TP_{12}^T = \Gamma^TP_2\Gamma, \tag{A.17}$$

$$\tau G^T = G^T,\quad \Gamma\tau = \Gamma, \tag{A.18}$$

$$\hat{Q} = \tau\hat{Q},\quad \hat{P} = \hat{P}\tau, \tag{A.19}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^T. \tag{A.20}$$

Using (A.14) and Sylvester's inequality, it follows that

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} Q_{12} = \operatorname{rank} P_{12} = n_c.$$

Now using (A.17) and Sylvester's inequality yields

$$n_c = \operatorname{rank} Q_{12} + \operatorname{rank} G - n_c \le \operatorname{rank} \hat{Q} \le \operatorname{rank} Q_{12} = n_c$$

which implies that rank $\hat{Q} = n_c$. Similarly, rank $\hat{P} = n_c$, and rank $\hat{Q}\hat{P} = n_c$ follows from (A.20).

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q},\quad P_1 = P + \hat{P}, \tag{A.21}$$

$$Q_{12} = \hat{Q}\Gamma^T,\quad P_{12} = -\hat{P}G^T, \tag{A.22}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T,\quad P_2 = G\hat{P}G^T. \tag{A.23}$$

Next note that by using (A.21)–(A.23) it can be shown that the right-hand coefficient of C_c in (A.16) is given by

$$\hat{S} \triangleq I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T.$$

To prove that $\hat{S}$ is invertible use (A.19) and (6.3) and note that

$$\begin{aligned} I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T &= I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}\tau^TPG^T \\ &= I_{n_c} + \beta^2\gamma^{-2}(\Gamma\hat{Q}\Gamma^T)(GPG^T). \end{aligned}$$

Since $\Gamma\hat{Q}\Gamma^T$ and GPG^T are nonnegative definite, their product has nonnegative eigenvalues (see Lemma 5.1). Thus, each eigenvalue of $I_{n_c} + \beta^2\gamma^{-2}\Gamma\hat{Q}PG^T$ is real and is greater than unity. Hence, $\hat{S}$ is invertible. Now note that by using (6.2) and (6.3) it can be shown that

$$G^T\hat{S}^{-1} = SG^T.$$

The expressions (6.11), (6.12), and (6.13) follow from (A.15), (A.16), and the definition of $\mathcal{Q}$. Next, computing either Γ(A.9)–(A.10) or G(A.12) + (A.13) yields (6.10). Substituting (A.21)–(A.23) into (A.8)–(A.13) and the expression for A_c into (A.9), (A.10), (A.12), and (A.13) it follows that (A.10) = Γ(A.9) and (A.13) = G(A.12). Thus, (A.10) and (A.13) are superfluous and can be omitted. Thus, (A.8)–(A.13) reduce to

$$\begin{aligned} 0 = {} & AQ + QA^T + V_1 + \gamma^{-2}(Q+\hat{Q})R_{1\infty}(Q+\hat{Q}) \\ & + \beta^2\gamma^{-2}\hat{Q}S^TP\Sigma PS\hat{Q} \\ & + (A - \Sigma PS)\hat{Q} + \hat{Q}(A - \Sigma PS)^T, \end{aligned} \tag{A.24}$$

$$0=[(A-\Sigma PS)\hat{Q}+\hat{Q}(A-\Sigma PS)^T+Q\hat{\Sigma}Q+\gamma^{-2}(Q+\hat{Q})R_{1\infty}(Q+\hat{Q})-\gamma^{-2}QR_{1\infty}Q+\beta^2\gamma^{-2}\hat{Q}S^TP\Sigma PS\hat{Q}]\Gamma^T, \quad (A.25)$$

$$0=(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})^TP+P(A+\gamma^{-2}[Q+\hat{Q}]R_{1\infty})+R_1+(A-Q\hat{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\hat{\Sigma}+\gamma^{-2}QR_{1\infty}), \quad (A.26)$$

$$0=[(A-Q\hat{\Sigma}+\gamma^{-2}QR_{1\infty})^T\hat{P}+\hat{P}(A-Q\hat{\Sigma}+\gamma^{-2}QR_{1\infty})+S^TP\Sigma PS]G^T. \quad (A.27)$$

Next, using (A.24) + $G^T\Gamma$(A.25)G − (A.25)G − [(A.25)G]^T and $G^T\Gamma$(A.25)G − (A.25)G − [(A.25)G]^T yields (6.5) and (6.7). Similarly, using (A.26) + Γ^TG(A.27)Γ − (A.27)Γ − [(A.27)Γ]^T and Γ^TG(A.27)Γ − (A.27)Γ − [(A.27)Γ]^T yields (6.6) and (6.8).

Finally, to prove the converse we use (6.5)-(6.13) to obtain (2.14) and (A.4)-(A.7). Let A_c, B_c, C_c, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 6.1 and define Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2 by (A.21)-(A.23). Using (6.2), (6.11), and (6.12) it is easy to verify (A.6) and (A.7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ, and τ into (6.5)-(6.8) using (6.2), (6.3), and (A.19) to obtain (2.14) and (A.4). Finally, note that

$$\mathcal{Q}=\begin{bmatrix} Q & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c} \end{bmatrix}+\begin{bmatrix} I_n \\ \Gamma \end{bmatrix}\hat{Q}[I_n \;\; \Gamma^T]$$

which shows that $\mathcal{Q} \geq 0$. □

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Dennis S. Bernstein (M'82) received the Sc.B. degree in applied mathematics from Brown University, Providence, RI, and the M.S.E. and Ph.D. degrees from the Computer, Information and Control Engineering Program at the University of Michigan, Ann Arbor.

After spending two years at Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, he joined the Controls Analysis and Synthesis Group of the Government Aerospace Systems Division, Harris Corporation, Melbourne, FL. At Harris Corporation his research interests are directed primarily toward the control of spacecraft with flexible appendages with particular emphasis on robust and nonlinear control techniques.

Dr. Bernstein is a member of SIAM.

Wassim M. Haddad (S'87–M'87) was born in Athens, Greece, on July 14, 1961. He received the B.S., M.S., and Ph.D. degrees in mechanical engineering in 1983, 1984, and 1987, respectively, from the Florida Institute of Technology, Melbourne.

Since 1987 he has been a consultant for the Controls Analysis and Synthesis Group of the Government Aerospace Systems Division, Harris Corporation, Melbourne, FL, and he is currently a faculty member in the Department of Mechanical and Aerospace Engineering, Florida Institute of Technology. His research interests are in the area of robust estimation and control for aerospace systems.

Dr. Haddad is a member of Tau Beta Pi.

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Generalized Riccati equations for the full- and reduced-order mixed-norm H_2/H_∞ standard problem *

Wassim M. HADDAD
Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

Dennis S. BERNSTEIN
Harris Corporation, Government Aerospace Systems Division, MS 22/4842, Melbourne, FL 32902, U.S.A.

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Abstract: This paper considers the mixed-norm H_2/H_∞ standard problem. Specifically, an LQG control design problem involving a constraint on H_∞ disturbance attenuation is addressed. It is shown that the H_2/H_∞ dynamic compensator gains are completely characterized via coupled Riccati/Lyapunov equations. The principal result involves sufficient conditions for characterizing full- and reduced-order controllers that satisfy bounds on both H_2 and H_∞ performance costs. As a special case of this unified result we obtain the full-order H_∞ solution to the standard control problem and the pure reduced-order H_∞ solution with no H_2 contribution. Further extensions include nonstrictly proper dynamics, a direct transmission term from disturbances to H_∞ performance variables, cross-weighting and sensor noise/plant disturbance correlation, and a treatment of the pure *reduced-order* H_∞ control problem.

Keywords: H_2/H_∞ design; mixed norm; H_∞ reduced-order controllers.

1. Introduction

In a recent paper [1] a unification of the H_2 (LQG) and H_∞ control-design problems was obtained in terms of modified coupled algebraic Riccati equations. Specifically, the results of [1] address a unified solution of the H_2/H_∞ standard problem for full- and reduced-order controllers. This mixed-norm problem thus permits design tradeoffs between H_2 performance and H_∞ disturbance rejection.

The goal of the H_2/H_∞ problem is to minimize an H_2 performance criterion subject to a prespecified H_∞ constraint on the closed-loop transfer function. The H_∞ constraint is embedded within the optimization process by replacing the closed-loop covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the H_2 performance. The key idea to this approach is to view this upper bound as an auxiliary cost and, for a fixed controller structure, seek compensator gains that minimize the H_2 bound and guarantee that the disturbance attenuation constraint is enforced. The principal result is a sufficient condition involving coupled modified Riccati equations whose solutions, when they exist, are used to explicitly construct feedback gains for characterizing full- and reduced-order controllers with bounded H_2 and H_∞ costs. Note that, strictly speaking, the problem addressed is suboptimal in both the H_2 sense and the H_∞ sense. However, solving the design equations for progressively smaller H_∞ disturbance attenuation constraints should, in the limit, yield an H_∞-optimal controller over the class of fixed-structure stabilizing controllers. Although our main result gives sufficient conditions, these conditions will also be necessary as long as the mixed-norm optimization problem possesses at least one extremal over the class of fixed-structure controllers (see Lemma 2.2 and [2]).

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The solution given in [1] however, was restricted to the case in which the plant was strictly proper and there was no direct transmission from disturbances to H_∞ performance variables. The main contribution of the present paper is to extend the results of [1] to remove these restrictions and to allow further generalizations. First, a direct transmission term in the state space plant dynamics is included within the problem formulation along with a direct feedthrough term from exogenous disturbances to H_∞ performance variables. Next, to allow for greater design flexibility we permit correlated plant and measurement noise. And, finally, we consider the dual design feature of cross weighting in both the H_2 and H_∞ performance criteria. These generalizations have been studied in [14] for full-state feedback and in [4,5,11] for dynamic compensation. However, the results of [4,5,11] are limited to the 'pure' full-order H_∞ standard problem without the H_2/H_∞ unification. Furthermore, the results given in [4,5,11] are obtained by indirect transformation methods. In the present paper we derive the solution a to mixed-norm H_2/H_∞ fixed-order (i.e., full-, and reduced-order) dynamic compensation problem without employing such transformations.

It should be noted that the approach developed in [4,5] is quite different from our fixed-structure optimization design approach. Specifically, the authors in [4,5] consider a general H_∞ optimization problem of the form $\|T-UQV\|_\infty$, where Q is a parameterization of all stabilizing controllers that give infinity norm better than γ. It is shown that the central member of this set minimizes an entropy functional at infinity and yields a set of decoupled Riccati equations that characterize *full-order* compensators satisfying an H_∞ norm bound [5,8]. Furthermore, the results of [4,5,11] are necessary as well as sufficient. In contrast, the approach of [1] and the present paper is based upon Lagrange multiplier methods which permit the fixed-order-constraints as well as different H_2 and H_∞ performance weights.

Finally, as a special case of the results given in the present paper we obtain the full-order H_2 solution (LQG), reduced-order H_2 solution [6], full-order H_∞ solution [3,4,5,11], and the 'pure' reduced-order H_∞ solution with no H_2 contribution. It is interesting to note that in the full-order H_∞ controller case with no H_2 contribution our results specialize to [3,4,5,11]. Since the results of [3,4,5,11] are necessary as well as sufficient, these connections show that our sufficient conditions (at least in this special case) are also necessary.

Notation. Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.
I_r, $(\ )^{\mathrm{T}}$, $(\ )^*$	$r\times r$ identity matrix, transpose, complex conjugate transpose.
$\rho(\ )$	spectral radius.
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r\times r$ symmetric, nonnegative-definite, positive-definite matrices.
x, u, y, x_c, $\tilde{x}$	n, m, l, n_c, $\tilde{n}$-dimensional vectors.
A, B, C, D	$n\times n$, $n\times m$, $l\times n$, $l\times m$ matrices.
A_c, B_c, C_c	$n_c\times n_c$, $n_c\times l$, $m\times n_c$ matrices.
$\tilde{x}$, $\tilde{A}$	$\begin{bmatrix} x \\ x_c \end{bmatrix}, \begin{bmatrix} A & BC_c \\ B_cC & A_c+B_cDC_c \end{bmatrix}$.
γ	positive constant.
E_∞	$q_\infty\times d$ matrix.
M	$I_{q_\infty}-\gamma^{-2}E_\infty E_\infty^{\mathrm{T}}$, $M\in\mathbb{P}^{q_\infty}$.
N	$I_d-\gamma^{-2}E_\infty^{\mathrm{T}}E_\infty$, $N\in\mathbb{P}^d$.
$w(\cdot)$	d-dimensional standard white noise or L_2 signal.
D_1, D_2	$n\times d$, $l\times d$ matrices.
V_1, V_2, V_{12}	$D_1D_1^{\mathrm{T}}$, $D_2D_2^{\mathrm{T}}$, $D_1D_2^{\mathrm{T}}$; $V_2\in\mathbb{P}^l$.
$V_{1\infty}$, $V_{2\infty}$, $V_{12\infty}$	$D_1N^{-1}D_1^{\mathrm{T}}$, $D_2N^{-1}D_2^{\mathrm{T}}$, $D_1N^{-1}D_2^{\mathrm{T}}$; $V_{2\infty}\in\mathbb{P}^l$.
$\tilde{D}$, $\tilde{V}$	$\begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}, \begin{bmatrix} V_1 & V_{12}B_c^{\mathrm{T}} \\ B_cV_{12}^{\mathrm{T}} & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}=\tilde{D}\tilde{D}^{\mathrm{T}}$.
E_1, E_2	$q\times n$, $q\times m$ matrices.
$\tilde{E}$, R_1, R_2	$[E_1\ \ E_2C_c]$, $E_1^{\mathrm{T}}E_1$, $E_2^{\mathrm{T}}E_2$; $R_2\in\mathbb{P}^m$.
R_{12}, $\tilde{R}$	$E_1^{\mathrm{T}}E_2$, $\tilde{E}^{\mathrm{T}}\tilde{E}$.

$E_{1\infty}$, $E_{2\infty}$	$q_\infty \times n$, $q_\infty \times m$ matrices.
$\tilde{E}_\infty$, $R_{1\infty}$, $R_{2\infty}$	$[E_{1\infty}\ E_{2\infty}C_c]$, $E_{1\infty}^{\mathrm{T}}M^{-1}E_{1\infty}$, $E_{2\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$.
$R_{12\infty}$, $\tilde{R}_\infty$	$E_{1\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$, $\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty$.
$R_{01\infty}$, $R_{02\infty}$	$E_\infty^{\mathrm{T}}M^{-1}E_{1\infty}$, $E_\infty^{\mathrm{T}}M^{-1}E_{2\infty}$.
α, β	nonnegative constants.

2. Statement of the problem

In this section we introduce the LQG dynamic output-feedback control problem with constrained H_∞ disturbance attenuation. Without the H_2 performance criterion the problem considered here is the standard H_∞ control problem [3,4,5]. For simplicity, the first part of the paper addresses controllers of order $n_c = n$ only, i.e., controllers whose order is equal to the dimension of the plant. This constraint is removed in Section 6 where controllers of reduced order are considered. Hence, throughout Sections 2–5 the controller dimension n_c and closed-loop plant dimension $\tilde{n} \triangleq n + n_c$ should be interpreted as n and $2n$, respectively.

H_∞-Constrained LQG Control Problem. Given the nth-order stabilizable and detectable plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) + D_2w(t), \tag{2.2}$$

determine an nth-order dynamic compensator

$$\dot{x}(t) = A_cx_c(t) + B_cy(t), \tag{2.3}$$

$$u(t) = C_cx_c(t), \tag{2.4}$$

that satisfies the following design criteria:

(i) the closed-loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;

(ii) the $q_\infty \times p$ nonstrictly proper transfer function

$$H(s) \triangleq \tilde{E}_\infty\left(sI_{\tilde{n}} - \tilde{A}\right)^{-1}\tilde{D} + E_\infty \tag{2.5}$$

from $w(t)$ to $z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t) + E_\infty w(t)$ satisfies the constraint

$$\| H(s) \|_\infty \le \gamma, \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

(iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left\{\int_0^t\left[x^{\mathrm{T}}(t)R_1x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2u(t)\right]\,\mathrm{d}t\right\} \tag{2.7}$$

is minimized.

Note that the closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t)$$

and that (2.7) becomes

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathbb{E}\left\{\left[\tilde{E}\tilde{x}(t)\right]^{\mathrm{T}}\left[\tilde{E}\tilde{x}(t)\right]\right\} = \lim_{t\to\infty} \mathbb{E}\left[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)\right]. \tag{2.8}$$

Furthermore, by defining the transfer function

$$\tilde{H}(s) \triangleq \tilde{E}\left(sI_{\tilde{n}} - \tilde{A}\right)^{-1}\tilde{D}, \tag{2.9}$$

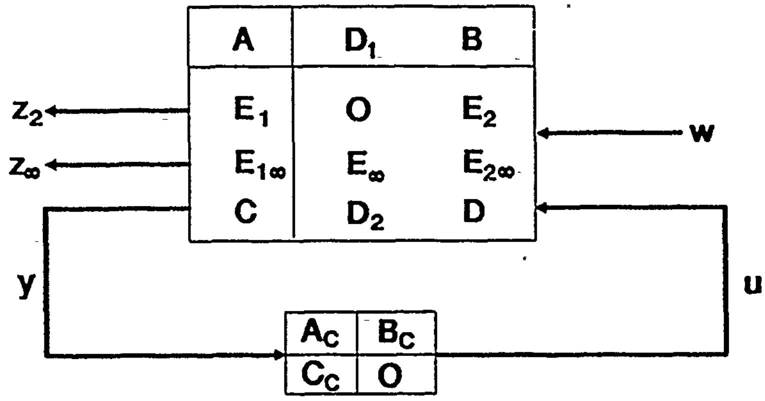

Fig. 1. The mixed-norm H_2/H_∞ standard problem includes the H_∞ standard problem and the LQG problem as special cases.

it can be shown that when $\tilde{A}$ is asymptotically stable, (2.8) is given by

$$J(A_c, B_c, C_c) = \| \tilde{H}(s) \|_2^2. \tag{2.10}$$

Note that the problem statement involves both H_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the H_2 weights for the state and control variables. By introducing the variables

$$z(t) = E_1 x(t), \qquad v(t) = E_2 u(t), \tag{2.11}$$

the H_2 cost (2.7) can be written as

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathbb{E}\left[z^{\mathrm{T}}(t) z(t) + 2z^{\mathrm{T}}(t) v(t) + v^{\mathrm{T}}(t) v(t) \right]. \tag{2.12}$$

For convenience we thus define $R_1 \triangleq E_1^{\mathrm{T}} E_1$ and $R_2 \triangleq E_2^{\mathrm{T}} E_2$ which appear in subsequent expressions. Note that $R_{12} \triangleq E_1^{\mathrm{T}} E_2$ is an H_2 cross-weighting term which is included for greater design flexibility.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$, $E_{2\infty}$, and E_∞ for the state, control, and disturbance variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^{\mathrm{T}} M^{-1} E_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}} M^{-1} E_{2\infty}$ are thus the H_∞ counterparts of the H_∞ weights R_1 and R_2. Here $M \triangleq I_{q_\infty} - \gamma^{-2} E_\infty E_\infty^{\mathrm{T}}$ arises due to the feedthrough term to the H_∞ performance variables. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2, we shall assume for simplicity that $R_2 = \alpha^2 \hat{R}_2$ and $R_{2\infty} = \beta^2 \hat{R}_2$, where the nonnegative scalars α, β are design variables such that $\alpha^2 + \beta^2 \neq 0$. As in the H_2 case we allow an H_∞ cross-weighting term $R_{12\infty} \triangleq E_{1\infty}^{\mathrm{T}} M^{-1} E_{2\infty}$. Finally, the dual design feature of plant disturbance and sensor noise correlation is also permitted. As in [1], $w(t)$ is interpreted as white noise for the H_2 design aspect and as an L_2 signal for the H_∞ design aspect. Note that without the H_2 performance criterion, i.e., $R_1 = 0$ and $\alpha = 0$, the problem considered here reduces to the 'pure' H_∞ standard problem (see Figure 1).

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given compensator (A_c, B_c, C_c) then the H_2 performance (2.8) is given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{Q} \tilde{R}, \tag{2.13}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}\left[\tilde{x}(t) \tilde{x}^{\mathrm{T}}(t) \right] \tag{2.14}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{2.15}$$

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.15) by an algebraic Riccati equation that overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1. *Let (A_c, B_c, C_c) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{n\times n}$ satisfying*

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.16}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)M^{-1}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)^{\mathrm{T}} + \tilde{V}. \tag{2.17}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.18}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.19}$$

In this case,

$$\| H(s) \|_\infty \le \gamma \tag{2.20}$$

and

$$\tilde{Q} \le \mathcal{Q}. \tag{2.21}$$

Consequently,

$$J(A_c, B_c, C_c) \le \mathcal{J}(A_c, B_c, C_c, \mathcal{Q}), \tag{2.22}$$

where

$$\mathcal{J}(A_c, B_c, C_c, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{2.23}$$

Proof. It follows from [13, Theorem 3.6] that (2.18) implies that

$$\left(\tilde{A}, \left[\gamma^{-2}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)M^{-1}\big(\tilde{D}E_\infty + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)^{\mathrm{T}} + \tilde{V}\right]^{1/2}\right)$$

is also stabilizable. Using the assumed existence of a nonnegative-definite solution to (2.17) and [13, Lemma 12.2] it now follows that $\tilde{A}$ is asymptotically stable. The converse is immediate. To prove (2.20), replace $\tilde{V}$ by $\tilde{D}\tilde{D}^{\mathrm{T}}$ and add and subtract $\mathrm{j}\omega I_{\tilde{n}}\mathcal{Q}$ to (2.17) so that (2.17) becomes

$$0 = \big(-\mathrm{j}\omega I_{\tilde{n}} + \tilde{A}\big)\mathcal{Q} + \mathcal{Q}\big(\mathrm{j}\omega I_{\tilde{n}} + \tilde{A}\big)^{\mathrm{T}} + \gamma^{-2}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)M^{-1}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)^{\mathrm{T}} + \tilde{D}\tilde{D}^{\mathrm{T}} \tag{2.24}$$

or, equivalently,

$$\tilde{D}\tilde{D}^{\mathrm{T}} = \big(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)\mathcal{Q} + \mathcal{Q}\big(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{\mathrm{T}} - \gamma^{-2}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}E_\infty^{\mathrm{T}}\big)M^{-1}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)^{\mathrm{T}}. \tag{2.25}$$

Next, forming

$$\tilde{E}_\infty\big(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-1}(2.25)\big(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}$$

yields

$$\begin{aligned}
&\tilde{E}_\infty\big(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-1}\tilde{D}\tilde{D}^{\mathrm{T}}\big(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}} \\
&\quad = \tilde{E}_\infty\big(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-1}\mathcal{Q}\tilde{E}_\infty^{\mathrm{T}} + \tilde{E}_\infty\mathcal{Q}\big(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}} \\
&\qquad - \gamma^{-2}\tilde{E}_\infty\big(\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-1}\Big[\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)M^{-1}\big(\tilde{D}E_\infty^{\mathrm{T}} + \mathcal{Q}\tilde{E}_\infty^{\mathrm{T}}\big)^{\mathrm{T}}\Big]\big(-\mathrm{j}\omega I_{\tilde{n}} - \tilde{A}\big)^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}.
\end{aligned} \tag{2.26}$$

Now adding $\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\tilde{D}E_\infty^{\mathrm{T}}+E_\infty\tilde{D}^{\mathrm{T}}(-\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}$ to both sides of (2.26) yields

$$\begin{aligned}&\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\tilde{D}\tilde{D}^{\mathrm{T}}(-\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}+\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\tilde{D}E_\infty^{\mathrm{T}}\\&\quad+E_\infty\tilde{D}^{\mathrm{T}}(-\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}\\&\quad=\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right]+\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right]^{\mathrm{T}}(-\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}+E_\infty E_\infty^{\mathrm{T}}\\&\quad+\gamma^{-2}\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\left[(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})M^{-1}(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})^{\mathrm{T}}\right](-\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-\mathrm{T}}\tilde{E}_\infty^{\mathrm{T}}.\end{aligned}\tag{2.27}$$

Note that the left hand side of (2.27) is equal $H(\mathrm{j}\omega)H^*(\mathrm{j}\omega)$ and the right hand side of (2.27) can be written as

$$S+S^*-\gamma^{-2}SM^{-1}S^*+\gamma^2(I_{q_\infty}-M)\tag{2.28}$$

where

$$S\triangleq\tilde{E}_\infty(\mathrm{j}\omega I_{\tilde{n}}-\tilde{A})^{-1}\left[\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}\right]$$

and $E_\infty E_\infty^{\mathrm{T}}$ is replaced by $\gamma^2(I_{q_\infty}-M)$. Hence, it follows from (2.27) and (2.28) that

$$H(\mathrm{j}\omega)H^*(\mathrm{j}\omega)=-\left[(\gamma M^{1/2}-\gamma^{-1}SM^{1/2})(\gamma M^{1/2}-\gamma^{-1}SM^{1/2})^*\right]+\gamma^2I_{q_\infty}\geq 0,\tag{2.29}$$

which implies $H(\mathrm{j}\omega)H^*(\mathrm{j}\omega)\leq\gamma^2I_{q_\infty}$. This proves (2.20). To prove (2.21), subtract (2.15) from (2.17) to obtain

$$0=\tilde{A}(\mathscr{Q}-\tilde{Q})+(\mathscr{Q}-\tilde{Q})\tilde{A}^{\mathrm{T}}+\gamma^{-2}(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})M^{-1}(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})^{\mathrm{T}}\tag{2.30}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathscr{Q}-\tilde{Q}=\int_0^\infty \mathrm{e}^{\tilde{A}t}\left[\gamma^{-2}(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})M^{-1}(\tilde{D}E_\infty^{\mathrm{T}}+\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}})^{\mathrm{T}}\right]\mathrm{e}^{\tilde{A}^{\mathrm{T}}t}\,\mathrm{d}t\geq 0.$$

Finally, (2.22) follows immediately from (2.21). □

Remark 2.1. An equivalent form of (2.17) is given by

$$0=(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty)\mathscr{Q}+\mathscr{Q}(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty)^{\mathrm{T}}+\gamma^{-2}\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\mathscr{Q}+\tilde{D}N^{-1}\tilde{D}^{\mathrm{T}}.\tag{2.31}$$

The equivalence of (2.17) and (2.31) is easily shown by noting that (2.17) can be rewritten as

$$\begin{aligned}0&=(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty)\mathscr{Q}+\mathscr{Q}(\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty)^{\mathrm{T}}\\&\quad+\gamma^{-2}\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty\mathscr{Q}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}E_\infty\tilde{D}^{\mathrm{T}}+\tilde{D}\tilde{D}^{\mathrm{T}}\end{aligned}\tag{2.32}$$

and noting that $\tilde{D}[\gamma^{-2}E_\infty^{\mathrm{T}}M^{-1}E_\infty+I_d]\tilde{D}^{\mathrm{T}}$ is equal to $\tilde{D}N^{-1}\tilde{D}^{\mathrm{T}}$ since $E_\infty^{\mathrm{T}}M^{-1}=N^{-1}E_\infty^{\mathrm{T}}$ and $N^{-1}(\gamma^{-2}E_\infty^{\mathrm{T}}E_\infty+N)=N^{-1}$.

Lemma 2.1 shows that H_∞ disturbance attenuation is automatically enforced when a nonnegative-definite solution to (2.17) is known to exist and $\tilde{A}$ is asymptotically stable. Furthermore, all such solutions provide upper bounds for the actual closed-loop state covariance $\tilde{Q}$ along with a bound on the H_2 performance criterion. Next, we present a partial converse of Lemma 2.1 that guarantees the existence of a unique minimal nonnegative-definite solution to (2.17) when (2.20) is satisfied. The minimal solution is desirable since it yields the tightest performance bound in (2.22). This was first pointed out in [7].

Lemma 2.2. *Let (A_c, B_c, C_c) be given, suppose $\tilde{A}$ is asymptotically stable, and assume the disturbance attenuation constraint* (2.20) *is satisfied. Then there exists a unique nonnegative-definite solution $\mathscr{Q}$ satisfying* (2.17) *and such that the eigenvalues of $\tilde{A}+\gamma^{-2}\tilde{D}E_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty+\gamma^{-2}\mathscr{Q}\tilde{E}_\infty^{\mathrm{T}}M^{-1}\tilde{E}_\infty$ lie in the closed left half plane. Furthermore, this solution is also minimal.*

Proof. The result is an immediate extension of [2, pp. 150 and 167], using Theorems 3 and 2. The proof of minimality of given in [12]. □

3. The Auxiliary Minimization Problem

As shown in the previous section, replacing (2.15) by (2.17) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the H_2 performance criterion. That is, given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.17), the *actual* H_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathscr{J}(A_c, B_c, C_c, \mathscr{Q})$. Hence, $\mathscr{J}(A_c, B_c, C_c, \mathscr{Q})$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, \mathscr{Q})$ which minimizes $\mathscr{J}(A_c, B_c, C_c, \mathscr{Q})$ subject to (2.17) with $\mathscr{Q} \in \mathbf{N}^{\tilde{n}}$.

It follows from Lemma 2.1 that the satisfaction of (2.16) and (2.17) along with the generic condition (2.18) lead to: (1) closed-loop stability; (2) prespecified H_∞ performance attenuation; (3) an upper bound for the H_2 performance criterion. Hence, it remains to determine $(A_c, B_c, C_c, \mathscr{Q})$ that minimizes $\mathscr{J}(A_c, B_c, C_c, \mathscr{Q})$, and thus provides an optimized bound for the actual H_2 performance $J(A_c, B_c, C_c)$.

4. Sufficient conditions for H_∞ disturbance attenuation

In this section we state sufficient conditions for characterizing full-order controllers guaranteeing closed-loop stability, constrained H_∞ disturbance attenuation, and an optimized H_2 performance bound. For arbitrary $Q, P, \hat{Q} \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \geq 0$ define the notation

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12\infty}, \qquad P_a \triangleq \left[B^{\mathrm{T}} + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}} + \gamma^{-2}R_{12\infty}^{\mathrm{T}}(Q + \hat{Q})\right]P + R_{12}^{\mathrm{T}},$$
$$S \triangleq \left(\alpha^2 I_n + \beta^2\gamma^{-2}\hat{Q}P\right)^{-1}$$

when the indicated inverse exists.

Theorem 4.1. *Suppose there exist $Q, P, \hat{Q} \in \mathbf{N}^n$ satisfying*

$$0 = \left(A + \gamma^{-2}D_1R_{01\infty}\right)Q + Q\left(A + \gamma^{-2}D_1R_{01\infty}\right)^{\mathrm{T}} + \gamma^{-2}QR_{1\infty}Q + V_{1\infty} - Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}, \tag{4.1}$$

$$\begin{aligned} 0 = {} & \left(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}}\right)^{\mathrm{T}}P \\ & + P\left(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}}\right) + R_1 - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS. \end{aligned} \tag{4.2}$$

$$\begin{aligned} 0 = {} & \left(A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS\right] + \gamma^{-2}\left[D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS\right]\right)\hat{Q} \\ & + \hat{Q}\left(A - B\hat{R}_2^{-1}P_aS + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS\right] + \gamma^{-2}\left[D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS\right]\right)^{\mathrm{T}} \\ & + \gamma^{-2}\hat{Q}\left(R_{1\infty} - R_{12\infty}\hat{R}_2^{-1}P_aS - S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_{12\infty}^{\mathrm{T}} + \beta^2S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}P_aS\right)\hat{Q} + Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}, \end{aligned} \tag{4.3}$$

and let $(A_c, B_c, C_c, \mathscr{Q})$ be given by

$$\begin{aligned} A_c = {} & A - B\hat{R}_2^{-1}P_aS - Q_aV_{2\infty}^{-1}C - Q_aV_{2\infty}^{-1}D\hat{R}_2^{-1}P_aS \\ & + \gamma^{-2}\left(QR_{1\infty} + D_1R_{01\infty} - D_1R_{02\infty}\hat{R}_2^{-1}P_aS - QR_{12\infty}\hat{R}_2^{-1}P_aS \right. \\ & \qquad \left. - Q_aV_{2\infty}^{-1}D_2R_{01\infty} + Q_aV_{2\infty}^{-1}D_2R_{02\infty}\hat{R}_2^{-1}P_aS\right), \end{aligned} \tag{4.4}$$

$$B_c = Q_a V_{2\infty}^{-1}, \qquad C_c = -\hat{R}_2^{-1} P_a S, \tag{4,5), (4,6}$$

$$\mathcal{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix}. \tag{4.7}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20) and the H_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}\big[(Q+\hat{Q})R_1 - 2R_{12}\hat{R}_2^{-1}P_a S\hat{Q} + S^{\mathrm{T}}P_a^{\mathrm{T}}\hat{R}_2^{-1}R_2\hat{R}_2^{-1}P_a S\hat{Q}\big]. \tag{4.8}$$

Proof. The proof follows as in the proof given in [1]. □

Remark 4.1. Theorem 4.1 presents sufficient conditions for designing controllers with a prespecified H_∞ constraint on the closed-loop transfer function. These sufficient conditions comprise a system of three modified algebraic Riccati equations in variables Q, P, and $\hat{Q}$. The Q and P equations are similar to the estimator and regulator Riccati equations of LQG theory, while the $\hat{Q}$ equation has no counterpart in the standard theory. Note that the Q equation is decoupled from the P and $\hat{Q}$ equations and thus can be solved independently. The P equation, however depends on Q. Thus, regulator/estimator separation holds in only one direction which clearly shows that the certainty equivalence principle is no longer valid for the mixed H_2/H_∞ design problem. Finally, note that if the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the P equation becomes decoupled from the $\hat{Q}$ equation and thus the $\hat{Q}$ equation becomes superfluous. Furthermore, the remaining Q and P equations separate and coincide with the standard LQG result. Alternatively, note that if both $\beta = 0$ and $R_{1\infty} = 0$, then Theorem 4.1 also specializes to the standard LQG result.

Remark 4.2. The results of [1] are a special case of Theorem 4.1. To see this set the plant/measurement noise correlation to zero ($V_{12} = 0$), set both the H_2 and H_∞ cross weighting terms to zero (R_{12}, $R_{12\infty} = 0$), set the direct transmission term in the plant dynamics to zero ($D = 0$) and set the feedthrough term from disturbances to H_∞ performance variables to zero ($E_\infty = 0$). This yields Theorem 3.1 of [1].

Remark 4.3. When solving (4.1)–(4.3) numerically, the H_∞ constraint can be adjusted to examine tradeoffs between H_2 performance and disturbance rejection. Specifically, γ can be varied systematically to determine the region of solvability of the design equations (4.1)–(4.3) and to study tradeoffs between the H_2/H_∞ performance criteria (see [1]).

5. The pure H_∞ standard problem

As shown in Theorem 4.1, the Riccati equations (4.1)–(4.3) provide sufficient conditions for explicitly synthesizing controllers (A_c, B_c, C_c) satisfying both H_2 and H_∞ performance bounds. The main purpose of this section is to completely eliminate the H_2 aspect in the design problem. This section also provides connections between our approach and the recent results obtained in [3,4,6]. In [1] it was shown that by equalizing the H_2/H_∞ weights the three coupled Riccati equation form could be transformed into two decoupled Riccati equations as in [3,7]. Furthermore, it was shown in [7] that the auxiliary cost (2.23) is equivalent to an entropy integral. However, it is important to note that, as noticed in Remark 2.1, the results of [7] cannot consider a general direct transmission term from disturbances to H_∞ performance variables in order to guarantee that the minimum value of the entropy evaluated at infinity is finite. In the present paper we utilize a simpler approach wherein we eliminate the H_2 contribution by letting R_1, R_{12}, α (and thus R_2) approach zero. By eliminating the H_2 contribution to the problem, the resulting setting

corresponds to the H_∞ standard problem. In order to state the main result we require some additional notation. For arbitrary $Y_\infty \in \mathbb{R}^{n\times n}$ define the notation

$$Y_{\infty a} \triangleq B^{\mathrm{T}}Y_\infty + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}}Y_\infty + R_{12\infty}^{\mathrm{T}}.$$

Theorem 5.1. *Suppose there exist $Q \in \mathbb{N}^n$ and $Y_\infty \in \mathbb{P}^n$ satisfying*

$$0 = (A+\gamma^{-2}D_1R_{01\infty})Q + Q(A+\gamma^{-2}D_1R_{01\infty})^{\mathrm{T}} + V_{1\infty} + \gamma^{-2}QR_{1\infty}Q - Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}, \tag{5.1}$$

$$0 = (A+\gamma^{-2}D_1R_{01\infty})^{\mathrm{T}}Y_\infty + Y_\infty(A+\gamma^{-2}D_1R_{01\infty}) + R_{1\infty} + \gamma^{-2}Y_\infty V_{1\infty}Y_\infty - Y_{\infty a}^{\mathrm{T}}R_{2\infty}^{-1}Y_{\infty a}, \tag{5.2}$$

$$\rho(QY_\infty) < \gamma^2, \tag{5.3}$$

and let (A_c, B_c, C_c) be given by

$$\begin{aligned} A_c = {}& A - BR_{2\infty}^{-1}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1} - Q_aV_{2\infty}^{-1}C + Q_aV_{2\infty}^{-1}DR_{2\infty}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1} \\ & + \gamma^{-2}\big[QR_{1\infty} + D_1R_{01\infty} - D_1R_{02\infty}R_{2\infty}^{-1}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1} \\ & \quad - QR_{12\infty}R_{2\infty}^{-1}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1} - Q_aV_{2\infty}^{-1}D_2R_{01\infty} \\ & \quad + Q_aV_{2\infty}^{-1}D_2R_{02\infty}R_{2\infty}^{-1}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1}\big], \end{aligned} \tag{5.4}$$

$$B_c = Q_aV_{2\infty}^{-1}, \qquad C_c = -R_{2\infty}^{-1}Y_{\infty a}(I_n-\gamma^{-2}QY_\infty)^{-1}. \tag{5.5), (5.6}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20).

Proof. First let R_1, R_{12}, $\alpha \to 0$ in equations (4.1)–(4.3) so that $S \to \beta^{-2}\gamma^2P^{-1}\hat{Q}^{-1}$. Next, note that $P_aS = \beta^{-2}\gamma^2\Sigma\hat{Q}^{-1}$, where

$$\Sigma \triangleq B^{\mathrm{T}} + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}} + \gamma^{-2}R_{12\infty}^{\mathrm{T}}(Q+\hat{Q}).$$

Now define $Z_\infty \triangleq \gamma^2\hat{Q}^{-1}$ and substitute into (4.3) to obtain

$$\begin{aligned} 0 = {}& (A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}D_1R_{01\infty})^{\mathrm{T}}Z_\infty + Z_\infty(A+\gamma^{-2}QR_{1\infty}+\gamma^{-2}D_1R_{01\infty}) + R_{1\infty} \\ & + \gamma^2Z_\infty Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}Z_\infty - Z_{\infty a}^{\mathrm{T}}R_{2\infty}^{-1}Z_{\infty a}, \end{aligned}$$

where $Z_{\infty a} \triangleq \Sigma Z_\infty$. Now note that (5.2) follows by forming $Y_\infty \triangleq (Z_\infty^{-1}+\gamma^2Q)^{-1}$. The gain expressions (5.4)–(5.6) follow as a direct consequence. □

Remark 5.1. The solutions Q and Y_∞ of (5.1) and (5.2) are analogous to the matrices S and P of [5] and Y_∞ and X_∞ of [4], while (5.3) corresponds to condition 5.2 (iii) of [4].

Remark 5.2. By setting $R_{12\infty}$, E_∞, and D to zero, the results of Theorem 5.1 specialize to Theorem 6 of [3] and Proposition 5.7 of [1] without the H_2 performance bound.

6. Mixed-norm reduced-order dynamic compensation

In this section we extend Theorem 4.1 by expanding the formulation of Sections 2 and 3 to allow the compensator to be of fixed dimension n_c which may be less than the plant order n. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \le n$. As in [1,6] this additional constraint leads to an oblique projection that introduces additional coupling in the design equations along with an additional equation. The following lemma is required for the statement of the main theorem (see [1].)

Lemma 6.1. *Let $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ and suppose* rank $\hat{Q}\hat{P} = n_c$. *Then there exist $n_c \times n$ G, Γ, and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \qquad \Gamma G^{\mathrm{T}} = I_{n_c}. \tag{6.1), (6.2}$$

Furthermore, the $n \times n$ matrices

$$\tau = G^{\mathrm{T}}\Gamma, \qquad \tau_\perp = I_n - \tau \tag{6.3), (6.4}$$

are idempotent and have rank n_c and $n - n_c$.

Theorem 6.1. *Let $n_c \le n$, suppose there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ satisfying*

$$\begin{aligned} 0 = {} & (A + \gamma^{-2} D_1 R_{01\infty})Q + Q(A + \gamma^{-2} D_1 R_{01\infty})^{\mathrm{T}} + \gamma^{-2} Q R_{1\infty} Q \\ & + V_{1\infty} - Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} + \tau_\perp Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} 0 = {} & \left(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2}\hat{Q} S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} R_{12\infty}^{\mathrm{T}}\right)^{\mathrm{T}} P \\ & + P\left(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2}\hat{Q} S^{\mathrm{T}} P_c \hat{R}_2^{-1} R_{12\infty}^{\mathrm{T}}\right) \\ & + R_1 - S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S + \tau_\perp^{\mathrm{T}} S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S \tau_\perp, \end{aligned} \tag{6.6}$$

$$\begin{aligned} 0 = {} & \left(A - B\hat{R}_2^{-1} P_a S + \gamma^{-2} Q[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1} P_a S] + \gamma^{-2} D_1 [R_{01\infty} - R_{02\infty}\hat{R}_2^{-1} P_a S]\right)\hat{Q} \\ & + \hat{Q}\left(A - B\hat{R}_2^{-1} P_a S + \gamma^{-2} Q[R_{1\infty} - R_{12\infty}\hat{R}_2^{-1} P_a S] + \gamma^{-2} D_1 [R_{01\infty} - R_{02\infty}\hat{R}_2^{-1} P_a S]\right)^{\mathrm{T}} \\ & + \gamma^{-2}\hat{Q}\left(R_{1\infty} - R_{12\infty}\hat{R}_2^{-1} P_a S - S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} R_{12\infty}^{\mathrm{T}} + \beta^2 S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S\right)\hat{Q} \\ & + Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} - \tau_\perp Q_a V_{2\infty}^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} 0 = {} & \left(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} + \gamma^{-2} Q R_{1\infty} - \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty}\right)^{\mathrm{T}} \hat{P} \\ & + \hat{P}\left(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} + \gamma^{-2} Q R_{1\infty} - \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty}\right) \\ & + S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S - \tau_\perp^{\mathrm{T}} S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S \tau_\perp, \end{aligned} \tag{6.8}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c, \tag{6.9}$$

and let $(A_c, B_c, C_c, \mathcal{Q})$ be given by

$$\begin{aligned} A_c = \Gamma\Big[& A - B\hat{R}_2^{-1} P_a S - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D \hat{R}_2^{-1} P_a S + \gamma^{-2}\big(Q R_{1\infty} + D_1 R_{01\infty} \\ & - D_1 R_{02\infty}\hat{R}_2^{-1} P_a S - Q R_{12\infty}\hat{R}_2^{-1} P_a S - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} + Q_a V_{2\infty}^{-1} D_2 R_{02\infty}\hat{R}_2^{-1} P_a S\big)\Big] G^{\mathrm{T}}, \end{aligned} \tag{6.10}$$

$$B_c = \Gamma Q_a V_{2\infty}^{-1}, \qquad C_c = -\hat{R}_2^{-1} P_a S G^{\mathrm{T}}, \tag{6.11), (6.12}$$

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}. \tag{6.13}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20) *and the H_2 performance criterion* (2.7) *satisfies the bound*

$$J(A_c, B_c, C_c) \le \operatorname{tr}\left[(Q + \hat{Q})R_1 - 2R_{12}\hat{R}_2^{-1} P_a S \hat{Q} + S^{\mathrm{T}} P_a^{\mathrm{T}} \hat{R}_2^{-1} P_a S \hat{Q}\right]. \tag{6.14}$$

Proof. The proof follows as in [1] with the additional terms arising due to cross weighting, disturbance/measurement noise correlation, and direct feedthrough terms. □

Remark 6.1. It is easy to see that Theorem 6.1 is a direct generalization of Theorem 4.1. To recover Theorem 4.1, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.5)–(6.8) can be deleted and (6.8) becomes superfluous. Furthermore, (6.5)–(6.7) now reduce to (4.1)–(4.3), as expected. Alternatively, setting $\gamma = \infty$ and retaining the reduced-order constraint $n_c < n$ yields the result of [6]. Finally, to recover Theorem 6.1 of [1] set $V_{12} = 0$, $R_{12} = 0$, $R_{12\infty} = 0$, $D = 0$, and $E_\infty = 0$.

Remark 6.2. As was noted earlier, the assumption that $R_2 = \alpha^2\hat{R}_2$ and $R_{2\infty} = \beta^2\hat{R}_2$ was made for simplicity. If it is desired that R_2 and $R_{2\infty}$ be independent then (6.12) is given by

$$C_c = -\mathrm{vec}^{-1}\left[\Omega\,\mathrm{vec}\left(P_a G^{\mathrm{T}}\right)\right],$$

where

$$\Omega \triangleq R_2 \otimes I_{n_c} + \gamma^{-2}R_{2\infty} \otimes \Gamma\hat{Q}PG^{\mathrm{T}},$$

'vec' is the column stacking operation, and $\otimes$ denotes Kronecker product. In this case, the compensator dynamics (6.10) along with the design equations (6.5)–(6.8) have to be changed accordingly. However, due to lack of space this result is not given. Similar remarks apply to the full-order mixed-norm problem given by Theorem 4.1.

7. The pure H_∞ reduced-order dynamic compensation problem

In this section we eliminate the H_2 aspect of the reduced-order design problem to obtain reduced-order controllers for the pure H_∞ standard problem. As in the full-order controller case (Section 5) we eliminate the H_2 contribution by letting R_1, R_{12}, α (and thus R_2) approach zero. In order to state the main result we require some additional notation. For arbitrary Q, $\hat{Q}$, $P \in \mathbf{N}^n$, and G, $\Gamma \in \mathbf{R}^{n_c \times n}$ define

$$P_{a\infty} \triangleq B^{\mathrm{T}}P + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}}P + \gamma^{-2}R_{12\infty}^{\mathrm{T}}(Q+\hat{Q})P, \tag{7.1}$$

$$M_\infty \triangleq \left(\Gamma\hat{Q}\Gamma^{\mathrm{T}}\right)^{-1}, \qquad N_\infty \triangleq \left(GPG^{\mathrm{T}}\right)^{-1}, \qquad (7.2),\ (7.3)$$

$$S_\infty \triangleq \gamma^2 N_\infty M_\infty, \quad W_\infty \triangleq \gamma^4 \Gamma^{\mathrm{T}}S_\infty^{\mathrm{T}}GP_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma. \qquad (7.4),\ (7.5)$$

Theorem 7.1. *Suppose there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbf{N}^n$ satisfying* (6.9), $GPG^{\mathrm{T}} > 0$, *and*

$$\begin{aligned} 0 = {} & \left(A + \gamma^{-2}D_1R_{01\infty}\right)Q + Q\left(A + \gamma^{-2}D_1R_{01\infty}\right)^{\mathrm{T}} \\ & + V_{1\infty} + \gamma^{-2}QR_{1\infty}Q - Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}} + \tau_\perp Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \end{aligned} \tag{7.6}$$

$$\begin{aligned} 0 = {} & \left(A + \gamma^{-2}\left[Q+\hat{Q}\right]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}\hat{Q}\Gamma^{\mathrm{T}}S_\infty^{\mathrm{T}}GP_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}R_{12\infty}^{\mathrm{T}} + \gamma^{-2}\hat{Q}W_\infty\right)^{\mathrm{T}}P \\ & + P\left(A + \gamma^{-2}\left[Q+\hat{Q}\right]R_{1\infty} + \gamma^{-2}D_1R_{01\infty} - \gamma^{-2}Q\Gamma^{\mathrm{T}}S_\infty^{\mathrm{T}}GP_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}R_{12\infty}^{\mathrm{T}} + \gamma^{-2}\hat{Q}W_\infty\right) \\ & + R_{1\infty} - P_{a\infty}R_{2\infty}^{-1}P_{a\infty} + \left(I_n - G^{\mathrm{T}}S_\infty\Gamma\right)^{\mathrm{T}}P_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}P_{a\infty}\left(I_n - G^{\mathrm{T}}S_\infty\Gamma\right), \end{aligned} \tag{7.7}$$

$$\begin{aligned} 0 = {} & \left(A - BR_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma\right]\right. \\ & \qquad \left. + \gamma^{-2}\left[D_1R_{01\infty} - R_{02\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma\right]\right)\hat{Q} \\ & + \hat{Q}\left(A - BR_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma + \gamma^{-2}Q\left[R_{1\infty} - R_{12\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma\right]\right. \\ & \qquad \left. + \gamma^{-2}\left[D_1R_{01\infty} - R_{02\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma\right]\right) \\ & + \gamma^{-2}\hat{Q}\left(R_{1\infty} - R_{12\infty}R_{2\infty}^{-1}P_{a\infty}G^{\mathrm{T}}S_\infty\Gamma - \Gamma^{\mathrm{T}}S_\infty^{\mathrm{T}}P_{a\infty}^{\mathrm{T}}R_{2\infty}^{-1}R_{12\infty}^{\mathrm{T}} + W_\infty\right)\hat{Q} \\ & + Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}} - \tau_\perp Q_aV_{2\infty}^{-1}Q_a^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \end{aligned} \tag{7.8}$$

$$0 = \left(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} + \gamma^{-2} Q R_{1\infty} - \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty}\right)^{\mathrm{T}} \hat{P}$$
$$+ \hat{P}\left(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} + \gamma^{-2} Q R_{1\infty} - \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty}\right)$$
$$+ P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} P_{a\infty} - \left(I_n - G^{\mathrm{T}} S_\infty \Gamma\right)^{\mathrm{T}} P_{a\infty}^{\mathrm{T}} R_{2\infty}^{-1} P_{a\infty}\left(I_n - G^{\mathrm{T}} S_\infty \Gamma\right) - \gamma^{-2}\left(W_\infty \hat{Q} P + P \hat{Q} W_\infty\right), \tag{7.9}$$

and let (A_c, B_c, C_c) be given by

$$A_c = \Gamma\left[A - R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right.$$
$$+ \gamma^{-2}\left(Q R_{1\infty} + D_1 R_{01\infty} - D_1 R_{02\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma - Q R_{12\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right.$$
$$\left.\left. - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} + Q_a V_{2\infty}^{-1} D_2 R_{02\infty} R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty \Gamma\right)\right] G^{\mathrm{T}}, \tag{7.10}$$

$$B_c = \Gamma Q_a V_{2\infty}^{-1}, \qquad C_c = -R_{2\infty}^{-1} P_{a\infty} G^{\mathrm{T}} S_\infty. \qquad (7.11), (7.12)$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the H_∞ disturbance attenuation constraint (2.20).

Proof. The proof follows from Theorem 6.1 by using the relation $G^{\mathrm{T}}\hat{S}\Gamma = S_\tau$, where $\hat{S} \triangleq (\alpha^2 I_{n_c} + \gamma^{-2}\beta^2 \Gamma \hat{Q} P G^{\mathrm{T}})^{-1}$ and letting R_1, R_{12}, $\alpha \to 0$. □

Remark 7.1. Theorem 7.1 presents sufficient conditions for designing *reduced-order* controllers with a prespecified H_∞ constraint on the closed-loop transfer function with no H_2 contribution. Thus, Theorem 7.1 addresses the pure reduced-order H_∞-standard problem. Note that considerable simplification can be achieved in the design equations by setting $R_{12\infty}$, E_∞, and D to zero.

8. Numerical solution of the design equations

Although the design equations appearing in Theorems 4.1, 6.1 and 7.1 appear formidable, they are, in fact, quite numerically tractable. One of the principal motivations of the Riccati equation approach to the mixed norm problem is the opportunity it provides for developing efficient computational algorithms for control design. In particular, the goal is to develop numerical methods that exploit the structure of these modified Riccati equations. It should be noted, however, that existing methods for solving standard Riccati equations cannot account for the additional terms that appear in the modified equations such as (6.5)–(6.8). Therefore, a new class of numerical algorithms has been developed based upon homotopic continuation methods. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path (homotopy path) that connects the starting problem to the original problem. The advantage of such algorithms is that they are based on theories which are global in nature. In particular, homotopy methods facilitate the finding of (multiple) solutions to a problem, and the convergence of the homotopy algorithms is generally not dependent upon having initial conditions which are in some sense close to the actual solution. These ideas have been illustrated for the H_2 reduced-order problem in [9] and the H_∞ constrained problem in [1] where the additional coupling terms preclude standard solution techniques. A complete description of the homotopy algorithm is given in [10].

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On the gap between H_2 and entropy performance measures in H_∞ control design *

Wassim M. HADDAD
Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

Dennis S. BERNSTEIN
Harris Corporation, Government Aerospace Systems Division, MS 22/4842, Melbourne, FL 32902, U.S.A.

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Abstract: Recent papers have considered the problem of minimizing an entropy functional subject to an H_∞ performance constraint. Since the entropy is an upper bound for the H_2 cost, there remains a gap between entropy minimization and H_2 minimization. In this paper we consider a generalized cost functional involving both H_2 and entropy aspects. This approach thus provides a means for optimizing H_2 performance within H_∞ control design.

Keywords: H_2 design; minimum entropy; mixed-norm H_2/H_∞ design.

1. Introduction

It was recently shown in [1] that suboptimal H_∞ controllers can be characterized by means of modified Riccati equations. These equations were obtained by minimizing an H_2 performance bound subject to a constraint on the H_∞ performance. Subsequently it was shown that, in the equalized H_2/H_∞ weight case, the H_2 performance bound coincides with an entropy functional [4,5]. Although less familiar than the H_2 objective, the entropy functional is mathematically tractable within the context of H_∞ control theory.

In many practical applications, however, it may be desirable to minimize the H_2 cost directly. That is, although the entropy functional bounds the H_2 cost (in the equalized weight case), there may exist a 'gap' between these performance measures. Thus, the control law that minimizes the entropy need not also minimize the H_2 performance.

The goal of the present paper is to extend the approach of [1] to include both H_2 and entropy performance measures within the context of constrained H_∞ design. This multiobjective problem is treated by forming a convex combination of both performance measures. This approach is reminiscent of scalarization techniques for Pareto optimization [4].

For simplicity the present paper is confined to static full-state feedback control. Full- and reduced-order dynamic compensation as in [1] will be considered in a future paper.

Notation. Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$.
I_r, $(\ )^{\mathrm{T}}$, tr	$r\times r$ identity matrix, transpose, trace.
n, m, d, q, q_∞	positive integers.
x, u	n, m-dimensional vectors.
A, B, K	$n\times n$, $n\times m$, $m\times n$ matrices.
$w(\cdot)$	L_2 disturbance signal in $\mathbb{R}^d$.

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D, V	$n\times d$, $n\times n$ matrices; $V=DD^{\mathrm{T}}$.
E_1, E_2	$q\times n$, $q\times m$ matrices; $E_1^{\mathrm{T}}E_2=0$.
R_1, R_2	$E_1^{\mathrm{T}}E_1$, $E_2^{\mathrm{T}}E_2$.
$\tilde R$	$R_1+K^{\mathrm{T}}R_2K$.
$E_{1\infty}$, $E_{2\infty}$	$q_\infty\times n$, $q_\infty\times m$ matrices; $E_{1\infty}^{\mathrm{T}}E_{2\infty}=0$.
$R_{1\infty}$, $R_{2\infty}$	$E_{1\infty}^{\mathrm{T}}E_{1\infty}$, $E_{2\infty}^{\mathrm{T}}E_{2\infty}$.
$\tilde R_\infty$	$R_{1\infty}+K^{\mathrm{T}}R_{2\infty}K$.
α, β; γ	real numbers; positive number.

2. Problem statement

Combined H_2/H_∞/Entropy Control Problem. Consider the nth-order dynamic system

$$\dot x(t)=Ax(t)+Bu(t)+Dw(t),\quad t\in[0,\infty),\tag{2.1}$$

with feedback law

$$u(t)=Kx(t),\tag{2.2}$$

and H_2 and H_∞ performance variables

$$z_2(t)=E_1x(t)+E_2u(t),\tag{2.3}$$

$$z_\infty(t)=E_{1\infty}x(t)+E_{2\infty}u(t).\tag{2.4}$$

Then determine $K\in\mathbb{R}^{m\times n}$ satisfying the following design criteria:

(i) the closed-loop system (2.1), (2.2) is asymptotically stable, i.e., $\tilde A\triangleq A+BK$ is Hurwitz;

(ii) for given $\gamma>0$, the $q_\infty\times d$ transfer function

$$G_\infty(s)\triangleq(E_{1\infty}+E_{2\infty}K)(sI_n-\tilde A)^{-1}D\tag{2.5}$$

from disturbances $w(\cdot)$ to H_∞ performance variables z_∞ satisfies the H_∞-norm constraint

$$\|G_\infty\|_\infty<\gamma;\tag{2.6}$$

(iii) for $\mu\in[0,1]$ the cost functional

$$J(K)\triangleq\mu\|G_2\|_2^2+(1-\mu)I(G_\infty,\gamma)\tag{2.7}$$

is minimized, where

$$G_2(s)\triangleq(E_1+E_2K)(sI_n-\tilde A)^{-1}D\tag{2.8}$$

is the $q\times d$ transfer function from disturbances w to H_2 performance variables z_2, and

$$I(G_\infty,\gamma)\triangleq-\lim_{s_0\to\infty}\left[\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\ln\left|\det\left(I_n-\gamma^{-2}G_\infty(\mathrm{j}\omega)G_\infty^*(\mathrm{j}\omega)\right)\right|\left[\frac{s_0^2}{s_0^2+\omega^2}\right]\mathrm{d}\omega\right]\tag{2.9}$$

is the entropy functional for the H_∞ performance variables z_∞.

Note that the problem statement involves both H_2 and H_∞ performance variables where for generality z_2 is not necessarily equal to z_∞. For convenience we omit H_2 and H_∞ cross weighting terms by assuming $E_1^{\mathrm{T}}E_2=0$ and $E_{1\infty}^{\mathrm{T}}E_{2\infty}=0$.

As discussed in [5,6], the entropy functional (2.9) can be viewed as a measure of the distance from $\|G_\infty\|_\infty$ to γ. Like the H_2 norm, but unlike the H_∞ norm, however, the entropy $I(G_\infty,\gamma)$ accounts for $G_\infty(\mathrm{j}\omega)$ at all frequencies. Furthermore, it can be shown [2] that the entropy functional at infinity is equivalent to the exponential-of-quadratic cost of the Risk-Sensitive LQG Control Problem [8].

Remark 2.1. Note that (2.7) involves a convex combination of two scalar costs. By varying $\mu \in [0, 1]$, (2.7) can be viewed as the scalar representation of a multiobjective cost (see, e.g., [4]). By setting $\mu = 0$ we obtain an entropy/H_∞ control problem as in [1]. However, it is important to stress that if $\mu = 1$ then the entropy functional is excluded from the cost functional (2.7) so that the optimization procedure is unable to enforce (2.9). In this case the bound (2.6) plays no role and the standard H_2 LQR problem is obtained. The practical value of this formulation is the case $\mu \approx 1$ in which the role of the entropy functional (2.9) is deemphasized and the optimization problem corresponds to minimizing the *actual* H_2 cost while *enforcing* the H_∞ constraint (2.6).

3. Reformulation of the control problem

In this section we reformulate the combined H_2/H_∞/Entropy Control Problem to facilitate the development of optimality conditions. First, we present a sufficient condition that enforces the disturbance attenuation constraint (2.6). For arbitrary $K \in \mathbb{R}^{m\times n}$ define the notation

$$\tilde{R} \triangleq R_1 + K^{\mathrm{T}}R_2K, \qquad \tilde{R}_\infty \triangleq R_{1\infty} + K^{\mathrm{T}}R_{2\infty}K, \qquad V \triangleq DD^{\mathrm{T}}.$$

Lemma 3.1. *Let $K \in \mathbb{R}^{m\times n}$ be given and assume there exists a nonnegative-definite matrix $\mathcal{Q} \in \mathbb{R}^{n\times n}$ satisfying*

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + V. \tag{3.1}$$

Then

$$(\tilde{A}, D) \text{ is stabilizable} \tag{3.2}$$

if and only if

$$\tilde{A} \text{ is Hurwitz.} \tag{3.3}$$

In this case, the following statements hold:

(i) *the transfer function G_∞ satisfies*

$$\| G_\infty \|_\infty \le \gamma; \tag{3.4}$$

(ii) *if $\| G_\infty \|_\infty < \gamma$ then*

$$I(G_\infty, \gamma) \le \operatorname{tr} \mathcal{Q}\tilde{R}_\infty; \tag{3.5}$$

(iii) *the transfer function G_2 is given by*

$$\| G_2 \|_2^2 = \operatorname{tr} Q\tilde{R}, \tag{3.6}$$

where the $n \times n$ matrix Q satisfies

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + V; \tag{3.7}$$

(iv) *the solution Q to (3.7) satisfies the bound*

$$Q \le \mathcal{Q} \tag{3.8}$$

and hence

$$\| G_2 \|_2^2 \le \operatorname{tr} \mathcal{Q}\tilde{R}; \tag{3.9}$$

(v) *all real symmetric solutions to (3.1) are nonnegative definite;*

(vi) *there exists a (unique) minimal solution to (3.1) in the class of real symmetric solutions;*

(vii) *$\mathcal{Q}$ is the minimal solution to* (3.1) *if and only if*

$$\operatorname{Re} \lambda_i(\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty) \le 0; \tag{3.10}$$

(viii) $\|G_\infty\|_\infty < \gamma$ *if and only if* $\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ *is Hurwitz, where $\mathcal{Q}$ is the minimal solution to* (3.1);
(ix) *if $\mathcal{Q}$ is the minimal solution to* (3.1) *and* $\|G_\infty\|_\infty < \gamma$, *then*

$$I(G_\infty, \gamma) = \operatorname{tr} \mathcal{Q}\tilde{R}_\infty. \tag{3.11}$$

Proof. The proof of (3.2)–(3.4) and (3.6)–(3.9) is similar to the proof of Lemma 2.1 given in [1]. Assuming $\tilde{A}$ is Hurwitz, (v) follows by writing $\mathcal{Q} = \int_0^\infty e^{\tilde{A}t}[\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + V]\,e^{\tilde{A}^{\mathrm{T}}t}\,dt$. Result (vi) is given by Theorem 2.1 of [3], while (vii) follows from Theorem 2.1 of [3] and Theorem 2 of [7]. Statement (viii) follows from [6]. Finally, (ix) is given in [5,6], while (3.5) follows from (3.11). □

Remark 3.1. Consider the equalized weight case $z_2 = z_\infty$ so that $G_2 = G_\infty$. In this case it follows from (3.9) and (3.11) that

$$\|G_2\|_2^2 \le I(G_2, \gamma), \tag{3.12}$$

i.e., the entropy is an upper bound for the H_2 cost (see also [5,6]). If the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then it can be shown [5,6] that the entropy functional (2.9) coincides with the H_2 cost, i.e.,

$$I(G_2, \infty) = \|G_2\|_2^2 = \operatorname{tr} Q\tilde{R}. \tag{3.13}$$

Remark 3.2. The treatment of the entropy functional appears to be difficult when $\|G_\infty\|_\infty = \gamma$. This case was not considered in [6].

Lemma 3.1 shows that the H_∞ disturbance attenuation constraint is enforced when a nonnegative-definite solution to (3.1) is known to exist and $\tilde{A}$ is Hurwitz. Furthermore, all such solutions provide upper bounds for the H_2 performance $\|G_2\|_2^2$. Also, if $\mathcal{Q}$ is the minimal solution to (3.1), then the entropy functional (2.9) is given by (3.11). Then, the combined H_2/H_∞/Entropy Control Problem can be recast as the following Auxiliary Optimization Problem. We shall say $K \in \mathbb{R}^{m\times n}$ is *admissible* if $\tilde{A}$ is Hurwitz and $\|G_\infty\|_\infty < \gamma$.

Auxiliary Optimization Problem. For $\mu \in [0, 1]$, determine admissible $K \in \mathbb{R}^{m\times n}$ that minimizes

$$J(K) = \mu \operatorname{tr} Q\tilde{R} + (1-\mu) \operatorname{tr} \mathcal{Q}\tilde{R}_\infty, \tag{3.14}$$

where Q, $\mathcal{Q} \ge 0$ satisfy (3.7) and (3.1).

4. Sufficient conditions for optimality

In this section we state sufficient conditions for characterizing full-state feedback controllers guaranteeing closed-loop stability and constrained H_∞ disturbance attenuation. For convenience in stating the main result we assume

$$R_2 = \alpha^2\hat{R}_2, \qquad R_{2\infty} = \beta^2\hat{R}_2, \tag{4.1}$$

where α, β are real numbers and $\hat{R} \in \mathbb{R}^{m\times m}$ is positive definite. The general case in which (4.1) does not hold is discussed later in Remark 4.1. Also define

$$\Sigma \triangleq B\hat{R}_2^{-1}B^{\mathrm{T}}. \tag{4.2}$$

Theorem 4.1. *Suppose there exist $n \times n$ nonnegative-definite matrices Q, P, $\mathcal{Q}$, $\mathcal{P}$ satisfying*

$$0=(A-\Sigma M)Q+Q(A-\Sigma M)^{\mathrm{T}}+V, \tag{4.3}$$

$$0=(A-\Sigma M)^{\mathrm{T}}P+P(A-\Sigma M)+\mu R_1+\mu\alpha^2 M^{\mathrm{T}}\Sigma M, \tag{4.4}$$

$$0=(A-\Sigma M)\mathcal{Q}+\mathcal{Q}(A-\Sigma M)^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}R_{1\infty}\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}M^{\mathrm{T}}\Sigma M\mathcal{Q}+V, \tag{4.5}$$

$$0=\left(A-\Sigma M+\gamma^{-2}\mathcal{Q}\left[R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right]\right)^{\mathrm{T}}\mathcal{P}+\mathcal{P}\left(A-\Sigma M+\gamma^{-2}\mathcal{Q}\left[R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right]\right)$$
$$+(1-\mu)R_{1\infty}+(1-\mu)\beta^2M^{\mathrm{T}}\Sigma M, \tag{4.6}$$

and

$$\mu\alpha^2Q+(1-\mu)\beta^2\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}\mathcal{P}\mathcal{Q}>0, \tag{4.7}$$

where

$$M \triangleq (PQ+\mathcal{P}\mathcal{Q})\left(\mu\alpha^2Q+(1-\mu)\beta^2\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}\mathcal{P}\mathcal{Q}\right)^{-1}, \tag{4.8}$$

and let K be given by

$$K=-\hat{R}_2^{-1}B^{\mathrm{T}}M. \tag{4.9}$$

Then $(\tilde{A}, D)$ is stabilizable if and only if $\tilde{A}$ is Hurwitz. In this case,

$$\|G_2\|_2^2=\operatorname{tr} Q\left(R_1+\alpha^2M^{\mathrm{T}}\Sigma M\right), \tag{4.10}$$

$$\|G_\infty\|_\infty \le \gamma, \tag{4.11}$$

and, if $\|G_\infty\|_\infty<\gamma$, then

$$I(G_\infty,\gamma)\le \operatorname{tr}\mathcal{Q}\left(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right). \tag{4.12}$$

If, in addition, $A-\Sigma M+\gamma^{-2}\mathcal{Q}(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M)$ is Hurwitz, then

$$I(G_\infty,\gamma)=\operatorname{tr}\mathcal{Q}\left(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right). \tag{4.13}$$

Proof. First we obtain necessary conditions for the Auxiliary Optimization Problem and then show, by construction, that these conditions serve as sufficient conditions for closed-loop stability and prespecified disturbance attenuation. Thus, to optimize (3.14) subject to (3.1) and (3.7), form the Lagrangian

$$\mathcal{L}(K,Q,\mathcal{Q},P,\mathcal{P}) \triangleq \operatorname{tr}\Big[\lambda\left[\mu Q\tilde{R}+(1-\mu)\mathcal{Q}\tilde{R}_\infty\right]$$
$$+(\tilde{A}Q+Q\tilde{A}^{\mathrm{T}}+V)P+\left(\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}+V\right)\mathcal{P}\Big], \tag{4.14}$$

where the Lagrange multipliers $\lambda \ge 0$ and P, $\mathcal{P}\in\mathbb{R}^{n\times n}$ are not all zero. By viewing K, Q, and $\mathcal{Q}$ as independent variables, we obtain

$$\frac{\partial\mathcal{L}}{\partial Q}=\tilde{A}^{\mathrm{T}}P+P\tilde{A}+\lambda\mu\tilde{R}, \tag{4.15}$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{Q}}=\left(\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right)^{\mathrm{T}}\mathcal{P}+\mathcal{P}\left(\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right)+\lambda(1-\mu)\tilde{R}_\infty. \tag{4.16}$$

If both $\tilde{A}$ and $\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ are Hurwitz, then $\lambda=0$ implies $P=0$ and $\mathcal{P}=0$. Hence, it can be assumed without loss of generality that $\lambda=1$. Furthermore, note that P and $\mathcal{P}$ are nonnegative-definite. Thus the

stationary conditions with $\lambda = 1$ are given by

$$\frac{\partial \mathscr{L}}{\partial Q} = \tilde{A}^{\mathrm{T}} P + P\tilde{A} + \mu \tilde{R} = 0, \tag{4.17}$$

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = \left(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty\right)^{\mathrm{T}}\mathscr{P} + \mathscr{P}\left(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty\right) + (1-\mu)\tilde{R}_\infty = 0, \tag{4.18}$$

$$\frac{\partial \mathscr{L}}{\partial K} = \mu R_2 K Q + (1-\mu) R_{2\infty} K \mathscr{Q} + \gamma^{-2} R_{2\infty} K \mathscr{Q}\mathscr{P}\mathscr{Q} + B^{\mathrm{T}}(PQ + \mathscr{P}\mathscr{Q}) = 0. \tag{4.19}$$

Assuming (4.1), (4.19) implies (4.9). Next, with K given by (4.9), (4.3)–(4.6) are equivalent to (3.7), (4.17), (3.1), and (4.18), respectively. It now follows from Lemma 3.1 that the stabilizability condition is equivalent to the stability of $\tilde{A}$. In this case the H_∞ disturbance attenuation constraint (4.11) holds, the entropy is bounded as in (4.12), and the H_2 cost is given by (4.10). If, finally, $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty$ is Hurwitz, then the entropy is given by (4.13), which is a restatement of (3.11). □

Remark 4.1. Condition (4.1) was assumed for convenience only. When (4.1) does not hold, K is given by

$$K = -\mathrm{vec}^{-1}\left\{\Omega^{-1}\,\mathrm{vec}\left[B^{\mathrm{T}}(PQ + \mathscr{P}\mathscr{Q})\right]\right\}, \tag{4.20}$$

where 'vec' denotes the column stacking operator, and Ω is defined by

$$\Omega \triangleq \mu R_2 \otimes Q + (1-\mu) R_{2\infty} \otimes \mathscr{Q} + \gamma^{-2} R_{2\infty} \otimes \mathscr{Q}\mathscr{P}\mathscr{Q}, \tag{4.21}$$

where $\otimes$ denotes Kronecker product. Since $\Omega \geq 0$, (4.20) is valid if $\Omega > 0$, which is a generalization of (4.7). When (4.1) does not hold, however, (4.3)–(4.6) cannot be used and must be replaced by (3.7), (4.17), (3.1) and (4.18), respectively.

5. Specializations of Theorem 4.1

To draw connections with the existing literature, a series of specializations of Theorem 4.1 is now given. We begin by considering the case of an entropy functional only, i.e., $\mu = 0$. In this case, set $R_1 = 0$, $\alpha = 0$ (i.e., $R_2 = 0$) so that (4.3) is superfluous and (4.4) implies $P = 0$. Furthermore, (4.9) becomes

$$K = -R_{2\infty}^{-1} B^{\mathrm{T}} \mathscr{P} S \tag{5.1}$$

and $\mathscr{Q}$, $\mathscr{P}$ satisfy

$$0 = (A - \Sigma_\infty \mathscr{P} S)\mathscr{Q} + \mathscr{Q}(A - \Sigma_\infty \mathscr{P} S)^{\mathrm{T}} + \gamma^{-2}\mathscr{Q} R_{1\infty}\mathscr{Q} + \gamma^{-2}\beta^2 \mathscr{Q} S^{\mathrm{T}} \mathscr{P} \Sigma_\infty \mathscr{P} S \mathscr{Q} + V, \tag{5.2}$$

$$0 = \left(A + \gamma^{-2}\mathscr{Q} R_{1\infty}\right)^{\mathrm{T}}\mathscr{P} + \mathscr{P}\left(A + \gamma^{-2}\mathscr{Q} R_{1\infty}\right) + R_{1\infty} - S^{\mathrm{T}}\mathscr{P}\Sigma_\infty\mathscr{P}S, \tag{5.3}$$

where

$$S \triangleq \left(I_n + \gamma^{-2}\beta^2\mathscr{Q}\mathscr{P}\right)^{-1}, \tag{5.4}$$

$$\Sigma_\infty \triangleq B R_{2\infty}^{-1} B^{\mathrm{T}}. \tag{5.5}$$

Next, by introducing the transformation $Z = \mathscr{P}S = (\mathscr{P}^{-1} + \gamma^{-2}\beta^2\mathscr{Q})^{-1}$ and forming $Z[\mathscr{P}^{-1}(5.3)\mathscr{P}^{-1} + \beta^2\gamma^{-2}(4.2)]Z$, (5.1)–(5.3) collapse to

$$K = -R_{2\infty}^{-1} B^{\mathrm{T}} Z, \tag{5.6}$$

$$0 = A^{\mathrm{T}} Z + ZA + R_{1\infty} + \gamma^{-2} Z V Z - Z\Sigma_\infty Z, \tag{5.7}$$

which is the result given in [9]. Furthermore, it can be shown that

$$\|G_2\|_2^2 \le I(G_\infty, \gamma) = \operatorname{tr} ZV. \tag{5.8}$$

Next, to recover the standard LQR result from Theorem 4.1, set $R_{1\infty}=0$, $\beta=0$ (i.e., $R_{2\infty}=0$) and $\mu=1$ or, effectively (see Remark 2.1), $\gamma\to\infty$. In this case (4.3) and (4.5) are superfluous while (4.6) implies $\mathscr{P}=0$. Furthermore, (4.9) becomes

$$K = -R_2^{-1}B^{\mathrm{T}}P, \tag{5.9}$$

where P satisfies the standard regulator Riccati equation

$$0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma_2 P, \tag{5.10}$$

where

$$\Sigma_2 \triangleq BR_2^{-1}B^{\mathrm{T}}. \tag{5.11}$$

Furthermore,

$$\|G_2\|_2^2 = I(G_\infty, \infty) = \operatorname{tr} PV. \tag{5.12}$$

Note that in this case the H_∞ performance bound (2.6) is not enforced since the entropy functional is excluded from the optimality criterion.

Finally, it is important to point out a generalization of (5.1)–(5.3). Specifically, suppose as in [1] we seek to minimize an *overbound* on the H_2 cost while enforcing the disturbance attenuation constraint with performance variables $z_2 \ne z_\infty$, i.e., (3.14) replaced by tr $\mathscr{Q}\tilde{R}$ and $\mu=0$ so that the actual H_2 cost is not considered. Note that in this case tr $\mathscr{Q}\tilde{R}$ is not generally equal to $I(G_\infty, \gamma)$ and the entropy interpretation of the performance is no longer valid. In this case, (4.3)–(4.6) and (4.8) become

$$K = -\hat{R}_2^{-1}B^{\mathrm{T}}\mathscr{P}\hat{S}, \tag{5.13}$$

where $\mathscr{Q}$, $\mathscr{P}$ satisfy

$$0 = (A - \Sigma\mathscr{P}\hat{S})\mathscr{Q} + \mathscr{Q}(A - \Sigma\mathscr{P}\hat{S})^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}R_{1\infty}\mathscr{Q} + \gamma^{-2}\beta^2\mathscr{Q}\hat{S}^{\mathrm{T}}\mathscr{P}\Sigma\mathscr{P}\hat{S}\mathscr{Q} + V, \tag{5.14}$$

$$0 = \left(A + \gamma^{-2}\mathscr{Q}R_{1\infty}\right)^{\mathrm{T}}\mathscr{P} + \mathscr{P}\left(A + \gamma^{-2}\mathscr{Q}R_{1\infty}\right) + R_1 - \hat{S}^{\mathrm{T}}\mathscr{P}\Sigma\mathscr{P}\hat{S}, \tag{5.15}$$

and

$$\hat{S} \triangleq \left(\alpha^2 I_n + \gamma^{-2}\beta^2\mathscr{Q}\mathscr{P}\right)^{-1}. \tag{5.16}$$

Furthermore,

$$\|G_2\|_2^2 \le \operatorname{tr} \mathscr{Q}\left(R_1 + \hat{S}^{\mathrm{T}}\mathscr{P}\Sigma\mathscr{P}\hat{S}\right). \tag{5.17}$$

It is interesting to note that the full state feedback overbound H_2/H_∞ unequalized weights case involves two coupled equations, one modified Riccati equation, and one modified Lyapunov equation, unlike the entropy/H_∞ (equalized weights) case, which involves one modified Riccati equation given by (5.7).

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Appendix C

Robust Stability and Performance Analysis for Linear Dynamic Systems

DENNIS S. BERNSTEIN AND WASSIM M. HADDAD

Abstract—In a recent paper Zhou and Khargonekar obtained sufficient conditions for robust stability over specified sets of matrix perturbations. In the present note these results are extended to include, in addition, performance bounds. Here performance is defined as the worst-case expected value of a quadratic functional involving the state variables when the system is subjected to white noise disturbances. The results are

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D. S. Bernstein is with Harris Corporation, Government Aerospace Systems Division, Melbourne, FL 32902.

W. M. Haddad is with the Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901.

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illustrated by considering the gain margin of both an LQG controller and a robustified design obtained by Bernstein and Greeley for Doyle's example.

I. Introduction

It is well known that unavoidable discrepancies between mathematical models and real-world systems can result in the degradation of control-system performance. Ideally, feedback control systems should be designed to be *robust* with respect to uncertainties in the plant characteristics. Thus, robustness *analysis* must play a key role in control-system design. That is, given an existing or proposed control system, determine the performance degradation due to variations in the plant. The most fundamental concern in this regard is clearly that of stability. For linear state-space systems with which the present note is concerned, this problem has received increasing attention over the past several years (see, e.g., [1]-[2]).

One of the principal techniques used to assess robust stability is based upon quadratic Lyapunov functions (see [1]-[4], [10]). Quadratic Lyapunov functions have also been used extensively for robust control-system *synthesis*; see [13] for relevant references. The problem of robust synthesis is, however, beyond the scope of the present note.

In addition to assessing robust stability, it is often desirable to quantify performance by considering the degradation of a cost functional as the plant parameters deviate from their nominal values. Although any robustly stable system over a compact set of parameters possesses a worst-case performance, it is desirable in practice to actually determine a bound for the worst-case performance. The concern for *both* robust stability and performance goes back to the early work of Michael and Merriam [14], while more recent references include the work of Chang and Peng [15], Noldus [16], and Petersen [17]. The results of [15]-[17] can be shown to depend upon a modified Lyapunov equation of the form

$$0=AQ+QA^T+\hat{\Omega}(Q)+V \tag{1.1}$$

where the operator $\hat{\Omega}(Q)$ is chosen to bound terms of the form $\Delta AQ + Q\Delta A^T$, where ΔA is an uncertain perturbation of the dynamics matrix A. Since robust performance per se was not discussed in [16], [17], the work most closely related to the present note is that of Chang and Peng [15]. They essentially show that consideration of (1.1) leads to a bound on worst-case performance. Although the development in [15] was carried out for full-state feedback, specialization of their approach to robust performance analysis is straightforward. A systematic, in-depth treatment of robust performance analysis involving the approach of [15] as well as other bounds is given in [18].

The starting point for the present note is the recent paper by Zhou and Khargonekar [10]. By analyzing the Lyapunov equation they obtain a series of stability robustness tests which improve significantly upon earlier work [2]-[4]. In the present note we extend the results of [10] to obtain, in addition, a bound on worst-case performance. As in (1.1) we consider a Lyapunov equation of the form

$$0=AQ+QA^T+\Omega+V \tag{1.2}$$

where Ω bounds uncertainty terms of the form $\Delta AQ + Q\Delta A^T$. The principal difference between (1.1) and (1.2) is that Ω in (1.2) is a constant matrix independent of the solution Q. The case considered in [15] in which Ω is a function of Q is discussed in [18].

The cost functional used in the present note to quantify robust performance is the trace of the output covariance of a system subjected to white noise disturbances. This measure of performance is identical in form to the standard performance criterion of LQG theory. Since we also obtain a bound for the state covariance matrix, our results yield bounds on the variances (mean square response levels) of system states. Although the results of [15] were obtained within a deterministic setting, it is easy to see that the performance criterion of [15] is also of this form.

The contents of the note are as follows. After introducing notation at the end of this section we consider the robust stability and performance problems in Section II. In Section III we present the main result (Theorem 3.1) which provides sufficient conditions for robust stability over a set of parameter variations along with a performance bound. In Section IV we present a dual result (Theorem 4.1) in terms of the dual matrix P. This result serves two purposes. First, it clarifies connections with the previous literature where results are presented in terms of the quadratic Lyapunov function $V(x) = x^TPx$. And, second, we show that the dual performance bound may be much better than the primal bound (and vice versa) for particular problems. The results of Theorems 3.1 and 4.1 are given in terms of a robustness set $\mathfrak{U}$ which is a subset of a maximal set $\hat{\mathfrak{U}}$. Since $\hat{\mathfrak{U}}$ is defined implicitly, we provide explicit characterizations of subsets $\mathfrak{U}$ in Section V. Here we restate the principal results of [2]-[4], [10] which, in our context, correspond to particular characterizations of subsets of $\hat{\mathfrak{U}}$. We also introduce an additional subset of $\hat{\mathfrak{U}}$ which provides a new robust stability result. Finally, in Section VI we consider a pair of illustrative examples. The first example, which was previously considered in [10], involves two uncertain parameters. It is shown that the new guaranteed stability region is considerably larger for certain parameter values than the regions given in [10] (see Fig. 1). Furthermore, we obtain a robust performance bound, a result which has no counterpart in [10]. The second example involves controllers for a second-order open-loop unstable plant originally considered in [19] to demonstrate the lack of a guaranteed stability margin for LQG controllers. We apply Theorems 3.1 and 4.1 to analyze both the LQG design and a robustified design obtained in [20]. We show that the new robust stability test is effective in the sense that the *guaranteed* gain margin for the robustified controller is a factor of 5 larger than the *actual* gain margin of the LQG design.

Notation

Note: All matrices have real entries

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation
I_r	$r \times r$ identity matrix
Asymptotically stable matrix	Matrix with eigenvalues in the open left-half plane
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \geq Z_2$, $Z_1 > Z_2$	$Z_1 - Z_2 \in \mathbb{N}^r$, $Z_1 - Z_2 \in \mathbb{P}^r$, Z_1, $Z_2 \in \mathbb{S}^r$
$\operatorname{tr} Z$, Z^T, co	Trace of Z, transpose of Z, convex hull
$\lambda_{\min}(Z)$, $\lambda_{\max}(Z)$	Smallest and largest eigenvalues of $Z \in \mathbb{S}^r$
$\\|Z\\|_s$	Spectral norm
$Z_{(i,j)}$	(i, j) element of matrix Z
$Z \geq\geq 0$	$Z_{(i,j)} \geq 0$, $i, j = 1, \cdots, r$, $Z \in \mathbb{R}^{r\times r}$
$Z \gg 0$	$Z_{(i,j)} > 0$, $i, j = 1, \cdots, r$, $Z \in \mathbb{R}^{r\times r}$
$\|Z\|_m$	$\{\|Z_{(i,j)}\|\}_{i,j=1}^r$, $Z \in \mathbb{R}^{r\times r}$ (matrix modulus).

II. Robust Stability and Performance Problems

Let $\mathfrak{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of the nominal dynamics matrix A. Throughout the note it is assumed that A is asymptotically stable. We begin by considering the question of whether or not $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathfrak{U}$.

Robust Stability Problem. Determine whether the linear system

$$\dot{x}(t)=(A+\Delta A)x(t), \qquad t \in [0, \infty) \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathfrak{U}$.

The problem of robust performance involves a quadratic form $x^T(t)Rx(t)$, where $R \in \mathbb{N}^n$, when the system is subjected to a white noise disturbance $w(t)$ with nonnegative-definite intensity V. The matrix R can be viewed as a means for selecting output variables of interest while the matrix V can be used to specify disturbance levels.

Robust Performance Problem: For the disturbed linear system

$$\dot{x}(t)=(A+\Delta A)x(t)+w(t), \qquad t \in [0, \infty) \tag{2.2}$$

determine a performance bound β satisfying

$$J(\mathfrak{U}) \triangleq \sup_{\Delta A\in\mathfrak{U}} \limsup_{t\to\infty} \mathbb{E}[x^T(t)Rx(t)] \leq \beta. \tag{2.3}$$

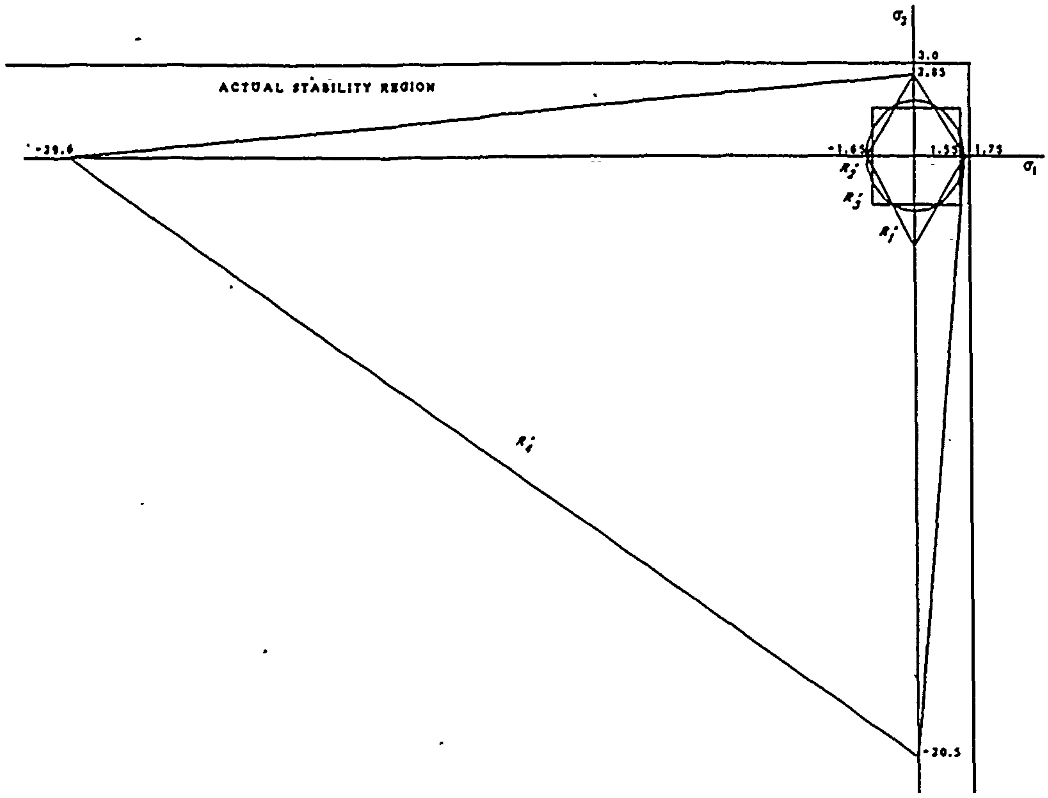

Fig. 1.

The system (2.2) may, for example, denote a control system in closed-loop configuration subjected to external white noise disturbances (see Section VI). Such specializations are not required for this development, however. Note that $J(\mathfrak{U})$ represents the worst case (over $\mathfrak{U}$) of the average (over the white noise statistics) of quadratically weighted steady-state deviations of the state from the origin. Thus, β represents an upper bound on selected output variances.

Of course, since R and V are only assumed to be nonnegative definite, there may be cases in which a finite performance bound β satisfying (2.3) exists while (2.1) is not asymptotically stable over $\mathfrak{U}$. In practice, however, robust performance is mainly of interest when (2.1) is robustly stable. In this case the performance $J(\mathfrak{U})$ is given in terms of the steady-state second moment of the state. The following result from linear system theory will be useful.

Lemma 2.1: Suppose (2.1) is asymptotically stable for all $\Delta A \in \mathfrak{U}$. Then

$$J(\mathfrak{U}) = \sup_{\Delta A \in \mathfrak{U}} \operatorname{tr} Q_{\Delta A} R \tag{2.4}$$

where $n \times n$ $Q_{\Delta A} \triangleq \lim_{t\to\infty} \mathfrak{E}[x(t)x^T(t)]$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V. \tag{2.5}$$

In the present note our approach is to obtain sufficient conditions for robust stability as a *consequence* of sufficient conditions for robust performance. Such conditions are developed in the following sections.

III. Sufficient Conditions for Robust Stability and Performance

The key step in obtaining robust stability and performance is to replace the uncertain terms in the Lyapunov equation (2.5) by a bounding matrix Ω. The nonnegative-definite solution Q of this bounding Lyapunov equation is then guaranteed to be an upper bound for $Q_{\Delta A}$. The uncertainty set $\mathfrak{U}$ over which robustness is guaranteed then depends upon Q. The following easily proved result is fundamental and forms the basis for all later developments. The hypotheses of this result are of a general nature and are not intended to be directly verifiable. Suitably verifiable specializations of the hypotheses are discussed in Section V.

Theorem 3.1: Let $\Omega \in \mathfrak{N}^n$, let $Q \in \mathfrak{N}^n$ be the unique solution to

$$0 = AQ + QA^T + \Omega + V \tag{3.1}$$

and let $\mathfrak{U}$ be a subset of

$$\hat{\mathfrak{U}} \triangleq \{\Delta A \in \mathbb{R}^{n\times n} : \Delta A Q + Q \Delta A^T \leq \Omega\}. \tag{3.2}$$

Then

$$(A + \Delta A, [V + \Omega - (\Delta A Q + Q\Delta A^T)]^{1/2}) \text{ is stabilizable}, \quad \Delta A \in \mathfrak{U} \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathfrak{U}. \tag{3.4}$$

In this case,

$$Q_{\Delta A} \leq Q, \quad \Delta A \in \mathfrak{U} \tag{3.5}$$

where $Q_{\Delta A} \in \mathfrak{N}^n$ is given by (2.5), and

$$J(\mathfrak{U}) \leq \operatorname{tr} QR. \tag{3.6}$$

If, in addition, there exists $\Delta A \in \hat{\mathfrak{U}}$ such that $(A + \Delta A, [V + \Omega - (\Delta A Q + Q\Delta A^T)]^{1/2})$ is controllable, then Q is positive definite.

Proof: This result is a minor variation of [21, Theorem 3.1] and hence the proof is omitted. □

To apply Theorem 3.1, one first chooses a nonnegative-definite matrix Ω and then solves (3.1) for Q. Next, as shown in Section IV, one examines $\hat{\mathfrak{U}}$ to determine subsets $\mathfrak{U}$ of perturbations ΔA over which robustness is guaranteed. Note that if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are subsets of $\hat{\mathfrak{U}}$, then so is the convex hull of their union. (To see this note that $\hat{\mathfrak{U}}$ is convex.) The set $\hat{\mathfrak{U}}$ is the largest set over which robustness can be guaranteed by Theorem 3.1 for the particular choice of Ω. One may also select several matrices Ω and determine subsets of each resulting $\hat{\mathfrak{U}}$ as a constructive approach to determining larger robustness sets. In the next section we

examine subsets $\mathfrak{U}$ of $\tilde{\mathfrak{U}}$ of specified structure. Before doing so, we have the following observations.

In applying Theorem 3.1 it may be convenient to replace condition (3.3) with stronger conditions which are easier to verify in practice. The following result is immediate.

Proposition 3.1: Consider the conditions

$$V>0 \tag{3.7}$$

$$(A+\Delta A,\ V^{1/2}) \text{ is stabilizable}, \qquad \Delta A \in \mathfrak{U}, \tag{3.8}$$

$$\Delta AQ+Q\Delta A^T<\Omega, \qquad \Delta A \in \mathfrak{U}, \tag{3.9}$$

$$\Delta AQ+Q\Delta A^T<\Omega+V, \qquad \Delta A \in \mathfrak{U}. \tag{3.10}$$

Then (3.7) ⇒ (3.8) ⇒ (3.3), (3.7) ⇒ (3.10) ⇒ (3.3), and (3.9) ⇒ (3.10) ⇒ (3.3).

If only robust stability is of interest, then the noise intensity V need not have physical significance. In this case one may either set $V = \epsilon I_n$, where $\epsilon > 0$ is small to satisfy (3.7), or set $V = 0$ and confine $\mathfrak{U}$ to perturbations ΔA for which (3.9) holds. This is the case in [3], [4], [10] where $V = 0$, $\Omega = 2I_n$, and the parametric robustness sets are characterized by strict inequality.

Remark 3.1: Since A is asymptotically stable, Q is given by

$$Q=\int_0^\infty e^{At}(\Omega+V)e^{A^Tt}\,dt=\int_0^\infty e^{At}\Omega e^{A^Tt}\,dt+Q_0 \tag{3.11}$$

where $Q_0 \in \mathfrak{N}^n$ is given by

$$0=AQ_0+Q_0A^T+V. \tag{3.12}$$

Note that $Q_0 \leq Q$ and that the nominal performance is given by $\operatorname{tr} Q_0R$.

Remark 3.2: Using (3.11) it is also useful to note that the bound for $J(\mathfrak{U})$ given by (3.6) can be written as

$$\operatorname{tr} QR=\operatorname{tr}\int_0^\infty e^{At}(\Omega+V)e^{A^Tt}\,dtR=\operatorname{tr} P_0(\Omega+V) \tag{3.13}$$

where $P_0 \in \mathfrak{N}^n$ is given by

$$0=A^TP_0+P_0A+R. \tag{3.14}$$

The bound $\operatorname{tr} P_0(\Omega + V)$ can be viewed as a dual formulation of the bound $\operatorname{tr} QR$ since the roles of A and A^T are reversed. Dual bounds are developed in the following section. Note that $\operatorname{tr} Q_0R = \operatorname{tr} P_0V$.

IV. Dual Sufficient Conditions for Robust Stability and Performance

As noted in Remark 3.2, the performance bound $\operatorname{tr} QR$ given by (3.6) can be expressed equivalently in terms of a dual variable P_0 for which the roles of A and A^T are reversed. Using a similar technique, additional conditions for robust stability and performance can be obtained by developing a dual version of Theorem 3.1. A prime motivation for developing such dual bounds is to draw direct connections with previous results in the literature relating to robust stability. Traditionally, the use of the quadratic Lyapunov function $V(x) = x^TPx$ for robust stability leads naturally to the dual formulation. In addition, the dual bounds may, for certain problems, be much sharper than the bounds introduced in the previous section. This point is illustrated at the end of this section by examining an extreme case and in Section VI by means of numerical examples. We note, in addition, that robust performance bounds are more difficult to motivate within the dual formulation without first developing the primal results. The following result is immediate.

Lemma 4.1: Suppose (2.1) is asymptotically stable for all $\Delta A \in \mathfrak{U}$. Then

$$J(\mathfrak{U})=\sup_{\Delta A\in\mathfrak{U}} \operatorname{tr} P_{\Delta A}V \tag{4.1}$$

where $n \times n$ $P_{\Delta A}$ is the unique, nonnegative-definite solution to

$$0=(A+\Delta A)^TP_{\Delta A}+P_{\Delta A}(A+\Delta A)+R. \tag{4.2}$$

The dual of Theorem 3.1 can now be stated.

Theorem 4.1: Let $\Lambda \in \mathfrak{N}^n$, let $P \in \mathfrak{N}^n$ be the unique solution to

$$0=A^TP+PA+\Lambda+R \tag{4.3}$$

and let $\mathfrak{U}$ be a subset of

$$\tilde{\mathfrak{U}}' \triangleq \{\Delta A \in \mathbb{R}^{n\times n}: \Delta A^TP+P\Delta A\leq\Lambda\}. \tag{4.4}$$

Then

$$([R+\Lambda-(\Delta A^TP+P\Delta A)]^{1/2},\ A+\Delta A) \text{ is detectable}, \qquad \Delta A \in \mathfrak{U}, \tag{4.5}$$

if and only if

$$A+\Delta A \text{ is asymptotically stable}, \qquad \Delta A \in \mathfrak{U}. \tag{4.6}$$

In this case,

$$P_{\Delta A}\leq P, \qquad \Delta A \in \mathfrak{U} \tag{4.7}$$

where $P_{\Delta A} \in \mathfrak{N}^n$ is given by (4.2), and

$$J(\mathfrak{U})\leq \operatorname{tr} PV. \tag{4.8}$$

If, in addition, there exists $\Delta A \in \tilde{\mathfrak{U}}'$ such that $([R + \Lambda - (\Delta A^TP + P\Delta A)]^{1/2}, A + \Delta A)$ is observable, then P is positive definite.

The usefulness of Theorem 4.1 resides in the fact that it provides stability and performance bounds which are generally different from those given by Theorem 3.1. Hence, depending upon Ω and Λ either bound (3.6) or bound (4.8) may be better for a particular problem. To illustrate how dual bounds can improve estimates of robust performance, consider the case in which $V = 0$, i.e., plant disturbances are absent. In this case $Q_{\Delta A} = 0$ satisfies (2.5) and thus $J(\mathfrak{U}) = 0$ as long as $A + \Delta A$ is stable for all $\Delta A \in \mathfrak{U}$. The performance bound $\operatorname{tr} QR$ given by (3.6) may, however, be arbitrarily large depending upon R since Q may be nonzero due to Ω. Hence, this performance bound may be arbitrarily conservative. The dual bound (4.8), on the other hand, is zero in this case, which completely eliminates the conservatism.

V. Characterization of Subsets of $\tilde{\mathfrak{U}}$ and $\tilde{\mathfrak{U}}'$

To apply Theorems 3.1 and 4.1 it is necessary to explicitly characterize subsets $\mathfrak{U}$ of $\tilde{\mathfrak{U}}$ and $\tilde{\mathfrak{U}}'$ over which robustness is guaranteed. In this section we provide several such characterizations by collecting together and extending known results from the literature.

For the following result let $\Omega = \omega I_n$, where $\omega > 0$, let $W \in \mathbb{R}^{n\times n}$, $W > 0$, and let $A_1, \cdots, A_p \in \mathbb{R}^{n\times n}$ be arbitrary. Furthermore, for $Q \in \mathfrak{P}^n$ satisfying (3.1) define for $i = 1, \cdots, p$:

$$\alpha_i \triangleq \lambda_{\min}(A_iQ+QA_i^T), \qquad \beta_i \triangleq \lambda_{\max}(A_iQ+QA_i^T),$$

$$\begin{aligned}
\mathcal{I}_i &\triangleq (-\infty, \infty) && \alpha_i=\beta_i=0,\\
&\triangleq (-\infty, \omega/\beta_i), && \alpha_i\geq 0,\ \beta_i>0,\\
&\triangleq (\omega/\alpha_i, \infty), && \alpha_i<0,\ \beta_i\leq 0,\\
&\triangleq (\omega/\alpha_i, \omega/\beta_i), && \alpha_i<0<\beta_i.
\end{aligned}$$

Finally, let $e_i^{(p)}$ denote the ith column of the $p \times p$ identity matrix.

Proposition 5.1: Let $Q \in \mathfrak{P}^n$ satisfy (3.1) with $\Omega = \omega I_n$, where $\omega > 0$. Then the following sets are subsets of $\tilde{\mathfrak{U}}$ which also satisfy (3.9):

$$\mathfrak{U}_1 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n}: \|\Delta A\|_s<\frac{\omega}{2}\|Q\|_s^{-1}\right\},$$

$$\mathfrak{U}_2 \triangleq \{\Delta A \in \mathbb{R}^{n\times n}: |\Delta A|_m \ll \omega\| W|Q|_m+|Q|_mW^T\|_s^{-1} W\},$$

$$\mathfrak{U}_3 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n}: \Delta A=\sum_{i=1}^{p} \sigma_iA_i,\ (\sigma_1, \cdots, \sigma_p)^T \in \mathfrak{R}\right\}$$

where $\mathcal{R}$ is one of the following regions in $\mathbb{R}^p$:

$$\mathcal{R}_1 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : \sum_{i=1}^{p} |\sigma_i| \, \|A_iQ+QA_i^T\|_s<\omega\right\},$$

$$\mathcal{R}_2 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : \sum_{i=1}^{p} \sigma_i^2<\omega^2 \left\|\sum_{i=1}^{p} (A_iQ+QA_i^T)^2\right\|_s^{-1}\right\},$$

$$\mathcal{R}_3 \triangleq \left\{(\sigma_1, \cdots, \sigma_p) : |\sigma_i|<\omega \left\|\sum_{i=1}^{p} |A_iQ+QA_i^T|_m\right\|_s^{-1}, \quad i=1, \cdots, p\right\},$$

$$\mathcal{R}_4 \triangleq \mathrm{co}\,\{\sigma_i e_i^{(p)} : \sigma_i \in \mathcal{I}_i, \; i=1, \cdots, p\}.$$

For the dual case we set $\Lambda = \lambda I_n$, where $\lambda > 0$, and define the dual sets $\bar{\mathcal{U}}_1'$, $\bar{\mathcal{U}}_2'$, $\bar{\mathcal{U}}_3'$, $\mathcal{I}_i'$, $\mathcal{R}_1'$, $\mathcal{R}_2'$, $\mathcal{R}_3'$, and $\mathcal{R}_4'$ in an analogous fashion.

Remark 5.1: The proof of Proposition 5.1 is omitted since the results are either known or are immediate. Specifically, $\mathcal{U}_1'$ can be found in [2] while $\mathcal{U}_2'$ appears in [3], [4]. The sets $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$ are given in [10]. The set $\mathcal{R}_4'$ has not appeared previously in the literature although the result is immediate. It is only necessary to diagonalize $A_i^TP + PA_i$ by means of an orthogonal transformation and compare diagonal elements to obtain $\mathcal{I}_i'$. Taking the convex hull over the intervals $\mathcal{I}_i'$ thus yields $\mathcal{R}_4'$. Of course, the required eigenproblem entails additional computation.

Remark 5.2: Although most of the dual of Proposition 5.1 has appeared previously, the primal result Proposition 5.1 has not been discussed in the literature. For robust stability this result can be obtained by considering the stability of A^T in place of A. As will be shown in Section VI, the primal and dual results lead in general to different robust stability regions and performance bounds. It should also be stressed that although most of the dual of Proposition 5.1 has appeared previously, the present note extends its applicability to the problem of robust performance in addition to robust stability.

Remark 5.3: As mentioned previously, the convex hull of the union of any collection of subsets of $\bar{\mathcal{U}}$ is also a subset of $\bar{\mathcal{U}}$ since $\bar{\mathcal{U}}$ is convex. This observation applies to $\mathcal{U}_3$ in the sense that if $\mathcal{U}_3$ is a subset of $\bar{\mathcal{U}}$ with regions $\mathcal{R} = \hat{\mathcal{R}}$ and $\mathcal{R} = \tilde{\mathcal{R}}$ separately, then $\mathcal{U}_3$ is also a subset with $\mathcal{R}$ equal to the convex hull of the union of $\hat{\mathcal{R}}$ and $\tilde{\mathcal{R}}$. Note that these observations follow from the convexity of $\bar{\mathcal{U}}$ and do not contradict the fact that the set of asymptotically stable matrices is not convex.

Remark 5.4: The requirement that Ω be of the form ωI_n is not a constraint in applying Proposition 5.1. Indeed, it is only required that Ω be positive definite. To see this let invertible $\phi \in \mathbb{R}^{n\times n}$ be such that $\phi\Omega\phi^T = I_n$. Then Proposition 5.1 can be applied with suitable transformations of ΔA, Q, W, and A_i.

Remark 5.5: As in [2]–[4], [10], the sets $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are defined in terms of strict inequalities. In this case $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$ consist of elements of $\bar{\mathcal{U}}$ satisfying $\Delta AQ + Q\Delta A^T < \Omega$ so that (3.9) is satisfied. Thus, by Proposition 3.1, the stabilizability condition (3.3) is automatically satisfied without reference to V.

Remark 5.6: In the special case $p = 1$ it is clear that $\mathcal{R}_1 = \mathcal{R}_2$. Furthermore, in this case $\mathcal{R}_3$ is always a subset of $\mathcal{R}_1$ and $\mathcal{R}_2$, and hence leads to a more conservative stability region. The largest possible set of perturbations ΔA of the form σ_iA_i contained in $\bar{\mathcal{U}}$ is given by $\mathcal{R}_4$.

Remark 5.7: It is shown in [10, Remark 2.12] that $\mathcal{U}_2$ can be obtained as a consequence of $\mathcal{U}_3$ with $\mathcal{R} = \mathcal{R}_3$ and a suitable choice of A_i. Hence, $\mathcal{U}_2$ need not actually be considered separately. Our assumption that $W > 0$ (and not $W \geq\geq 0$) is for convenience only.

Remark 5.8: Note that all of the subsets of $\bar{\mathcal{U}}$ given by Proposition 5.1 are symmetric except for $\mathcal{U}_3$ with $\mathcal{R} = \mathcal{R}_4$. When the actual stability region is highly asymmetric, it follows that a symmetric robust stability region is necessarily highly conservative. This observation is illustrated by an example in Section VI.

Remark 5.9: The regions given by $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ correspond, respectively, to 1-norm 2-norm, and ∞-norm neighborhoods. These results can easily be extended to include more general regions. For example, in the definition of $\mathcal{R}_2$ replace σ_i by σ_i/a_i and $A_iQ + QA_i^T$ by $a_i(A_iQ + QA_i^T)$, where a_i is an arbitrary positive constant, $i = 1, \cdots, p$. With this modification $\mathcal{R}_2$ corresponds to an elliptical robust stability region. Detailed investigation of such regions is beyond the scope of this note.

Remark 5.10: When each interval $\mathcal{I}_i$ is finite, or when only a finite interval is of interest, $\mathcal{R}_4$ can be expressed as the convex hull of a finite number of points. Specifically, letting $\mathcal{I}_i = [a_i, b_i]$, $i = 1, \cdots, p$, it follows that

$$\mathcal{R}_4=\mathrm{co}\,\{a_1e_1^{(p)}, b_1e_1^{(p)}, \cdots, a_pe_p^{(p)}, b_pe_p^{(p)}\}.$$

This set is illustrated by means of an example in the next section.

VI. Examples

As a first example we adopt Example 2 of [10]. This example, which involves two uncertain parameters, was used in [10] to illustrate the robust stability regions $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$. The problem was originally cast in the form of a static output feedback controller with uncertain gains. Here for convenience in discussing robust performance we reformulate the example to involve uncertainty in the control input matrix. Hence, consider the control system

$$\dot{x}(t)=A_0x(t)+B_0u(t), \tag{6.1}$$

$$y(t)=C_0x(t), \tag{6.2}$$

$$u(t)=Ky(t) \tag{6.3}$$

where

$$A_0=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B_0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_0 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$K=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the uncertainty ΔB_0 in B_0 is given by

$$\Delta B_0=\begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_2 \\ -\sigma_1 & -\sigma_2 \end{bmatrix}.$$

The closed-loop dynamics matrix is then given by

$$A+\Delta A=\begin{bmatrix} -2+\sigma_1 & 0 & -1+\sigma_1 \\ 0 & -3+\sigma_2 & 0 \\ -1+\sigma_1 & -1+\sigma_2 & -4+\sigma_1 \end{bmatrix}$$

where $\Delta A = \sigma_1A_1 + \sigma_2A_2$ and A_1, A_2 have the evident definitions. It can easily be shown that the exact stability region is given by $\sigma_1 \in (-\infty, 1.75)$ and $\sigma_2 \in (-\infty, 3)$. Thus, the nominal dynamics matrix corresponding to $\sigma_1 = \sigma_2 = 0$ lies in the upper right-hand corner of the exact stability region so that, as noted in Remark 5.8, a high degree of conservatism can be expected using symmetric robustness regions. To consider robust stability alone, set $V = R = 0$ and $\omega = \lambda = 2$. In this case regions $\mathcal{R}_1'$, $\mathcal{R}_2'$, and $\mathcal{R}_3'$, as computed in [10], are shown in Fig. 1. Region $\mathcal{R}_4'$ for this problem is given (see Remark 5.10) by

$$\mathcal{R}_4'=\mathrm{co}\left\{\begin{pmatrix}-29.6\\0\end{pmatrix}, \begin{pmatrix}1.65\\0\end{pmatrix}, \begin{pmatrix}0\\-20.5\end{pmatrix}, \begin{pmatrix}0\\2.85\end{pmatrix}\right\}$$

which accounts somewhat better for the asymmetry of the stability region. The regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ were found to be smaller than the corresponding dual regions, while $\mathcal{R}_4$ is given by

$$\mathcal{R}_4=\mathrm{co}\left\{\begin{pmatrix}-31.1\\0\end{pmatrix}, \begin{pmatrix}1.64\\0\end{pmatrix}, \begin{pmatrix}0\\-10.4\end{pmatrix}, \begin{pmatrix}0\\2.63\end{pmatrix}\right\}$$

which yields slight improvement in σ_1.

To evaluate robust performance replace (6.1) by

$$\dot{x}(t)=A_0x(t)+B_0u(t)+w(t) \tag{6.4}$$

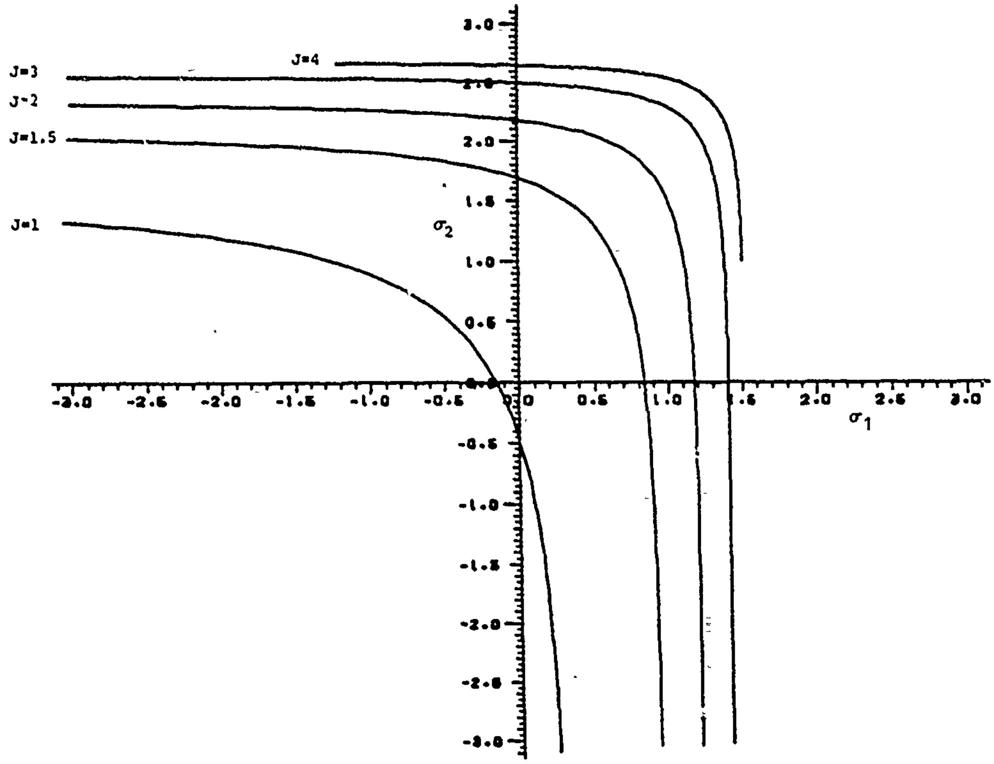

Fig. 2.

and define

$$J=\lim_{t\to\infty} \mathcal{E}[x^T(t)R_1x(t)+u^T(t)R_2u(t)]$$

which corresponds to (2.3) with $R = R_1 + C_0^TK^TR_2KC_0$. Hence, setting $R_1 = I_3$ and $R_2 = I_2$ yields

$$R=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

We also set $V = I_3$ and $\omega = \lambda = 2$. The resulting stability region for these values of V and R is given by

$$\mathcal{R}_1' = \{(\sigma_1, \sigma_2) : |\sigma_1|/0.70+|\sigma_2|/1.46<1\},$$

$$\mathcal{R}_2' = \{(\sigma_1, \sigma_2) : \sigma_1^2+\sigma_2^2<(0.70)^2\},$$

$$\mathcal{R}_3' = \{(\sigma_1, \sigma_2) : |\sigma_i|<0.68,\ i=1, 2\},$$

$$\mathcal{R}_4' = \mathrm{co}\left\{\begin{pmatrix} -20.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.70 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -13.7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.46 \end{pmatrix}\right\}.$$

Over these combined regions the performance bound was computed to be tr $PV = 2.26$. The primal result produced the regions

$$\mathcal{R}_1 = \{(\sigma_1, \sigma_2) : |\sigma_1|/1.09+|\sigma_2|/1.75<1\},$$

$$\mathcal{R}_2 = \{(\sigma_1, \sigma_2) : \sigma_1^2+\sigma_2^2<(1.08)^2\},$$

$$\mathcal{R}_3 = \{(\sigma_1, \sigma_2) : |\sigma_i|<1.0,\ i=1, 2\},$$

$$\mathcal{R}_4 = \mathrm{co}\left\{\begin{pmatrix} -20.8 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.09 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -6.93 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.75 \end{pmatrix}\right\}.$$

Over these regions the performance bound was computed to be tr $QR = 3.18$. Contour plots of actual performance for perturbed values of σ_1 and σ_2 are shown in Fig. 2. Note that when determining robust performance Theorems 3.1 and 4.1 yield performance bounds over robust stability regions which are generally smaller than the robust stability regions determined with $R = 0$ and $V = 0$. This mechanism represents the natural tradeoff between stability and performance. In general, to determine the largest stability regions, V and R should be set to zero initially.

As a second example we consider the control system given in [19] to demonstrate the lack of a guaranteed gain margin for LQG controllers. Hence, consider

$$\dot{x}_0(t)=A_0x_0(t)+B_0u(t)+w_1(t), \tag{6.5}$$

$$y(t)=C_0x_0(t)+w_2(t) \tag{6.6}$$

with controller

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t), \tag{6.7}$$

$$u(t)=C_cx_c(t) \tag{6.8}$$

and performance

$$J=\lim_{t\to\infty} \mathcal{E}[x_0^T(t)R_1x_0(t)+u^T(t)R_2u(t)]. \tag{6.9}$$

The data are

$$A_0=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B_0=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0=[1\ \ 0],$$

$$V_1=R_1=\rho\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ V_2=R_2=1$$

where V_1 and V_2 are the intensities of $w_1(t)$ and $w_2(t)$, respectively. Uncertainty ΔB_0 in B_0 is thus represented by σ_1B_1, where $B_1 = [0,\ 1]^T$. Thus, the closed-loop system corresponds to

$$A=\begin{bmatrix} A_0 & B_0C_c \\ B_cC_0 & A_c \end{bmatrix},\ A_1=\begin{bmatrix} 0 & B_1C_c \\ 0 & 0 \end{bmatrix},$$

$$R=\begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix},\ V=\begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^T \end{bmatrix}$$

where the zero in the (2, 2) block of R denotes the fact that we are considering the robust performance bound for the state regulation cost

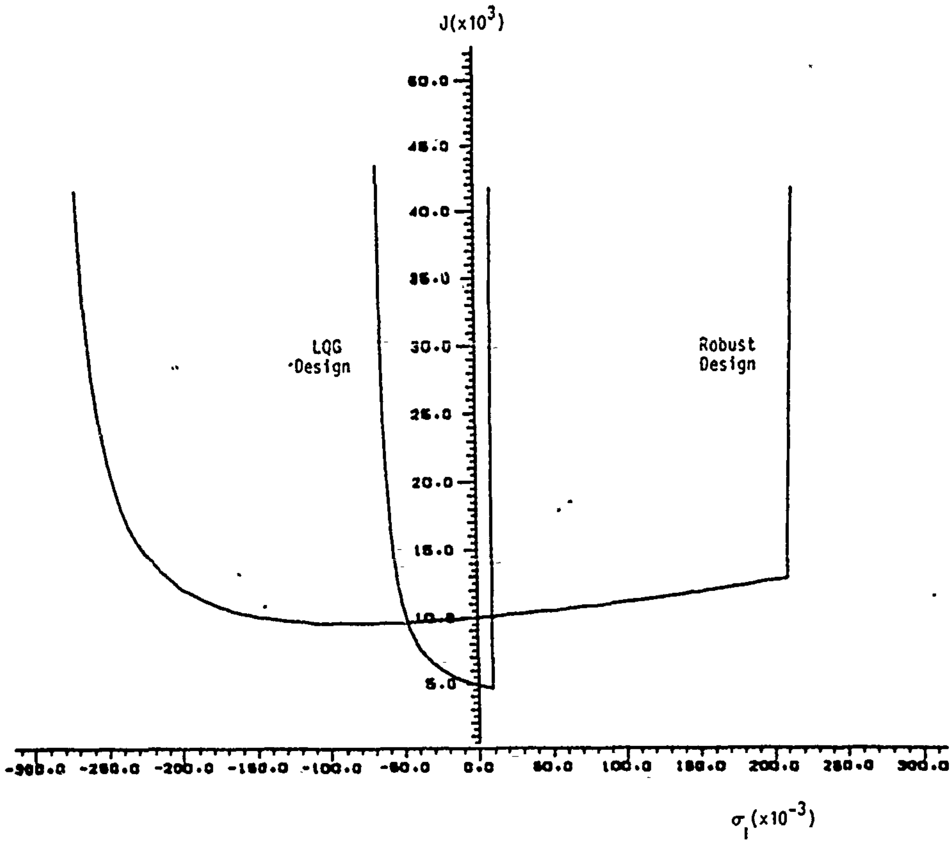

Fig. 3.

only. Choosing $\rho = 60$, it follows that the LQG gains are given by

$$A_c=\begin{bmatrix} -9 & 1 \\ -20 & -9 \end{bmatrix}, \; B_c=\begin{bmatrix} 10 \\ 10 \end{bmatrix}, \; C_c=[-10 \;\; -10].$$

For this controller the actual stability region corresponds to $\sigma_1 \in (-0.07, 0.01)$ (see Fig. 3). Applying the results of Section V with $V = R = 0$ (for robust stability only) and $\omega = \lambda = 2$, we obtain

$$\mathcal{R}_1=\mathcal{R}_2=\mathcal{R}_3=(-0.000242, 0.000242), \; \mathcal{R}_4=(-0.000242, 0.000728),$$

$$\mathcal{R}_1'=\mathcal{R}_2'=(-0.0000247, 0.0000247), \; \mathcal{R}_3'=(-0.0000219, 0.0000219),$$

$$\mathcal{R}_4'=(-0.0000247, 0.0000265).$$

Note that although the primal results are better than the dual results by an order of magnitude, they are conservative by two orders of magnitude with respect to the actual gain margin. For robust performance we again set $\omega = \lambda = 2$ and, using R and V given above, we obtained the bound tr $QR = 7633$ over the stability region $\mathcal{R}_4 = (-0.000192, 0.000613)$. The nominal performance was given by tr Q_0R = tr $P_0V = 4875$, while the dual performance bound was tr $PV = 10510$ over $\mathcal{R}_4' = (-0.0000222, 0.0000238)$.

Robustified controllers for the example of [19] were obtained in [20] using the approach discussed in [13]. As shown in Fig. 3 (see also [20]), the closed-loop system with the controller

$$A_c=\begin{bmatrix} -10.69 & 1 \\ -32.97 & -5.295 \end{bmatrix}, \; B_c=\begin{bmatrix} 11.69 \\ 26.67 \end{bmatrix}, \; C_c=[-6.245 \;\; -6.245]$$

is stable over the range $\sigma_1 \in (-0.28, 0.21)$. Hence, we wish to determine whether the robust stability tests are capable of detecting this increase in gain margin. Applying Theorems 3.1 and 4.1 with $\omega = \lambda = 2$ and $V = R = 0$ yields stability for σ_1 in the regions $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = (-0.0115, 0.0115)$ and $\mathcal{R}_4 = (-0.0115, 0.057)$. This guarantee of stability is two orders of magnitude greater than the guarantee for the LQG design but is still an order of magnitude conservative with respect to the actual stability region for this controller. Note, however, that for $\sigma_1 > 0$ the guaranteed gain margin for the robustified design given by $\mathcal{R}_4$ (i.e., 0.057) is greater than the *actual* gain margin of the LQG design (0.01). Hence, the robustness test given by the $\mathcal{R}_4$ was able to detect a factor of 5 stability augmentation provided by the robustified design compared to the LQG controller. Finally, the robust performance bound for this controller was computed to be tr $QR = 11185$ over the region $\mathcal{R}_4 = (-0.00165, 0.00493)$, while the dual bound was found to be tr $PV = 11223$ over $\mathcal{R}_4' = (-0.000724, 0.00123)$. For this problem the nominal performance is tr Q_0R = tr $P_0V = 9997$.

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007

ROBUST STABILITY AND PERFORMANCE ANALYSIS FOR STATE-SPACE SYSTEMS VIA QUADRATIC LYAPUNOV BOUNDS*

DENNIS S. BERNSTEIN† AND WASSIM M. HADDAD‡

Abstract. For a given asymptotically stable linear dynamic system it is often of interest to determine whether stability is preserved as the system varies within a specified class of uncertainties. If, in addition, there also exist associated performance measures (such as the steady-state variances of selected state variables), it is desirable to assess the worst-case performance over a class of plant variations. These are problems of robust stability and performance analysis. In the present paper, quadratic Lyapunov bounds used to obtain a simultaneous treatment of both robust stability and performance are considered. The approach is based on the construction of modified Lyapunov equations, which provide sufficient conditions for robust stability along with robust performance bounds. In this paper, a wide variety of quadratic Lyapunov bounds are systematically developed and a unified treatment of several bounds developed previously for feedback control design is provided.

Key words. robust analysis, stability, performance, Lyapunov equations, structured uncertainty

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1. Introduction. Unavoidable discrepancies between mathematical models and real-world systems can result in degradation of control-system performance including instability [1], [2]. Ideally, feedback control systems should be designed to be *robust* with respect to uncertainties, or perturbations, in the plant characteristics. Such uncertainties may arise either due to limitations in performing system identification prior to control-system implementation or because of unpredictable plant changes that occur during operation. Thus robustness *analysis* must play a key role in control-system design. That is, given an existing or proposed control system, determine the performance degradation due to variations in the plant.

In performing robustness analysis there are two principal concerns, namely, stability robustness and performance robustness. Stability robustness addresses the qualitative question as to whether or not the system remains stable for all plant perturbations within a specified class of uncertainties. A related problem involves determining the largest class of plant perturbations under which stability is preserved. Once robust stability has been ascertained, it is of interest to investigate *quantitatively* the performance degradation within a given robust stability range. In practice it is often desirable to determine the *worst-case* performance as a measure of degradation.

The concern for both robust stability and performance can be traced back to the earliest developments in control theory. Design specifications such as gain and phase margin have traditionally been used to gauge system reliability in the face of uncertainty. In the modern control literature considerable effort has focused on rigorous robustness analysis and design techniques in a variety of settings. Analysis and synthesis results have been developed for both state-space and frequency-domain plant models to address structured parameter variations as well as normed-neighborhood uncertainty [3]–[7].

The present paper is concerned solely with the analysis of structured real-valued parameter uncertainty within the context of state-space models. One motivation for such

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† Harris Corporation, Government Aerospace Systems Division, MS 22/4842, Melbourne, Florida 32902.

‡ Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, Florida 32901.

problems is illustrated by the examples given in [1] and [2]. These examples show that standard linear-quadratic methods used to design either full-state feedback controllers or dynamic compensators may result in closed-loop systems that are arbitrarily sensitive to structured real-valued plant parameter variations. A particularly effective technique for analyzing robust stability is to construct a quadratic Lyapunov function $V(x) = x^TPx$, which guarantees stability of the system as the uncertain parameters vary over a specified range. This technique has been extensively developed for both analysis and synthesis (see, e.g., [8]–[37]).

Although both robust stability and performance are of interest in practice, most of the literature involving quadratic Lyapunov functions is confined to the problem of robust stability. A notable exception is the early work of Chang and Peng [9], which also provides bounds on worst-case quadratic performance within the context of full-state-feedback control design. In the present paper, we further extend the approach of [9] to obtain a series of results for analyzing both robust stability and performance. As will be seen, these results also provide substantial unification of more recent results pertaining to robust stability alone.

To illustrate the basis for our approach, consider the system

$$\dot{x}(t) = (A+\Delta A)x(t) + Dw(t), \qquad t \in [0,\infty), \quad x(0)=0, \tag{1.1}$$

$$y(t) = Ex(t), \tag{1.2}$$

where $x(t)$ is an n-vector, A is an $n \times n$ matrix denoting the nominal dynamics matrix, ΔA denotes an uncertain perturbation of A belonging to a specified set $\mathcal{U}$, $Dw(t)$ is (for now) a white noise signal of intensity $V \triangleq DD^T$, and $y(t)$ is a q-vector of outputs. System (1.1), (1.2) may, for example, denote a control system in closed-loop configuration.

For the system (1.1) the performance measure involves the steady-state second moment of the outputs $y(t)$. In practice the diagonal elements of the second moment are measures of the ability of the external disturbances $Dw(t)$ to excite specified states. In the presence of uncertainties ΔA, it is of interest to determine the *worst-case* steady-state values of the second moments of selected states. Thus, we define the scalar performance criterion

$$J_S(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \mathbb{E}\{y^T(t)y(t)\}, \tag{1.3}$$

where $\mathbb{E}$ denotes expectation and lim sup is a technicality to ensure that $J_S(\mathcal{U})$ is a well-defined quantity even when $A + \Delta A$ has eigenvalues in the closed right half plane. To evaluate (1.3) define the second-moment matrix

$$Q(t) \triangleq \mathbb{E}[x(t)x^T(t)],$$

which satisfies the Lyapunov differential equation

$$\dot{Q}_{\Delta A}(t) = (A+\Delta A)Q_{\Delta A}(t) + Q_{\Delta A}(t)(A+\Delta A)^T + V, \tag{1.4}$$

so that (1.3) becomes

$$J_S(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \operatorname{tr} Q_{\Delta A}(t)R, \tag{1.5}$$

where $R \triangleq E^TE$. To guarantee both robust stability and performance we consider modified algebraic Lyapunov equations of the form

$$0 = AQ + QA^T + \Omega(Q) + V, \tag{1.6}$$

where $\Omega(\cdot)$ is a matrix operator satisfying

$$\Delta A Q + Q \Delta A^T \leqq \Omega(Q) \tag{1.7}$$

for all $\Delta A \in \mathcal{U}$ and all nonnegative-definite matrices Q. The ordering in (1.7) is defined with respect to the cone of nonnegative-definite matrices. Our results are based on the following robust stability and performance result (for convenience, assume that V is positive definite). If there exists a positive-definite solution Q to (1.6), where $\Omega(\cdot)$ satisfies (1.7), then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$ and, furthermore,

$$J_S(\mathcal{U}) \leqq \operatorname{tr} QR. \tag{1.8}$$

The robust stability result is a direct consequence of Lyapunov theory, while the performance bound (1.8) follows from the fact that since $A + \Delta A$ is asymptotically stable, $Q_{\Delta A} \triangleq \lim_{t \to \infty} Q_{\Delta A}(t)$ exists, is independent of $Q_{\Delta A}(0)$, and satisfies

$$0 = (A + \Delta A) Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V. \tag{1.9}$$

Now subtracting (1.9) from (1.6) yields

$$0 = (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^T + \Omega(Q) - (\Delta A Q + Q \Delta A^T) + V,$$

which, by (1.7) and the fact that $A + \Delta A$ is stable, implies

$$Q_{\Delta A} \leqq Q. \tag{1.10}$$

Now (1.5) and (1.10) yield the bound (1.8).

Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, it should not be expected that there exists an operator $\Omega(\cdot)$ satisfying (1.7), which is a least upper bound. Indeed, there are many alternative definitions for the bound $\Omega(\cdot)$. To illustrate some of these alternatives, assume for convenience that ΔA is of the form

$$\Delta A = \sigma_1 A_1, \qquad |\sigma_1| \leqq \delta_1, \tag{1.11}$$

where σ_1 is an uncertain real scalar parameter assumed only to satisfy the stated bounds, and A_1 is a known matrix denoting the structure of the parametric uncertainty. The bound $\Omega(\cdot)$ utilized in [9] and [12] for full-state-feedback design was chosen to be

$$\Omega(Q) = \delta_1 | A_1 Q + Q A_1^T |, \tag{1.12}$$

where $|\cdot|$ denotes the nonnegative-definite matrix obtained by replacing each eigenvalue by its absolute value. More recently, the quadratic (in Q) bound

$$\Omega(Q) = \delta_1 [A_L A_L^T + Q A_R^T A_R Q] \tag{1.13}$$

has been considered, where A_L, A_R are a factorization of A_1 of the form $A_1 = A_L A_R$. Bound (1.13) was studied in [29] for robustness analysis and in [17], [25], [28], [30], [33], and [36] for robust controller synthesis. A third bound that has also been considered is the linear (in Q) bound

$$\Omega(Q) = \delta_1 [\alpha Q + \alpha^{-1} A_1 Q A_1^T], \tag{1.14}$$

where α is an arbitrary positive scalar. As shown in [33], bound (1.14) arises from a multiplicative white noise model with exponential disturbance weighting. Control-design applications of bound (1.14) are given in [23], [27], [33]–[35]. The principal contribution of the present paper is thus a unified development of bounds (1.12)–(1.14) for both robust stability and performance analysis. In addition, we present a systematic

approach that pays careful attention to the structure of the uncertainty set $\mathcal{U}$. For example, we show that bound (1.12) guarantees stability over a rectangular uncertainty set while (1.14) is most naturally associated with an ellipsoidal region. Furthermore, to provide a methodical development, we identify three classes of bounds (Types I, II, and III) that operate by exploiting, respectively, the symmetry of $\Delta AQ + Q\Delta A^T$, the structure of Q, and the structure of ΔA. This approach clarifies the relationships among different bounds and suggests several new bounds. The principal goal in this regard is to demonstrate the richness of quadratic Lyapunov bounds to stimulate future developments.

Finally, the present paper also considers an alternative cost functional for robust performance analysis. Specifically, in place of white noise disturbances, we reinterpret $w(t)$ in (1.1) as a deterministic L_2 signal as in H_∞ theory [6]. By imposing an L_∞ norm on the output $y(t)$ (rather than an L_2 norm as in H_∞ theory), the corresponding performance measure is given by (see [38])

$$J_D(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \limsup_{t \to \infty} \lambda_{\max}(Q_{\Delta A}(t)R), \tag{1.15}$$

in contrast to (1.5). Both performance measures $J_S(\mathcal{U})$ and $J_D(\mathcal{U})$ are considered in the paper.

The contents of the paper are as follows. After summarizing notation later in this section, the Robust Stability Problem, Stochastic Robust Performance Problem, and Deterministic Robust Performance Problem are introduced in § 2. In § 3 the basic result guaranteeing robust stability and performance (Theorem 3.1) is stated. This result is easily stated and forms the basis for all later developments. A dual version of Theorem 3.1 (Theorem 4.1) provides additional sufficient conditions and clarifies connections to traditional robust stability results. The bound $\Omega(\cdot)$ and its dual $\Lambda(\cdot)$ are given concrete forms in § 5. In § 6, the bounds of § 5 are merged with Theorem 3.1 to yield the main results guaranteeing robust stability and performance (Theorems 6.1–6.5) via modified Lyapunov equations. In § 7 we analyze the modified Lyapunov equations with regard to existence, uniqueness, and monotonicity of solutions. Additional bounds are derived in § 8 by utilizing a recursive substitution technique, while both upper and lower bounds are obtained in § 9. Finally, illustrative examples are considered in §§ 10 and 11.

Notation. Note: All matrices have real entries.

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation,
I_r	$r \times r$ identity matrix,
asymptotically stable matrix	matrix with eigenvalues in open left half plane,
$\mathbb{S}^r$	$r \times r$ symmetric matrices,
$\mathbb{N}^r$	$r \times r$ symmetric nonnegative-definite matrices,
$\mathbb{P}^r$	$r \times r$ symmetric positive-definite matrices,
$Z_1 \geqq Z_2$	$Z_1 - Z_2 \in \mathbb{N}^r$, $Z_1, Z_2 \in \mathbb{S}^r$,
$Z_1 > Z_2$	$Z_1 - Z_2 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$,
$\operatorname{tr} Z, Z^T$	trace of Z, transpose of Z,
$\lambda_i(Z)$	eigenvalue of matrix Z,
$\lambda_{\max}(Z)$	maximum eigenvalue of matrix Z having real spectrum,
$\lVert\cdot\rVert_2$	Euclidean vector norm,
$\lVert\cdot\rVert_s$	spectral matrix norm (largest singular value),
$\lVert\cdot\rVert_F$	Frobenius matrix norm.

2. Robust stability and performance problems. Let $\mathcal{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n\times n}$. Throughout the paper it is assumed that A is asymptotically stable and that $0 \in \mathcal{U}$. We begin by considering the question of whether or not $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

ROBUST STABILITY PROBLEM. Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A)x(t), \qquad t \in [0, \infty), \tag{2.1}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

To consider the problem of robust performance it is necessary to introduce external disturbances. In this paper we consider both stochastic and deterministic disturbance models. The stochastic disturbance model involves white noise signals as in standard LQG theory, whereas the deterministic disturbance model involves L_2 signals as in H_∞ theory [6]. By defining an appropriate performance measure for each disturbance class it turns out that we can provide a simultaneous treatment of both cases.

We first consider the case of stochastic disturbances. In this case the robust performance problem concerns the worst-case magnitude of the expected value of a quadratic form involving outputs $y(t) = Ex(t)$, where $E \in \mathbb{R}^{q\times n}$, when the system is subjected to a standard white noise disturbance $w(t) \in \mathbb{R}^d$ with weighting $D \in \mathbb{R}^{n\times d}$.

STOCHASTIC ROBUST PERFORMANCE PROBLEM. For the disturbed linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \qquad t \in [0, \infty), \quad x(0) = 0, \tag{2.2}$$

$$y(t) = Ex(t), \tag{2.3}$$

where $w(\cdot)$ is a zero-mean d-dimensional white noise signal with intensity I_d, determine a performance bound β_S satisfying

$$J_S(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t\to\infty} \mathbb{E}\{\|y(t)\|_2^2\} \leq \beta_S. \tag{2.4}$$

The system (2.2), (2.3) may denote, for example, a control system in closed-loop configuration subjected to external white noise disturbances for which $y(t)$ may be the state regulation error. Such specializations are not required for this development, however.

Of course, since D and E may be rank deficient, there may be cases in which a finite performance bound β_S satisfying (2.4) exists while (2.1) is not asymptotically stable over $\mathcal{U}$. In practice, however, robust performance is mainly of interest when (2.1) is robustly stable. In this case the performance $J_S(\mathcal{U})$ is given in terms of the steady-state second moment of the state. The following result from linear system theory will be useful. For convenience define the $n \times n$ nonnegative-definite matrices

$$R \triangleq E^T E, \qquad V \triangleq DD^T.$$

LEMMA 2.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J_S(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} Q_{\Delta A} R, \tag{2.5}$$

where the $n \times n$ matrix $Q_{\Delta A} \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)x^T(t)]$ is given by

$$Q_{\Delta A} = \int_0^\infty e^{(A+\Delta A)t} V e^{(A+\Delta A)^T t}\, dt, \tag{2.6}$$

which is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^T + V. \tag{2.7}$$

To state the Deterministic Robust Performance Problem some additional notation is required. For a measurable function z: $[0, \infty) \to \mathbb{R}^r$ define

$$\|z(\cdot)\|_{2,2} \triangleq \left\{ \int_0^\infty \|z(t)\|_2^2 \, dt \right\}^{1/2}, \tag{2.8}$$

which is an L_2 function norm with a Euclidean spatial norm, and define

$$\|z(\cdot)\|_{\infty,2} \triangleq \text{ess.} \sup_{t \in [0,\infty)} \|z(t)\|_2,$$

which is an L_∞ function norm with a Euclidean spatial norm. We now reconsider (2.2) with $w(\cdot)$ interpreted as a square-integrable function. In this case the robust performance problem concerns the worst-case L_∞ norm of the output $y(t)$.

DETERMINISTIC ROBUST PERFORMANCE PROBLEM. For the disturbed linear system (2.2), (2.3), where $\|w(\cdot)\|_{2,2} \leqq 1$, determine a performance bound β_D satisfying

$$J_D(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \sup_{\|w(\cdot)\|_{2,2} \leq 1} \|y(\cdot)\|_{\infty,2}^2 \leqq \beta_D. \tag{2.9}$$

The performance measure $J_D(\mathcal{U})$ in (2.9) is given by the following result.

LEMMA 2.2. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J_D(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \lambda_{\max}(Q_{\Delta A} R), \tag{2.10}$$

where $Q_{\Delta A}$ is the unique, nonnegative-definite solution to (2.7).

Proof. The result is an immediate consequence of Theorem 1(b) of [38]. □

Remark 2.1. Although $J_S(\mathcal{U})$ and $J_D(\mathcal{U})$ arise from different mathematical settings they are quite similar in form. Note that in general $J_D(\mathcal{U}) \leqq J_S(\mathcal{U})$, and $J_D(\mathcal{U}) = J_S(\mathcal{U})$ if rank $R = 1$.

Remark 2.2. In Lemma 2.2 $Q_{\Delta A}$ can be viewed as the controllability Gramian for the pair $(A + \Delta A, D)$ rather than the state covariance. Note that $Q_{\Delta A}$ is independent of $x(0)$ and $Q_{\Delta A}(0)$.

Remark 2.3. The stochastic performance measure $J_S(\mathcal{U})$ given by (2.5) can also be written as

$$J_S(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \int_0^\infty \|Ee^{(A+\Delta A)t} D\|_F^2 \, dt, \tag{2.11}$$

which involves the L_2 norm of the impulse response of (2.2), (2.3). This stochastic performance measure can thus also be given a deterministic interpretation by letting $w(t)$ denote impulses at time $t = 0$. For details of this formulation see [46, p. 331].

In the present paper our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are developed in the following sections.

3. Sufficient conditions for robust stability and performance. The key step in obtaining robust stability and performance is to bound the uncertain terms $\Delta AQ + Q\Delta A^T$ in the Lyapunov equation (2.7) by means of a function $\Omega(Q)$. The nonnegative-definite solution Q of this modified Lyapunov equation is then guaranteed to be an upper bound for $Q_{\Delta A}$. The following easily proved result is fundamental and forms the basis for all later developments. The result is based on Lyapunov function theory as applied to linear systems. For our purposes, a suitable statement of this result is given by Lemma 12.2 of [39]. Essentially this result states that if the matrix equation $0 = \Phi F + F\Phi^T + SS^T$ has a solution $F \geqq 0$ and (Φ, S) is stabilizable, then Φ is an asymptotically stable matrix. Of

course, (Φ, S) is stabilizable (regardless of Φ) if S has full row rank, and we note (see [39, Thm. 3.6]) that if (Φ, S) is stabilizable then so is $(\Phi, [SS^T + H]^{1/2})$ for all nonnegative-definite matrices H.

THEOREM 3.1. *Let* $\Omega : \mathbb{N}^n \to \mathbb{N}^n$ *be such that*

$$\Delta AQ + Q\Delta A^T \leqq \Omega(Q), \qquad \Delta A \in \mathcal{U}, \quad Q \in \mathbb{N}^n, \tag{3.1}$$

and suppose there exists $Q \in \mathbb{N}^n$ *satisfying*

$$0 = AQ + QA^T + \Omega(Q) + V. \tag{3.2}$$

Then

$$(A + \Delta A, D) \text{ is stabilizable}, \qquad \Delta A \in \mathcal{U}, \tag{3.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{3.4}$$

In this case,

$$Q_{\Delta A} \leqq Q, \qquad \Delta A \in \mathcal{U}, \tag{3.5}$$

where $Q_{\Delta A} \in \mathbb{N}^n$ *is given by* (2.7), *and*

$$J_S(\mathcal{U}) \leqq \operatorname{tr} QR, \tag{3.6}$$

$$J_D(\mathcal{U}) \leqq \lambda_{\max}(QR). \tag{3.7}$$

In addition, if there exists $\Delta A \in \mathcal{U}$ *such that* $(A + \Delta A, D)$ *is controllable, then* Q *is positive definite.*

Proof. We stress that in (3.1), Q denotes an arbitrary element of $\mathbb{N}^n$, whereas in (3.2) Q denotes a specific solution of the modified Lyapunov equation. This minor abuse of notation considerably simplifies the presentation. Now note that for all $\Delta A \in \mathbb{R}^{n \times n}$, (3.2) is equivalent to

$$0 = (A + \Delta A)Q + Q(A + \Delta A)^T + \Omega(Q) - (\Delta AQ + Q\Delta A^T) + V. \tag{3.8}$$

Hence, by assumption, (3.8) has a solution $Q \in \mathbb{N}^n$ for all $\Delta A \in \mathbb{R}^{n \times n}$. If ΔA is restricted to the set $\mathcal{U}$ then, by (3.1), $\Omega(Q) - (\Delta AQ + Q\Delta A^T)$ is nonnegative definite. Thus if the stabilizability condition (3.3) holds for all $\Delta A \in \mathcal{U}$, then it follows from Theorem 3.6 of [39] that $(A + \Delta A, [V + \Omega(Q) - (\Delta AQ + Q\Delta A^T)]^{1/2})$ is stabilizable for all $\Delta A \in \mathcal{U}$. It now follows from (3.8) and Lemma 12.2 of [39] that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then (3.3) is immediate. Next, subtracting (2.7) from (3.8) yields

$$0 = (A + \Delta A)(Q - Q_{\Delta A}) + (Q - Q_{\Delta A})(A + \Delta A)^T + \Omega(Q) - (\Delta AQ + Q\Delta A^T), \qquad \Delta A \in \mathcal{U},$$

or, equivalently, since $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$

$$Q - Q_{\Delta A} = \int_0^\infty e^{(A+\Delta A)t}[\Omega(Q) - (\Delta AQ + Q\Delta A^T)]e^{(A+\Delta A)^T t}\, dt \geqq 0, \qquad \Delta A \in \mathcal{U}, \tag{3.9}$$

which implies (3.5). The performance bound (3.6) is now an immediate consequence of (2.5) and (3.5). To prove (3.7) we note that if $0 \leqq M_1 \leqq M_2$ then $\lambda_{\max}(M_1) \leqq \lambda_{\max}(M_2)$ (see, e.g., Corollary 7.7.4 of [40]). Thus

$$\begin{aligned} J_D(\mathcal{U}) &= \sup_{\Delta A \in \mathcal{U}} \lambda_{\max}(Q_{\Delta A}R) = \sup_{\Delta A \in \mathcal{U}} \lambda_{\max}(EQ_{\Delta A}E^T) \\ &\leqq \lambda_{\max}(EQE^T) = \lambda_{\max}(QR). \end{aligned} \tag{3.10}$$

Finally, it follows from (3.8) that if $(A + \Delta A, D)$ is controllable for some $\Delta A \in \mathcal{U}$, then the controllability Gramian Q for the pair

$$(A+\Delta A, [V+\Omega(Q)-(\Delta AQ+Q\Delta A^T)]^{1/2})$$

is positive definite. □

For convenience we shall say that $\Omega(\cdot)$ *bounds* $\mathcal{U}$ if (3.1) is satisfied. To apply Theorem 3.1, we first specify a function $\Omega(\cdot)$ and an uncertainty set $\mathcal{U}$ such that $\Omega(\cdot)$ bounds $\mathcal{U}$. If the existence of a nonnegative-definite solution Q to (3.2) can be determined analytically or numerically and (3.3) is satisfied, then robust stability is guaranteed and the performance bounds (3.6), (3.7) can be computed. We can then enlarge $\mathcal{U}$, modify $\Omega(\cdot)$, and again attempt to solve (3.2). If, however, a nonnegative-definite solution to (3.2) cannot be determined, then $\mathcal{U}$ must be decreased in size until (3.2) is solvable. For example, $\Omega(\cdot)$ can be replaced by $\varepsilon\Omega(\cdot)$ to bound $\varepsilon\mathcal{U}$, where $\varepsilon > 1$ enlarges $\mathcal{U}$ and $\varepsilon < 1$ shrinks $\mathcal{U}$. Of course, the actual range of uncertainty that can be bounded depends on the nominal matrix A, the function $\Omega(\cdot)$, and the structure of $\mathcal{U}$. In § 5 the uncertainty set $\mathcal{U}$ and bound $\Omega(\cdot)$ satisfying (3.1) are given concrete forms. We complete this section with several observations.

Remark 3.1. If only robust stability is of interest, then the noise intensity V need not have physical significance. In this case we may set $D = I_n$ to satisfy (3.3).

Remark 3.2. Since A is asymptotically stable, Q satisfying (3.2) is given by

$$Q=\int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dt, \tag{3.11}$$

or, equivalently,

$$Q=\int_0^\infty e^{At}\bar{\Omega}(Q)e^{A^Tt}\,dt+Q_0, \tag{3.12}$$

where $Q_0 \in \mathbb{N}^n$ is defined by

$$Q_0 \triangleq \int_0^\infty e^{At}Ve^{A^Tt}\,dt \tag{3.13}$$

and satisfies

$$0=AQ_0+Q_0A^T+V. \tag{3.14}$$

Note that $Q_0 \leqq Q$ and that the nominal performances $J_S(\{0\})$ and $J_D(\{0\})$ are given by $\operatorname{tr} Q_0R$ and $\lambda_{\max}(Q_0R)$, respectively.

Remark 3.3. Using (3.11) it is also useful to note that the bound for $J_S(\mathcal{U})$ given by (3.6) can be written as

$$\operatorname{tr} QR=\operatorname{tr}\int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dtR=\operatorname{tr} P_0[\Omega(Q)+V], \tag{3.15}$$

where $P_0 \in \mathbb{N}^n$ is defined by

$$P_0 \triangleq \int_0^\infty e^{A^Tt}Re^{At}\,dt \tag{3.16}$$

and satisfies

$$0=A^TP_0+P_0A+R. \tag{3.17}$$

The bound $\operatorname{tr} P_0[\Omega(Q) + V]$ can be viewed as a dual formulation of the bound $\operatorname{tr} QR$ since the roles of A and A^T are reversed. Dual bounds are developed in the following section. Note that $\operatorname{tr} Q_0 R = \operatorname{tr} P_0 V$.

Remark 3.4. If $\Omega(\cdot)$ bounds $\mathcal{U}$ then clearly $\Omega(\cdot)$ bounds the convex hull of $\mathcal{U}$. Hence, only convex uncertainty sets $\mathcal{U}$ need be considered. Next, we shall later use the obvious fact that if $\Omega'(\cdot)$ bounds $\mathcal{U}'$ and $\Omega''(\cdot)$ bounds $\mathcal{U}''$, then $\Omega'(\cdot) + \Omega''(\cdot)$ bounds $\mathcal{U}' + \mathcal{U}''$. Hence if $\mathcal{U}$ can be decomposed additively then it suffices to bound each component separately. Finally, if $\Omega(\cdot)$ bounds $\mathcal{U}$ and there exists $\Omega' : \mathbb{N}^n \to \mathbb{N}^n$ such that $\Omega(Q) \leqq \Omega'(Q)$ for all $Q \in \mathbb{N}^n$, then $\Omega'(\cdot)$ also bounds $\mathcal{U}$. That is, any *overbound* $\Omega'(\cdot)$ for $\Omega(\cdot)$ also bounds $\mathcal{U}$. Of course, as we shall see, it is quite possible that an overbound $\Omega'(\cdot)$ for $\Omega(\cdot)$ may actually bound a set $\mathcal{U}'$ that is larger than the "original" uncertainty set $\mathcal{U}$.

4. Dual sufficient conditions for robust stability and performance. As noted in Remark 3.3, the performance bound $\operatorname{tr} QR$ given by (3.6) can be expressed equivalently in terms of a dual variable P_0 for which the roles of A and A^T are reversed. Using a similar technique, additional conditions for robust stability and performance can be obtained by developing a dual version of Theorem 3.1. A prime motivation for developing such dual bounds is to draw connections with previous results in the literature relating to robust stability. Specifically, we shall show that traditional robust stability techniques based on the quadratic Lyapunov function $V(x) = x^T P x$ correspond to dual conditions. Robust performance bounds within the dual formulation, however, are difficult to motivate without first developing the primal performance bounds as was done in the previous section. In addition, the dual bounds may, for certain problems, yield larger stability regions and sharper performance bounds than the primal bounds.

LEMMA 4.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then*

$$J_S(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_{\Delta A} V, \tag{4.1}$$

where $P_{\Delta A} \in \mathbb{R}^{n \times n}$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)^T P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R. \tag{4.2}$$

Proof. It need only be noted that

$$\operatorname{tr} Q_{\Delta A} R = \operatorname{tr} \int_0^\infty e^{(A+\Delta A)t} V e^{(A+\Delta A)^T t} \, dt R = \operatorname{tr} P_{\Delta A} V,$$

where

$$P_{\Delta A} \triangleq \int_0^\infty e^{(A+\Delta A)^T t} R e^{(A+\Delta A)t} \, dt$$

satisfies (4.2). □

The proof of Lemma 4.1 relies on the fact that $\operatorname{tr} Q_{\Delta A} R = \operatorname{tr} P_{\Delta A} V$. However, it is not necessarily true that $\lambda_{\max}(Q_{\Delta A} R) = \lambda_{\max}(P_{\Delta A} V)$ even when $\Delta A = 0$. For example, if

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad R = I_2, \quad V = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

then

$$Q_0 R = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad P_0 V = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

and thus $\lambda_{\max}(Q_0R) = (15 + \sqrt{145})/24$ and $\lambda_{\max}(P_0V) = (5 + \sqrt{17})/8$. Thus to obtain a suitable dual version of $J_D(\mathscr{U})$ we need to define a dual deterministic cost $\hat{J}_D(\mathscr{U})$, which is distinct from $J_D(\mathscr{U})$. This can be done if the disturbance signals are taken to be integrable rather than square integrable. Thus, for measurable $z:[0,\infty)\to\mathbb{R}^r$ define

$$\|z(\cdot)\|_{1,2} \triangleq \int_0^\infty \|z(t)\|_2\,dt, \tag{4.3}$$

which is an L_1 function norm with a Euclidean spatial norm. The dual deterministic cost $\hat{J}_D(\mathscr{U})$ is thus defined by

$$\hat{J}_D(\mathscr{U}) \triangleq \sup_{\Delta A\in\mathscr{U}} \sup_{\|w(\cdot)\|_{1,2}\leq 1} \|y(\cdot)\|_{2,2}^2. \tag{4.4}$$

The following dual result follows from Theorem 1(a) of [38].

LEMMA 4.2. *Suppose $A+\Delta A$ is asymptotically stable for all $\Delta A\in\mathscr{U}$. Then*

$$\hat{J}_D(\mathscr{U}) = \lambda_{\max}(P_{\Delta A}V), \tag{4.5}$$

where $P_{\Delta A}\in\mathbb{R}^{n\times n}$ is the unique, nonnegative-definite solution to (4.2).

The dual version of Theorem 3.1 can now be stated.

THEOREM 4.1. *Let $\Lambda:\mathbb{N}^n\to\mathbb{N}^n$ be such that*

$$\Delta A^TP + P\Delta A \leq \Lambda(P), \qquad \Delta A\in\mathscr{U}, \quad P\in\mathbb{N}^n, \tag{4.6}$$

and suppose there exists $P\in\mathbb{N}^n$ satisfying

$$0 = A^TP + PA + \Lambda(P) + R. \tag{4.7}$$

Then

$$(E, A+\Delta A) \text{ is detectable}, \qquad \Delta A\in\mathscr{U}, \tag{4.8}$$

if and only if

$$A+\Delta A \text{ is asymptotically stable}, \qquad \Delta A\in\mathscr{U}. \tag{4.9}$$

In this case,

$$P_{\Delta A}\leq P, \qquad \Delta A\in\mathscr{U}, \tag{4.10}$$

where $P_{\Delta A}$ is given by (4.2), *and*

$$J_S(\mathscr{U}) \leq \operatorname{tr} PV, \tag{4.11}$$

$$\hat{J}_D(\mathscr{U}) \leq \lambda_{\max}(PV). \tag{4.12}$$

In addition, if there exists $\Delta A\in\mathscr{U}$ such that $(E, A+\Delta A)$ is observable, then P is positive definite.

Proof. The proof is completely analogous to the proof of Theorem 3.1. □

Remark 4.1. Note that $\hat{J}_D(\mathscr{U}) \leq J_S(\mathscr{U})$ and that $\hat{J}_D(\mathscr{U}) = J_S(\mathscr{U})$ if rank $V = 1$. Combining this fact with Remark 2.1, it follows that $J_D(\mathscr{U}) = \hat{J}_D(\mathscr{U})$ if both rank $R = 1$ and rank $V = 1$. In general, however, we should not expect that $J_D(\mathscr{U}) = \hat{J}_D(\mathscr{U})$.

It is quite possible that the *bounds* tr QR and tr PV for $J_S(\mathscr{U})$ given by (3.6) and (4.11) may be different in spite of the fact, as shown in the proof of Lemma 4.1, that $\operatorname{tr} Q_{\Delta A}R = \operatorname{tr} P_{\Delta A}V$. That is, depending on $\Omega(\cdot)$ and $\Lambda(\cdot)$ either bound (3.6) or bound (4.11) may be better for a particular problem. In general, we have the following result.

PROPOSITION 4.1. *Let* $\Omega(\cdot)$, $\Lambda(\cdot)$, Q, *and* P *be as in Theorems* 3.1 *and* 4.1, *and let* Q_0 *and* P_0 *be given by* (3.13) *and* (3.16), *respectively. Then*

$$\operatorname{tr} Q_0\Lambda(P) < \operatorname{tr} P_0\Omega(Q) \Leftrightarrow \operatorname{tr} QR > \operatorname{tr} PV, \tag{4.13}$$

$$\operatorname{tr} Q_0\Lambda(P) = \operatorname{tr} P_0\Omega(Q) \Leftrightarrow \operatorname{tr} QR = \operatorname{tr} PV, \tag{4.14}$$

$$\operatorname{tr} Q_0\Lambda(P) > \operatorname{tr} P_0\Omega(Q) \Leftrightarrow \operatorname{tr} QR < \operatorname{tr} PV. \tag{4.15}$$

Proof. Note that

$$\operatorname{tr} QR = \int_0^\infty e^{At}[\Omega(Q)+V]e^{A^Tt}\,dt\,R = \operatorname{tr} P_0\Omega(Q) + \operatorname{tr}\int_0^\infty e^{At}Ve^{A^Tt}\,dt\,R$$

and

$$\operatorname{tr} PV = \operatorname{tr}\int_0^\infty e^{A^Tt}[\Lambda(P)+R]e^{At}\,dt\,V = \operatorname{tr} Q_0\Lambda(P) + \operatorname{tr}\int_0^\infty e^{A^Tt}Re^{At}\,dt\,V$$

so that

$$\operatorname{tr} QR - \operatorname{tr} PV = \operatorname{tr} P_0\Omega(Q) - \operatorname{tr} Q_0\Lambda(P),$$

which yields (4.13)–(4.15). □

Remark 4.2. To draw connections with traditional Lyapunov theory, let R and V be positive definite and assume that there exists a positive-definite solution to (4.7). Then $V(x) \triangleq x^TPx$ satisfies $\dot V(x(t)) < 0$ for $x(\cdot)$ satisfying (2.1) and for all $\Delta A \in \mathcal{U}$. Thus $V(\cdot)$ is a Lyapunov function for (2.1) that guarantees robust asymptotic stability over $\mathcal{U}$.

5. Construction of the bounds $\Omega(\cdot)$ and $\Lambda(\cdot)$. As discussed in § 1, we consider three distinct classes of bounds $\Omega(\cdot)$ denoted by Type I, Type II, and Type III. Roughly speaking, these bounds exploit, respectively, the symmetry of the Lyapunov terms $\Delta AQ + Q\Delta A^T$, the structure of Q, and the structure of ΔA. The dual bounds $\Lambda(\cdot)$ can be constructed similarly by replacing Q and ΔA by P and ΔA^T. Hence these bounds will not be discussed separately. For convenience in discussing the set $\mathcal{U}$, we shall use the terms *rectangle* and *ellipse* to refer to closed regions bounded by such figures in multiple dimensions. As usual, a polytope is the convex hull of a finite number of points.

5.1. Type I bounds. We begin by constructing bounds $\Omega(\cdot)$ that exploit only the symmetry of the Lyapunov terms $\Delta AQ + Q\Delta A^T$. First we require the following well-known definition of a function of a symmetric matrix as an extension of a real-valued function (see, e.g., [40, p. 300]). Specifically, if $f: \mathbb{R} \to \mathbb{R}$, then (with a minor abuse of notation) $f: \mathbb{S}^n \to \mathbb{S}^n$ can be defined by setting

$$f(S) \triangleq Uf(D)U^T,$$

where $S = UDU^T$, U is orthogonal, D is real diagonal, and $f(D)$ is the diagonal matrix obtained by applying f to each diagonal element of D. Note that if f is the polynomial $f(x) = \sum_{i=0}^r a_ix^i$ then $f(S) = \sum_{i=0}^r a_iS^i$. Note also that if $f(x) = |x|$ then $f(S) = (S^2)^{1/2}$, where $(\cdot)^{1/2}$ denotes the (unique) nonnegative-definite square root. As in [41, p. 262], we use the notation $|S|$ to denote $(S^2)^{1/2}$. Finally, note that if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are such that $f(x) \leq g(x)$, $x \in \mathbb{R}$, then $f(S) \leq g(S)$, $S \in \mathbb{S}^n$.

As a concretization of the uncertainty set $\mathcal{U}$, consider the set

$$\mathcal{U}_1 \triangleq \left\{\Delta A \in \mathbb{R}^{n\times n}: \Delta A = \sum_{i=1}^p \sigma_iA_i,\ |\sigma_i| \leq \delta_i,\ i=1,\cdots,p\right\}, \tag{5.1}$$

where, for $i = 1, \cdots, p: A_i \in \mathbb{R}^{n \times n}$ is a given matrix denoting the structure of the parametric uncertainty, σ_i is a real uncertain parameter, and δ_i denotes the range of parameter uncertainty. Clearly, the multidimensional set of uncertain parameters $(\sigma_1, \cdots, \sigma_p)$ is the rectangle $[-\delta_1, \delta_1] \times \cdots \times [-\delta_p, \delta_p]$ and $\mathcal{U}_1$ is a symmetric polytope of matrices in $\mathbb{R}^{n \times n}$. Note that the symmetry of the uncertainty interval $[-\delta_i, \delta_i]$ entails no loss of generality since the nominal value of A can be redefined if necessary. Furthermore, it is also possible without loss of generality, to define $\delta_i = 1$ by replacing A_i by $\delta_i A_i$. For clarity, however, we choose not to employ this scaling. We begin by considering the bound utilized by Chang and Peng in [9].

PROPOSITION 5.1. *The function*

$$\Omega_1(Q) \triangleq \sum_{i=1}^{p} \delta_i |A_i Q + Q A_i^T| \tag{5.2}$$

bounds $\mathcal{U}_1$.

Proof. For $i = 1, \cdots, p$ and $|\sigma_i| \leqq \delta_i$,

$$\sigma_i(A_i Q + Q A_i^T) \leqq |\sigma_i(A_i Q + Q A_i^T)| = |\sigma_i|\,|A_i Q + Q A_i^T| \leqq \delta_i |A_i Q + Q A_i^T|.$$

Summing over i yields

$$\Delta A Q + Q \Delta A^T = \sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \leqq \sum_{i=1}^{p} \delta_i |A_i Q + Q A_i^T|,$$

which implies (3.1) with $\Omega(\cdot) = \Omega_1(\cdot)$ and $\mathcal{U} = \mathcal{U}_1$. $\square$

Remark 5.1. It is tempting to prove Proposition 5.1 by writing

$$\sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \leqq \left| \sum_{i=1}^{p} \sigma_i(A_i Q + Q A_i^T) \right| \leqq \sum_{i=1}^{p} |\sigma_i(A_i Q + Q A_i^T)|.$$

However, counterexamples show that the inequality $|M_1 + M_2| \leqq |M_1| + |M_2|$ is not generally true for arbitrary symmetric matrices M_1, M_2.

Remark 5.2. Because of its simplicity it is tempting to conjecture that $\Omega_1(\cdot)$ is the best bound for $\Delta A Q + Q \Delta A^T$ over the set $\mathcal{U}_1$. To show that this is not the case, let $Q = \frac{1}{2} I_2$, $p = 1$, $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and $\delta_1 = 1$. Then $\sigma_1(A_1 Q + Q A_1^T) \leqq \delta_1 |A_1 Q + Q A_1^T| = I_2$, $|\sigma_1| \leqq 1$. However, it is also true that

$$\sigma_1(A_1 Q + Q A_1^T) \leqq \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix}, \qquad |\sigma_1| \leqq 1.$$

Neither bound, however, is an overbound for the other. This is a consequence of the fact that the nonnegative-definite matrix ordering is only a partial order.

As mentioned earlier, an overbound for $\Omega_1(\cdot)$ will also bound $\mathcal{U}_1$. The following result is immediate.

LEMMA 5.1. *For* $i = 1, \cdots, p$, *let* $f_i: \mathbb{R} \to \mathbb{R}$ *satisfy*

$$f_i(x) \geqq |x|, \qquad x \in \mathbb{R}. \tag{5.3}$$

Then the function

$$\Omega_2(Q) \triangleq \sum_{i=1}^{p} \delta_i f_i(A_i Q + Q A_i^T) \tag{5.4}$$

is an overbound for $\Omega_1(\cdot)$ *and hence also bounds* $\mathcal{U}_1$.

One particular choice of f_i satisfying (5.3) will be considered here, namely, the polynomial

$$f_i(x)=\tfrac{1}{4}\beta_i+\beta_i^{-1}x^2, \tag{5.5}$$

where β_i is an arbitrary positive constant. Thus $\Omega_2(\cdot)$ has the following specialization.

COROLLARY 5.1. *Let $\beta_1, \cdots, \beta_p$ be arbitrary positive constants. Then the function*

$$\Omega_3(Q)\triangleq\frac{1}{4}\sum_{i=1}^{p}\delta_i\beta_i I_n+\sum_{i=1}^{p}\left(\frac{\delta_i}{\beta_i}\right)(A_iQ+QA_i^T)^2 \tag{5.6}$$

is an overbound for $\Omega_1(\cdot)$ and hence also bounds $\mathcal{U}_1$.

Although overbounding $\Omega_1(\cdot)$ by $\Omega_3(\cdot)$ results in a looser bound for $\mathcal{U}_1$, it turns out that $\Omega_3(\cdot)$ actually bounds a set that is larger than $\mathcal{U}_1$. Specifically, in place of $\mathcal{U}_1$ consider

$$\mathcal{U}_2\triangleq\left\{\Delta A\in\mathbb{R}^{n\times n}:\Delta A=\sum_{i=1}^{p}\sigma_iA_i,\ \sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2}\leqq 1\right\}, \tag{5.7}$$

where $\alpha_1, \cdots, \alpha_p$ are given positive constants. Note that (5.7) replaces the rectangle of uncertain parameters $(\sigma_1, \cdots, \sigma_p)$ by an ellipse. Thus the set $\mathcal{U}_2$ of matrix perturbations is an ellipse of matrices in $\mathbb{R}^{n\times n}$ in contrast to the polytope $\mathcal{U}_1$. Of course, $\mathcal{U}_1=\mathcal{U}_2$ if $p=1$ and $\alpha_1=\delta_1$. Again it is possible to take $\alpha_i=1$ without loss of generality by replacing A_i by α_iA_i. We again choose not to do this, however. The following result provides a convenient characterization of the relationship between the rectangle $\mathcal{U}_1$ and the ellipse $\mathcal{U}_2$.

PROPOSITION 5.2. *Suppose $\mathcal{U}_1$ is defined by the positive constants $\delta_1, \cdots, \delta_p$, and let $\mathcal{U}_2$ be characterized by*

$$\alpha_i=\left(\frac{\alpha\delta_i}{\beta_i}\right)^{1/2}, \qquad i=1, \cdots, p, \tag{5.8}$$

where α is defined by

$$\alpha=\sum_{i=1}^{p}\delta_i\beta_i \tag{5.9}$$

and $\beta_1, \cdots, \beta_p$ are arbitrary positive constants. Then the ellipse

$$\left\{(\sigma_1, \cdots, \sigma_p): \sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2}\leqq 1\right\}$$

circumscribes the rectangle $\{(\sigma_1, \cdots, \sigma_p): |\sigma_i|\leqq\delta_i, i=1, \cdots, p\}$ and thus $\mathcal{U}_2$ contains $\mathcal{U}_1$. Furthermore, $\Omega_3(\cdot)$ actually bounds $\mathcal{U}_2$.

Proof. If $|\sigma_i|\leqq\delta_i$, $i=1, \cdots, p$, then it follows from (5.8) and (5.9) that

$$\sum_{i=1}^{p}\frac{\sigma_i^2}{\alpha_i^2}=\alpha^{-1}\sum_{i=1}^{p}\frac{\beta_i\sigma_i^2}{\delta_i}\leqq\alpha^{-1}\sum_{i=1}^{p}\ {}_i\delta_i=1.$$

Thus the ellipse contains the rectangle. If, in addition, $(\sigma_1, \cdots, \sigma_p)$ is a vertex of the rectangle, i.e., $|\sigma_i| = \delta_i$, $i = 1, \cdots, p$, then $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 = 1$, which corresponds to a point on the boundary of the ellipse. To show that $\Omega_3(\cdot)$ actually bounds $\mathcal{U}_2$ note that

$$0 \leq \sum_{i=1}^p \left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)I_n - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)\right]^2$$

$$= \frac{\alpha}{4}\sum_{i=1}^p \left(\frac{\sigma_i^2}{\alpha_i^2}\right)I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2 - (\Delta AQ + Q\Delta A^T).$$

Since $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 \leq 1$ in $\mathcal{U}_2$, it follows that

$$\Delta AQ + Q\Delta A^T \leq \frac{\alpha}{4}I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2.$$

Utilizing (5.8) and (5.9) to substitute for α and α_i yields (3.1) with $\Omega(\cdot) = \Omega_3(\cdot)$ and $\mathcal{U} = \mathcal{U}_2$. □

Proposition 5.2 shows that each choice of constants $\beta_1, \cdots, \beta_p > 0$ leads to a particular ellipse $\mathcal{U}_2$ that contains the polytope $\mathcal{U}_1$. Furthermore, $\Omega_3(\cdot)$, which by Corollary 5.1 bounds $\mathcal{U}_1$, actually bounds the larger set $\mathcal{U}_2$. For convenience, we now dispense with the constants $\beta_1, \cdots, \beta_p$ that relate the rectangle $\mathcal{U}_1$ to the ellipse $\mathcal{U}_2$ and we characterize $\Omega_3(\cdot)$ entirely in terms of $\alpha, \alpha_1, \cdots, \alpha_p$.

COROLLARY 5.2. *Let α be an arbitrary positive constant. Then the function*

$$\Omega_4(Q) \triangleq \frac{\alpha}{4}I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)^2 \tag{5.10}$$

bounds $\mathcal{U}_2$.

Remark 5.3. Within the context of Corollary 5.2, the positive constant α plays no role in defining the set $\mathcal{U}_2$, although $\Omega_4(\cdot)$ is guaranteed to bound $\mathcal{U}_2$ for all choices of α. It can be expected, however, that certain choices of α provide better bounds than other choices. This will be seen by example in § 10.

The following variation of $\Omega_4(\cdot)$ was suggested by D. C. Hyland.

PROPOSITION 5.3. *Let α be an arbitrary positive constant. Then, for $Q > 0$,*

$$\Omega_{4'}(Q) \triangleq \frac{\alpha}{2}Q + \frac{\alpha^{-1}}{2}\sum_{i=1}^p \alpha_i^2[A_i^2Q + A_iQA_i^T + QA_i^TQ^{-1}A_iQ + QA_i^{2T}] \tag{5.10$'$}$$

bounds $\mathcal{U}_2$.

Proof. Note that

$$0 \leq \sum_{i=1}^p \left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q^{1/2} - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)Q^{-1/2}\right]$$

$$\times\left[\frac{1}{2}\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q^{1/2} - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)(A_iQ+QA_i^T)Q^{-1/2}\right]^T$$

$$= \frac{\alpha}{4}\sum_{i=1}^p \left(\frac{\sigma_i^2}{\alpha_i^2}\right)Q + \alpha^{-1}\sum_{i=1}^p \alpha_i^2(A_iQ+QA_i^T)Q^{-1}(A_iQ+QA_i^T) - (\Delta AQ + Q\Delta A^T),$$

which yields the desired result. □

Remark 5.4. The bound $\Omega_{4'}(Q)$ is of interest since it involves terms that arise from a multiplicative white noise model with a Stratonovich correction. Specifically, the term

$A_iQA_i^T$ arises from an Ito model [33], whereas the terms A_i^2Q and QA_i^{2T} can be viewed as the shift $A \to A + \frac{1}{2}\sum_{i=1}^p A_i^2$ due to the Stratonovich interpretation of stochastic integration [43]. These terms have interesting ramifications in designing controllers for flexible structures [23].

5.2. Type II bounds. We now consider additional bounds for $\mathscr{U}$ that exploit the structure of Q. For these bounds the natural uncertainty set is given by $\mathscr{U}_2$.

PROPOSITION 5.4. *Let α be an arbitrary positive number and, for each $Q \in \mathbb{N}^n$, let $Q_1 \in \mathbb{R}^{n\times m}$ and $Q_2 \in \mathbb{R}^{m\times n}$ satisfy*

$$Q = Q_1Q_2. \tag{5.11}$$

Then the function

$$\Omega_5(Q) \triangleq \alpha Q_2^TQ_2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2 A_iQ_1Q_1^TA_i^T \tag{5.12}$$

bounds $\mathscr{U}_2$.

Proof. Note that

$$0 \leq \sum_{i=1}^p \left[\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q_2^T - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)A_iQ_1\right]\left[\left(\frac{\alpha^{1/2}\sigma_i}{\alpha_i}\right)Q_2^T - \left(\frac{\alpha_i}{\alpha^{1/2}}\right)A_iQ_1\right]^T$$

$$= \alpha\sum_{i=1}^p\left(\frac{\sigma_i^2}{\alpha_i^2}\right)Q_2^TQ_2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQ_1Q_1^TA_i^T - \sum_{i=1}^p \sigma_i(A_iQ + QA_i^T),$$

which, since $\sum_{i=1}^p \sigma_i^2/\alpha_i^2 \leq 1$, yields (3.1) with $\Omega(\cdot) = \Omega_5(\cdot)$ and $\mathscr{U} = \mathscr{U}_2$. □

We consider three specializations of $\Omega_5(\cdot)$. Specifically, we set $m = n$ and define

$$Q_1 = Q, \qquad Q_2 = I_n, \tag{5.13}$$

$$Q_1 = Q_2 = Q^{1/2}, \tag{5.14}$$

$$Q_1 = I_n, \qquad Q_2 = Q. \tag{5.15}$$

COROLLARY 5.3. *Let α be an arbitrary positive number. Then the functions*

$$\Omega_6(Q) \triangleq \alpha I_n + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQ^2A_i^T, \tag{5.16}$$

$$\Omega_7(Q) \triangleq \alpha Q + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iQA_i^T, \tag{5.17}$$

$$\Omega_8(Q) \triangleq \alpha Q^2 + \alpha^{-1}\sum_{i=1}^p \alpha_i^2A_iA_i^T \tag{5.18}$$

bound $\mathscr{U}_2$.

Remark 5.5. Note that the term $A_iQ^2A_i^T$ appearing in $\Omega_6(\cdot)$ also appears in $\Omega_4(\cdot)$. Furthermore, both $\Omega_4(\cdot)$ and $\Omega_6(\cdot)$ involve a term proportional to I_n. Despite these similarities, neither bound $\Omega_4(\cdot)$ nor $\Omega_6(\cdot)$ is an overbound for the other. Furthermore, the term $A_iQA_i^T$ appears in both $\Omega_7(\cdot)$ and $\Omega_4(\cdot)$. However, neither $\Omega_7(\cdot)$ nor $\Omega_4(\cdot)$ is an overbound for the other.

Remark 5.6. The bound $\Omega_7(\cdot)$ given by (5.17) has t' tinction that it is linear in Q. This bound was originally studied in [27] for syste , with multiplicative white noise and was shown to yield robust stability and r mance in [33] and [35]. A similar bound was studied in [34].

Remark 5.7. By using (5.11) additional bounds can be developed. For example, by setting

$$Q_1 = Q^{1/4}, \qquad Q_2 = Q^{3/4}, \tag{5.19}$$

$\Omega_5(\cdot)$ becomes

$$\Omega_9(Q) = \alpha Q^{3/2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i Q^{1/2} A_i^T. \tag{5.20}$$

Remark 5.8. When $p = 1$ and α is replaced by $\alpha\alpha_1$, $\Omega_7(\cdot)$ becomes

$$\Omega_7'(Q) = \alpha_1[\alpha Q + \alpha^{-1} A_1 Q A_1^T].$$

A sum of such terms with $\alpha_i = \delta_i$ can be used to bound the smaller rectangular set $\mathcal{U}_1$. Similar remarks apply to $\Omega_6(\cdot)$, $\Omega_8(\cdot)$, and $\Omega_9(\cdot)$.

5.3. Type III bounds. We now consider bounds that exploit the structure of ΔA itself. It turns out that these bounds permit consideration of an uncertainty set $\mathcal{U}$ that is larger than $\mathcal{U}_2$. Specifically, define

$$\mathcal{U}_3 \triangleq \{\Delta A \in \mathbb{R}^{n\times n}: \Delta A = A_L A_R, A_L A_L^T \leqq M, A_R^T A_R \leqq N\}, \tag{5.21}$$

where $A_L \in \mathbb{R}^{n\times r}$ and $A_R \in \mathbb{R}^{r\times n}$ are uncertain matrices, r is an arbitrary positive integer, and M, $N \in \mathbb{N}^n$ are given uncertainty bounds. The bound $\Omega_{10}(\cdot)$ for $\mathcal{U}_3$ is given by the following result.

PROPOSITION 5.5. *Let α be an arbitrary positive constant. Then the function*

$$\Omega_{10}(Q) \triangleq \alpha^{-1} M + \alpha Q N Q \tag{5.22}$$

bounds $\mathcal{U}_3$.

Proof. Note that

$$\begin{aligned} 0 &\leqq [\alpha^{-1/2} A_L - \alpha^{1/2} Q A_R^T][\alpha^{-1/2} A_L - \alpha^{1/2} Q A_R^T]^T \\ &= \alpha^{-1} A_L A_L^T + \alpha Q A_R^T A_R Q - [A_L A_R Q + Q(A_L A_R)^T] \\ &\leqq \alpha^{-1} M + \alpha Q N Q - (\Delta A Q + Q \Delta A^T), \end{aligned}$$

which yields (3.1) with $\Omega(\cdot) = \Omega_{10}(\cdot)$ and $\mathcal{U} = \mathcal{U}_3$. □

Remark 5.9. The bound $\Omega_{10}(\cdot)$ was developed in [29] for robust analysis and independently in [25] and [28] for robust full-state feedback. Applications to fixed-order dynamic compensation are given in [36].

Remark 5.10. Without loss of generality we can set $\alpha = 1$ in (5.22) by replacing M and N by $\alpha^{-1}M$ and αN, respectively. Again for clarity we choose not to employ this scaling.

Note that $\Omega_8(\cdot)$ is of the form $\Omega_{10}(\cdot)$ with $M = \sum_{i=1}^{p} \alpha_i^2 A_i A_i^T$ and $N = I_n$. Thus $\Omega_8(\cdot)$ also bounds $\mathcal{U}_3$ for this choice of M and N. It turns out in this case that $\mathcal{U}_3$ is actually larger than $\mathcal{U}_2$. To see this consider the more general case in which M and N satisfy

$$\sum_{i=1}^{p} \alpha_i^2 A_i A_i^T \leqq M, \qquad I_n \leqq N. \tag{5.23}$$

In this case $\Omega_{10}(\cdot)$ is an overbound for $\Omega_8(\cdot)$ and thus bounds $\mathcal{U}_2$. As in the case of $\Omega_3(\cdot)$ overbounding $\Omega_1(\cdot)$, we should not be surprised to find that $\Omega_{10}(\cdot)$ with (5.23) actually bounds a set that is larger than $\mathcal{U}_2$. Indeed, we now show that $\mathcal{U}_2$ is actually a very special subset of $\mathcal{U}_3$ when M and N defining $\mathcal{U}_2$ satisfy (5.23).

PROPOSITION 5.6. *If M and N satisfy* (5.23) *then $\mathscr{U}_2$ is a subset of $\mathscr{U}_3$. Hence $\Omega_{10}(\cdot)$ also bounds $\mathscr{U}_2$.*

Proof. If $\Delta A \in \mathscr{U}_2$ then $\Delta A = \sum_{i=1}^{p} \sigma_i A_i$, where $\sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \leqq 1$. Alternatively, we can write $\Delta A = A_L A_R$, where $r = pn$ and

$$A_L = [\alpha_1 A_1 \cdots \alpha_p A_p], \qquad A_R = \begin{bmatrix} (\sigma_1/\alpha_1) I_n \\ \vdots \\ (\sigma_p/\alpha_p) I_n \end{bmatrix}. \tag{5.24}$$

Note that with M and N satisfying (5.23) and A_L and A_R defined by (5.24), it follows that $A_L A_L^T \leqq M$ and $A_R^T A_R \leqq N$. Thus $\Delta A \in \mathscr{U}_3$. □

The following result provides further conditions under which $\Omega_{10}(\cdot)$ bounds $\mathscr{U}_2$.

PROPOSITION 5.7. *Suppose $A_i = D_i E_i$, $i = 1, \cdots, p$, where $D_i \in \mathbb{R}^{n \times n_i}$ and $E_i \in \mathbb{R}^{n_i \times n}$, and suppose that*

$$\sum_{i=1}^{p} \alpha_i^2 D_i D_i^T \leqq M, \qquad \sum_{i=1}^{p} E_i^T E_i \leqq N. \tag{5.25}$$

Then $\mathscr{U}_2$ is a subset of $\mathscr{U}_3$ and thus $\Omega_{10}(\cdot)$ also bounds $\mathscr{U}_2$.

Proof. The result follows as in the proof Proposition 5.6. □

Remark 5.11. When $p = 1$, $A_1 = D_1 E_1$, $M = \alpha_1^2 D_1 D_1^T$, and $N = E_1^T E_1$, it is convenient to replace α by $\alpha\alpha_1$ so that $\Omega_{10}(\cdot)$ becomes

$$\Omega_{10'}(Q) = \alpha_1 [\alpha^{-1} D_1 D_1^T + \alpha Q E_1^T E_1 Q]. \tag{5.26}$$

In certain situations it is desirable to consider subsets of $\mathscr{U}_3$ of special structure. For example, define

$$\mathscr{U}_4 \triangleq \{ \Delta A \in \mathbb{R}^{n \times n} : \Delta A = D_0 A_L A_R E_0, \|A_L\|_s \leqq 1, \|A_R\|_s \leqq 1 \},$$

where $D_0 \in \mathbb{R}^{n \times n_1}$ and $E_0 \in \mathbb{R}^{n_2 \times n}$ are known matrices denoting the structure of the uncertainty, and $A_L \in \mathbb{R}^{n_1 \times r}$ and $A_R \in \mathbb{R}^{r \times n_2}$ are uncertain matrices [28]. Finer structure can be included within $\mathscr{U}_4$ by replacing $D_0 M N E_0$ by a sum of terms $D_i M_i N_i E_i$, where D_i, E_i are known and M_i, N_i are uncertain [36]. Note, however, that even though $\mathscr{U}_4$ is a proper subset of $\mathscr{U}_3$, the *form* of the bound $\Omega_{10}(\cdot)$ does not change. Thus such refinements render the bound $\Omega_{10}(\cdot)$ conservative with respect to $\mathscr{U}_4$ since the *larger* uncertainty set $\mathscr{U}_3$ is actually being bounded.

6. Robust stability and performance via modified Lyapunov equations. We now combine the principal results of §§ 3, 4, and 5 to obtain a series of conditions guaranteeing robust stability and performance. In particular, we focus on bounds Ω_1, Ω_4, Ω_6, Ω_7, and Ω_{10}. For simplicity we shall frequently assume that V is positive definite so that (3.3) is satisfied. In this case it follows that the solution Q of (3.2) is positive definite. Our first result is a corollary of Theorem 3.1 with $\Omega(\cdot) = \Omega_1(\cdot)$ and $\mathscr{U} = \mathscr{U}_1$.

THEOREM 6.1. *Let $V \in \mathbb{P}^n$, $\delta_1, \cdots, \delta_p > 0$, and suppose there exists $Q \in \mathbb{P}^n$ satisfying*

$$0 = AQ + QA^T + \sum_{i=1}^{p} \delta_i |A_i Q + Q A_i^T| + V. \tag{MLE1}$$

Then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}_1$, and

$$J_S(\mathscr{U}_1) \leqq \operatorname{tr} QR, \tag{6.1}$$

$$J_D(\mathscr{U}_1) \leqq \lambda_{\max}(QR). \tag{6.2}$$

For the next result define

$$A_\alpha \triangleq A+\frac{\alpha}{2}I_n \tag{6.3}$$

and

$$\gamma_i \triangleq \frac{\alpha_i^2}{\alpha}, \qquad i=1,\cdots,p. \tag{6.4}$$

Setting $\Omega(\cdot)=\Omega_4(\cdot)$, $\Omega_6(\cdot)$, $\Omega_7(\cdot)$ and $\mathcal{U}=\mathcal{U}_2$ yields the following corollary of Theorem 3.1.

THEOREM 6.2. *Let $V\in\mathbb{P}^n$, $\alpha, \alpha_1,\cdots,\alpha_p>0$, and suppose there exists $Q\in\mathbb{P}^n$ satisfying either*

$$0=AQ+QA^T+\sum_{i=1}^p \gamma_i(A_iQ+QA_i^T)^2+\frac{\alpha}{4}I_n+V, \tag{MLE2}$$

$$0=AQ+QA^T+\sum_{i=1}^p \gamma_iA_iQ^2A_i^T+\alpha I_n+V, \tag{MLE3}$$

or

$$0=A_\alpha Q+QA_\alpha^T+\sum_{i=1}^p \gamma_iA_iQA_i^T+V. \tag{MLE4}$$

Then $A+\Delta A$ is asymptotically stable for all $\Delta A\in\mathcal{U}_2$, and

$$J_S(\mathcal{U}_2)\leqq \operatorname{tr} QR, \tag{6.5}$$

$$J_D(\mathcal{U}_2)\leqq \lambda_{\max}(QR). \tag{6.6}$$

Next we set $\Omega(\cdot)=\Omega_{10}(\cdot)$ and $\mathcal{U}=\mathcal{U}_3$.

THEOREM 6.3. *Let $V\in\mathbb{P}^n$, $\alpha>0$, $M\in\mathbb{N}^n$, and $N\in\mathbb{N}^n$, and suppose there exists $Q\in\mathbb{P}^n$ satisfying*

$$0=AQ+QA^T+\alpha QNQ+\alpha^{-1}M+V. \tag{MLE5}$$

Then $A+\Delta A$ is asymptotically stable for all $\Delta A\in\mathcal{U}_3$, and

$$J_S(\mathcal{U}_3)\leqq \operatorname{tr} QR, \tag{6.7}$$

$$J_D(\mathcal{U}_3)\leqq \lambda_{\max}(QR). \tag{6.8}$$

Remark 6.1. Note that (MLE5) is a Riccati equation. This is precisely the equation studied in [29].

Additional sufficient conditions can be obtained by considering "mixed" bounds. That is, we can construct modified Lyapunov equations by combining two or more different bounds. Although mixed bounds will not be considered further in this paper, we present one such result for illustrative purposes.

THEOREM 6.4. *Let $V\in\mathbb{P}^n$, $\alpha, \delta_1,\cdots,\delta_p>0$, $M\in\mathbb{N}^n$, and $N\in\mathbb{N}^n$, and suppose there exists $Q\in\mathbb{P}^n$ satisfying*

$$0=AQ+QA^T+\sum_{i=1}^p \delta_i|A_iQ+QA_i^T|+\alpha QNQ+\alpha^{-1}M+V. \tag{MLE1, 5}$$

Then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_1 + \mathcal{U}_3$, and

$$J_S(\mathcal{U}_1+\mathcal{U}_3) \leq \operatorname{tr} QR, \tag{6.9}$$

$$J_D(\mathcal{U}_1+\mathcal{U}_3) \leq \lambda_{\max}(QR). \tag{6.10}$$

As noted previously, the bound $\Lambda(\cdot)$ can readily be constructed by replacing ΔA by ΔA^T in the definitions of $\Omega_1(\cdot)$ through $\Omega_{10}(\cdot)$. Denote these bounds by $\Lambda_1(\cdot)$ through $\Lambda_{10}(\cdot)$, respectively. For illustration we state the dual of Theorem 6.1 involving $\Lambda_1(\cdot)$. The dual versions of (MLE1)–(MLE5) will be denoted by (MLED1)–(MLED5).

THEOREM 6.5. *Let $R \in \mathbb{P}^n$, $\delta_1, \cdots, \delta_p > 0$, and suppose there exists $P \in \mathbb{P}^n$ satisfying*

$$0 = A^TP + PA + \sum_{i=1}^{p} \delta_i |A_i^TP + PA_i| + R. \tag{MLED1}$$

Then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}_1$, and

$$J_S(\mathcal{U}_1) \leq \operatorname{tr} PV, \tag{6.11}$$

$$\hat{J}_D(\mathcal{U}_1) \leq \lambda_{\max}(PV). \tag{6.12}$$

It is reasonable to expect that the sufficient conditions given by Theorems 3.1 and 4.1 are generally different. For example, the modified Lyapunov equations and their duals need not both possess a solution, while the bounds tr QR and tr PV need not be equal. An exception is the case in which $\Omega(\cdot) = \Omega_7(\cdot)$ and $\Lambda(\cdot) = \Lambda_7(\cdot)$. Note that the dual of (MLE4) is given by

$$0 = A_\alpha^TP + PA_\alpha + \sum_{i=1}^{p} \gamma_i A_i^TPA_i + V. \tag{MLED4}$$

PROPOSITION 6.1. *Let $\alpha, \alpha_1, \cdots, \alpha_p > 0$ and assume there exist $Q, P \in \mathbb{N}^n$ satisfying* (MLE4) *and* (MLED4). *Then*

$$\operatorname{tr} QR = \operatorname{tr} PV. \tag{6.13}$$

Proof. Note that

$$\begin{aligned}\operatorname{tr} QR &= -\operatorname{tr} Q\left(A_\alpha^TP + PA_\alpha + \sum_{i=1}^{p} \gamma_i A_i^TPA_i\right)\\ &= -\operatorname{tr} P\left(A_\alpha Q + QA_\alpha^T + \sum_{i=1}^{p} \gamma_i A_i Q A_i^T\right)\\ &= \operatorname{tr} PV. \qquad \square\end{aligned}$$

Remark 6.2. By setting $\Omega(\cdot) = \Omega_7(\cdot)$ and $\Lambda(\cdot) = \Lambda_7(\cdot)$ it follows from (4.14) that

$$\operatorname{tr} Q_0\left(\alpha P + \sum_{i=1}^{p} \gamma_i A_i^TPA_i\right) = \operatorname{tr} P_0\left(\alpha Q + \sum_{i=1}^{p} \gamma_i A_i Q A_i^T\right). \tag{6.14}$$

7. Existence, uniqueness, and monotonicity of solutions to the modified Lyapunov equations. It is important to stress that the sufficient conditions for robustness given by Theorems 6.1–6.5 assume only that there exist nonnegative-definite solutions Q, P sat-

isfying the modified Lyapunov equations. Indeed, no *explicit* assumptions on the problem data A, V, R, and $\mathscr{U}$ were utilized for assuring robust stability and performance. In applying Theorems 6.1–6.5 to specific problems it thus suffices to show that a nonnegative-definite solution Q *exists* in order to obtain robust stability, while, for robust performance, the bounds (6.1), (6.2), (6.5)–(6.8) require explicit knowledge of Q. Thus, any computational method that yields a nonnegative-definite solution will suffice to guarantee both robust stability and performance.

Before considering the numerical solution of the modified Lyapunov equations, several relevant issues require discussion. For example, before seeking to compute solutions to (MLE1)–(MLE5) it would be desirable to determine a priori whether these equations actually possess nonnegative-definite solutions. For example, it may be useful to obtain sufficient and/or necessary conditions for the *existence* of nonnegative-definite solutions. Thus, if the sufficient conditions are satisfied then existence (and hence robustness) is assured, whereas if the necessary conditions are *not* satisfied then existence is ruled out. If, on the other hand, either the sufficient conditions are not satisfied or the necessary conditions *are* satisfied, then nothing can be surmised. Finally, such conditions need to be easily verifiable and reasonably nonconservative since otherwise it would be more prudent to attempt to numerically solve the modified Lyapunov equations themselves.

It is quite possible that at least some of the modified Lyapunov equations possess multiple nonnegative-definite solutions. In this case we may seek the minimal solution (i.e., the smallest with respect to the nonnegative-definite matrix ordering) to minimize the performance bounds. If multiple solutions exist, none of which is minimal, then the best bound would depend on the matrix R.

Since the matrix Q determines the performance bound, it is reasonable to expect Q to be *monotonic* in $\mathscr{U}$. That is, if $\mathscr{U}$ decreases in size, then the solution Q is more likely to exist while decreasing in the nonnegative-definite matrix ordering. For example, consider $\mathscr{U}'_1$ characterized by δ'_i, where $\delta'_i \leqq \delta_i$, $i = 1, \cdots, p$. Then we might expect $Q' \leqq Q$, where Q' is the solution to (MLE1) with δ_i replaced by δ'_i. Finally, monotonicity with respect to V should also be expected. Because of linearity, the analysis of bound $\Omega_7(\cdot)$ is simplest and it is possible to obtain necessary and sufficient conditions for the existence of solutions to (MLE4). The basic tool required is the Kronecker matrix algebra [42]. For convenience, define

$$\mathscr{A} \triangleq A_\alpha \oplus A_\alpha + \sum_{i=1}^{p} \gamma_i A_i \otimes A_i, \tag{7.1}$$

where $\otimes$ denotes the Kronecker product and $A_\alpha \oplus A_\alpha \triangleq A_\alpha \otimes I_n + I_n \otimes A_\alpha$ is the Kronecker sum.

PROPOSITION 7.1. *If* $V \in \mathbb{N}^n$ *and* $\mathscr{A}$ *is asymptotically stable, then there exists a unique* $Q \in \mathbb{R}^{n \times n}$ *satisfying* (MLE4), *and* $Q \geqq 0$. *Conversely, if for all* $V \in \mathbb{N}^n$ *there exists* $Q \geqq 0$ *satisfying* (MLE4), *then* $\mathscr{A}$ *is asymptotically stable.*

Proof. Since (MLE4) is equivalent to

$$Q = -\mathrm{vec}^{-1}[\mathscr{A}^{-1}\,\mathrm{vec}\,V], \tag{7.2}$$

existence and uniqueness hold. Here, vec and vec^{-1} denote the column-stacking operation [42] and its inverse. To prove that Q is nonnegative definite, we rewrite (7.2) as

$$Q = \int_0^\infty \mathrm{vec}^{-1}[e^{\mathscr{A}t}\,\mathrm{vec}\,V]\,dt \tag{7.3}$$

and show that the integrand is nonnegative-definite for all $t \in [0, \infty)$. (Note that the following argument for fixed $t \geqq 0$ does not require that $\mathscr{A}$ be stable.) Using the exponential product formula,[1] the exponential in (7.3) can be written as

$$(7.4) \qquad e^{\mathscr{A}t} = \lim_{k \to \infty} \left\{ \exp\left[\frac{1}{k}(A_\alpha \oplus A_\alpha)t\right] \exp\left[\frac{1}{k}\sum_{i=1}^{p} \gamma_i (A_i \otimes A_i)t\right]\right\}^k.$$

For convenience, let S and N be $r \times r$ matrices with $N \geqq 0$. Since (see [42])

$$(7.5) \qquad \mathrm{vec}^{-1}[(S \otimes S)\,\mathrm{vec}\, N] = SNS^T \geqq 0$$

and

$$(7.6) \qquad (S \otimes S)^k = S^k \otimes S^k,$$

it follows that

$$(7.7) \qquad \mathrm{vec}^{-1}[e^{S \otimes S}\,\mathrm{vec}\, N] = \sum_{k=0}^{\infty} (k!)^{-1} S^k N S^{kT} \geqq 0.$$

Furthermore,

$$(7.8) \qquad \mathrm{vec}^{-1}[e^{S \oplus S}\,\mathrm{vec}\, N] = \mathrm{vec}^{-1}[(e^S \otimes e^S)\,\mathrm{vec}\, N] = e^S N e^{S^T} \geqq 0.$$

Applying (7.7) and (7.8) alternately with (7.4) and using induction on k, it follows that the integrand of (7.3) is nonnegative definite. To prove the converse, note that it follows from (MLE4) that Q satisfies

$$(7.9) \qquad Q = \mathrm{vec}^{-1}[e^{\mathscr{A}t}\,\mathrm{vec}\, Q] + \int_0^t \mathrm{vec}^{-1}[e^{\mathscr{A}s}\,\mathrm{vec}\, V]\, ds, \qquad t \in [0, \infty).$$

Since the integral term on the right-hand side of (7.9) is nonnegative definite, is bounded from above by Q, and $V \in \mathbb{N}^n$ is arbitrary, it follows that $\mathscr{A}$ is asymptotically stable. □

We now show that if $\mathscr{A}$ is asymptotically stable then actually A_α (and thus A) is asymptotically stable. This shows that the assumption that $\mathscr{A}$ is asymptotically stable is consistent with the original hypothesis that A is asymptotically stable.

PROPOSITION 7.2. *Assume $\mathscr{A}$ is asymptotically stable, let $\alpha_i' \in [0, \alpha_i]$, $i = 1, \cdots, p$, and define*

$$\mathscr{A}' \triangleq A_\alpha \oplus A_\alpha + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i \otimes A_i.$$

Then $\mathscr{A}'$ is also asymptotically stable. In particular, A_α and A are asymptotically stable.

Proof. Let $V \in \mathbb{N}^n$ be arbitrary and let Q be the unique, nonnegative-definite solution of (MLE4). Equivalently, Q satisfies

$$0 = A_\alpha Q + Q A_\alpha^T + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i Q A_i^T + V',$$

where

$$V' \triangleq \sum_{i=1}^{p} \alpha^{-1}(\alpha_i^2 - \alpha_i'^2) A_i Q A_i^T + V.$$

[1] The exponential product formula is essential to the proof here since (1) $A_\alpha \oplus A_\alpha$ cannot be expressed as a Kronecker product $S \otimes S$, and (2) $A_\alpha \oplus A_\alpha$ and $\sum_{i=1}^{p} \gamma_i A_i \otimes A_i$ do not generally commute.

Since $V' \in \mathbb{N}^n$, the stability of $\mathscr{A}'$ now follows as in the proof of the converse of Proposition 7.1. Finally, if V is chosen to be positive definite then $\sum_{i=1}^{p} (\alpha_i'^2/\alpha) A_i Q A_i^T + V'$ is also positive definite and it follows from Lemma 12.2 of [39] that A_α, and hence A, is asymptotically stable. □

Hence it follows from Proposition 7.2 that a necessary condition for $\mathscr{A}$ to be asymptotically stable is that

$$\alpha < 2 \max_{i=1,\cdots,n} \operatorname{Re} \lambda_i(A). \tag{7.10}$$

We now have the following monotonicity result.

PROPOSITION 7.3. *Let $\mathscr{U}_2' \subset \mathscr{U}_2$, where $\mathscr{U}_2'$ is defined as in (5.7) with α_i replaced by $\alpha_i' \in [0, \alpha_i]$, $i = 1, \cdots, p$. Furthermore, let $V \in \mathbb{P}^n$, assume $\mathscr{A}$ is asymptotically stable, and let $Q \in \mathbb{P}^n$ satisfy* (MLE4). *Then there exists $Q' \in \mathbb{P}^n$ satisfying*

$$0 = A_\alpha Q' + Q' A_\alpha^T + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i Q' A_i^T + V, \tag{7.11}$$

and, furthermore,

$$Q' \leq Q. \tag{7.12}$$

Consequently,

$$\operatorname{tr} Q'R \leq \operatorname{tr} QR, \tag{7.13}$$

$$\lambda_{\max}(Q'R) \leq \lambda_{\max}(QR). \tag{7.14}$$

Proof. Subtracting (7.11) from (MLE4) yields

$$0 = A_\alpha(Q - Q') + (Q - Q')A_\alpha^T + \sum_{i=1}^{p} \left(\frac{\alpha_i'^2}{\alpha}\right) A_i (Q - Q') A_i^T + V',$$

where V' is defined in the proof of Proposition 7.2. Since, by the converse portion of Proposition 7.1, $\mathscr{A}'$ is asymptotically stable, $Q - Q' \geqq 0$, which yields (7.12) and thus (7.13) and (7.14). □

Returning now to the existence question, Proposition 7.1 shows that a solution to (MLE4) exists so long as $\alpha_1, \cdots, \alpha_p$ are sufficiently small such that $\mathscr{A}$ remains asymptotically stable for some $\alpha > 0$. To this end we can treat this as a stability perturbation problem and apply results from [3]. Within our modified Lyapunov equation approach we have the following related result. For this and the following result let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^{n^2}$ and the corresponding induced matrix norm.

PROPOSITION 7.4. *If*

$$\left\| (A \oplus A)^{-1} \left(\alpha I_{n^2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i \otimes A_i \right) \right\| < 1, \tag{7.15}$$

then for all $V \in \mathbb{N}^n$ there exists $Q \in \mathbb{N}^n$ satisfying (MLE4) *and hence $\mathscr{A}$ is asymptotically stable.*

Proof. Define $\{Q_k\}_{k=0}^{\infty}$ where Q_0 satisfies (3.14) and Q_{k+1} satisfies

$$0 = AQ_{k+1} + Q_{k+1}A^T + \Omega_7(Q_k) + V.$$

Note that $Q_k \geqq 0$, $k = 1, 2, \cdots$. Hence it follows that

$$\operatorname{vec} Q_{k+1} - \operatorname{vec} Q_k = -(A \oplus A)^{-1}[\operatorname{vec} \Omega_7(Q_k) - \operatorname{vec} \Omega_7(Q_{k-1})]$$

and thus

$$\|\operatorname{vec} Q_{k+1} - \operatorname{vec} Q_k\| \leqq \left\| (A \oplus A)^{-1}\left(\alpha I_{n^2} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 A_i \otimes A_i\right) \right\| \|\operatorname{vec} Q_k - \operatorname{vec} Q_{k-1}\|.$$

Using (7.15) it follows that $Q \triangleq \lim_{k\to\infty} Q_k$ exists. Thus $Q \geqq 0$ and satisfies (MLE4). Finally, by the converse of Proposition 7.1, $\mathscr{A}$ is asymptotically stable. □

Since (MLE5) is nonlinear, a slightly different approach is required for existence. For the following result let $\kappa, \beta > 0$ satisfy

$$\|e^{At}\| \leqq \kappa e^{-\beta t}, \qquad t \geqq 0, \tag{7.16}$$

where $\|\cdot\|$ denotes an arbitrary submultiplicative matrix norm that is monotonic on $\mathbb{N}^n$, and define $\rho \triangleq 2\beta/\kappa^2$.

PROPOSITION 7.5. *Suppose* $V \in \mathbb{N}^n$ *and*

$$4\alpha\|N\|\|\alpha^{-1}M + V\| < \rho^2. \tag{7.17}$$

Then there exists $Q \in \mathbb{N}^n$ *satisfying* (MLE5).

Proof. Consider the sequence $\{Q_k\}_{k=0}^{\infty}$ where Q_0 satisfies (3.14) and Q_{k+1} is given by

$$0 = AQ_{k+1} + Q_{k+1}A^T + \alpha Q_k N Q_k + \alpha^{-1}M + V.$$

Clearly, $Q_k \geqq 0$, $k = 0, 1, \cdots$. Next we have

$$Q_{k+1} = \int_0^\infty e^{At}[\alpha Q_k N Q_k + \alpha^{-1}M + V]e^{A^T t}\, dt, \tag{7.18}$$

which yields

$$\|Q_{k+1}\| \leqq \alpha\rho^{-1}\|N\|\|Q_k\|^2 + \rho^{-1}\|\alpha^{-1}M + V\|. \tag{7.19}$$

Similarly, from (3.14) we obtain

$$\|Q_0\| \leqq \rho^{-1}\|V\| \leqq \rho^{-1}\|\alpha^{-1}M + V\|.$$

Now suppose that

$$\|Q_k\| \leqq 2\rho^{-1}\|\alpha^{-1}M + V\|.$$

Then (7.17) and (7.19) imply

$$\begin{aligned}\|Q_{k+1}\| &\leqq \alpha\rho^{-1}\|N\|[2\rho^{-1}\|\alpha^{-1}M + V\|]^2 + \rho^{-1}\|\alpha^{-1}M + V\| \\ &< 2\rho^{-1}\|\alpha^{-1}M + V\|.\end{aligned}$$

Thus $\|Q_k\| \leqq 2\rho^{-1}\|\alpha^{-1}M + V\|$, $k = 0, 1, \cdots$. Next, (7.18) yields

$$\begin{aligned}Q_{k+1} - Q_k &= \alpha \int_0^\infty e^{At}[Q_k N Q_k - Q_{k-1} N Q_{k-1}]e^{A^T t}\, dt \\ &= \alpha \int_0^\infty e^{At}[Q_k N(Q_k - Q_{k-1}) + (Q_k - Q_{k-1}) N Q_{k-1}]e^{A^T t}\, dt\end{aligned}$$

and thus

$$\begin{aligned}\|Q_{k+1}-Q_k\| &\leq \alpha\rho^{-1}\|N\|(\|Q_k\|+\|Q_{k-1}\|)\|Q_k-Q_{k-1}\|\\ &\leq 4\alpha\rho^{-2}\|N\|\|\alpha^{-1}M+V\|\|Q_k-Q_{k-1}\|\\ &\leq \varepsilon\|Q_k-Q_{k+1}\|,\end{aligned}$$

where $\varepsilon \triangleq 4\alpha\rho^{-2}\|N\|\|\alpha^{-1}M+V\|$. Since by (7.17) $\varepsilon<1$, $\lim_{k\to\infty} Q_k$ exists, is nonnegative definite, and satisfies (MLE5). □

8. Additional upper bounds via recursive substitution. In this section we obtain additional upper bounds for $J_S(\mathcal{U})$ and $J_D(\mathcal{U})$ by utilizing a recursive substitution technique. The main idea involves rewriting (2.7) as

$$Q_{\Delta A}=-\mathrm{vec}^{-1}\{(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\,\mathrm{vec}\,Q_{\Delta A}\}+Q_0 \tag{8.1}$$

and substituting this expression into the terms $\Delta A Q_{\Delta A}+Q_{\Delta A}\Delta A^T$ appearing in (2.7). This technique yields an equation that is, as expected, equivalent to (2.7) but that permits the development of additional bounds. As will be seen, the ability to develop new bounds exploits the fact that the substitution technique leads to terms that are quadratic in ΔA. We begin the development with the following technical result that does not require that A be asymptotically stable.

PROPOSITION 8.1. *Suppose $A\oplus A$ is invertible and let $\Delta A\in\mathbb{R}^{n\times n}$. If $Q_{\Delta A}$ satisfies (2.7), then $Q_{\Delta A}$ also satisfies*

$$\begin{aligned}0=AQ_{\Delta A}+Q_{\Delta A}A^T-\mathrm{vec}^{-1}[(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\,\mathrm{vec}\,Q_{\Delta A}\\ +(\Delta A\oplus\Delta A)(A\oplus A)^{-1}\,\mathrm{vec}\,V]+V.\end{aligned} \tag{8.2}$$

Conversely, if $Q_{\Delta A}$ satisfies (8.2) and $(A-\Delta A)\oplus(A-\Delta A)$ is invertible, then $Q_{\Delta A}$ also satisfies (2.7).

Proof. To obtain (8.2) substitute (8.1) into (2.7) as noted above. Conversely, adding the zero term $(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(A\oplus A)\,\mathrm{vec}\,Q_{\Delta A}-(\Delta A\oplus\Delta A)\,\mathrm{vec}\,Q_{\Delta A}$ to (8.2), it follows that (8.2) can be written as

$$0=[(A-\Delta A)\oplus(A-\Delta A)](A\oplus A)^{-1}[(A+\Delta A)\oplus(A+\Delta A)\,\mathrm{vec}\,Q_{\Delta A}+\mathrm{vec}\,V],$$

which, under the invertibility assumption, implies that $Q_{\Delta A}$ satisfies (2.7). □

The following result is analogous to Theorem 3.1. We shall say that $\mathcal{U}$ is symmetric if $\Delta A\in\mathcal{U}$ implies $-\Delta A\in\mathcal{U}$.

THEOREM 8.1. *Suppose $\mathcal{U}$ is symmetric, let $\Omega_0\in\mathbb{N}^n$ satisfy*

$$\Delta A Q_0+Q_0\Delta A^T\leq\Omega_0,\qquad \Delta A\in\mathcal{U}, \tag{8.3}$$

where Q_0 satisfies (3.14), let $\hat{\Omega}:\mathbb{N}^n\to\mathbb{N}^n$ satisfy

$$-\mathrm{vec}^{-1}[(\Delta A\oplus\Delta A)(A\oplus A)^{-1}(\Delta A\oplus\Delta A)\,\mathrm{vec}\,Q]\leq\hat{\Omega}(Q),\qquad \Delta A\in\mathcal{U},\quad Q\in\mathbb{N}^n, \tag{8.4}$$

and suppose there exists $Q\in\mathbb{N}^n$ satisfying

$$0=AQ+QA^T+\hat{\Omega}(Q)+\Omega_0+V. \tag{8.5}$$

Then

$$(A+\Delta A, D)\ \textit{is stabilizable},\qquad \Delta A\in\mathcal{U}, \tag{8.6}$$

if and only if

(8.7) $$A+\Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathscr{U}.$$

In this case,

(8.8) $$Q_{\Delta A} \leqq Q, \quad \Delta A \in \mathscr{U},$$

where $Q_{\Delta A}$ satisfies (2.7), *and*

(8.9) $$J_S(\mathscr{U}) \leqq \operatorname{tr} QR,$$

(8.10) $$J_D(\mathscr{U}) \leqq \lambda_{\max}(QR).$$

Proof. The equivalence of (8.6) and (8.7) follows from (8.5) as in the proof of Theorem 3.1. Next (8.8) follows by comparing (8.5) and (8.2) while using (8.3) and (8.4). Since $\mathscr{U}$ is assumed to be symmetric, it follows from (8.7) that $A-\Delta A$ is asymptotically stable, $\Delta A \in \mathscr{U}$, and hence $(A-\Delta A) \ominus (A-\Delta A)$ is invertible, $\Delta A \in \mathscr{U}$. Thus, the converse portion of Proposition 8.1 implies that $Q_{\Delta A}$ satisfying (8.2) also satisfies (2.7). Thus, the bound (8.8) can be used to obtain (8.9) and (8.10). □

The principal difference between (8.4) and (3.1) is that ΔA appears linearly in (3.1), whereas it appears quadratically in (8.4). By exploiting this structure we can obtain new bounds for $Q_{\Delta A}$. To simplify matters, we now consider the bound in (8.4) in two special cases. In the first case we set $\mathscr{U}=\mathscr{U}_1$ and $p=1$ so that $\Delta A=\sigma_1 A_1$, $|\sigma_1| \leq \delta_1$. In this case (8.4) becomes

(8.11)
$$-\sigma_1^2 \operatorname{vec}^{-1}[(A_1 \ominus A_1)(A \ominus A)^{-1}(A_1 \ominus A_1)\operatorname{vec} Q] \leqq \hat{\Omega}(Q), \qquad |\sigma_1| \leqq \delta_1, \quad Q \in \mathbb{N}^n.$$

One choice of $\hat{\Omega}(\cdot)$ that immediately suggests itself can be obtained by defining the matrix function $|\cdot|_+$ on the set of symmetric matrices by

(8.12) $$|S|_+ \triangleq \tfrac{1}{2}(S+|S|),$$

which effectively replaces the negative eigenvalues of S by zeros. We shall thus utilize the fact that

(8.13) $$\sigma_1^2 S \leqq \delta_1^2 |S|_+, \qquad |\sigma_1| \leqq \delta_1,$$

for all symmetric S.

COROLLARY 8.1. *Let $V \in \mathbb{P}^n$, $\mathscr{U}=\mathscr{U}_1$, $p=1$, let $\Omega_0 \in \mathbb{N}^n$ satisfy* (8.3), *and suppose there exists $Q \in \mathbb{N}^n$ satisfying*

(8.14) $$0=AQ+QA^T+\delta_1^2|-\operatorname{vec}^{-1}[(A_1 \ominus A_1)(A \ominus A)^{-1}(A_1 \ominus A_1)\operatorname{vec} Q]|_+ + \Omega_0 + V.$$

Then (8.7)–(8.10) *are satisfied.*

For the next specialization we shall assume that

(8.15) $$(\Delta A)A=A(\Delta A), \qquad \Delta A \in \mathscr{U},$$

which holds, for example, for modal systems with frequency uncertainty (see § 10). It thus follows that $(A \ominus A)^{-1}(\Delta A \ominus \Delta A)=(\Delta A \ominus \Delta A)(A \ominus A)^{-1}$ and thus (8.4) can be rewritten as

(8.16) $$\Delta A^2 \hat{Q} + 2\Delta A \hat{Q} \Delta A^T + \hat{Q} \Delta A^{2T} \leqq \hat{\Omega}(Q), \qquad \Delta A \in \mathscr{U}, \quad Q \in \mathbb{N}^n,$$

where $\hat{Q} \in \mathbb{N}^n$ satisfies

(8.17) $$0=A\hat{Q}+\hat{Q}A^T+Q.$$

Assuming in addition to (8.15) that $\Delta A = \sigma_1 A_1$, $|\sigma_1| \leqq \delta_1$, (8.14) becomes

$$0 = AQ + QA^T + \delta_1^2 |A_1^2\hat{Q} + 2A_1\hat{Q}A_1^T + \hat{Q}A_1^{2T}|_+ + \Omega_0 + V. \tag{8.18}$$

Remark 8.1. It is interesting to note that the left-hand side of (8.16) is of the same form as $\Omega_{4'}(\cdot)$. Specifically, the term $\Delta A^2\hat{Q} + \hat{Q}\Delta A^{2T}$ is analogous to $A_i^2 Q + QA_i^{2T}$ whereas $2\Delta A\hat{Q}\Delta A^T$ is similar to $A_i Q A_i^T$.

9. An alternative approach yielding upper and lower bounds. In this section we develop a variation on the results of § 3 that has the additional benefit of yielding both upper and lower performance bounds. The basic approach was suggested by results obtained in [44]. To simplify the presentation we assume as in the preceding section that $\mathcal{U}$ is symmetric. This symmetry assumption of course holds for all of the uncertainty sets considered in previous sections. The underlying idea involves bounding the deviation of $Q_{\Delta A}$ from Q_0 rather than bounding $Q_{\Delta A}$ directly.

THEOREM 9.1. *Let $\Omega_0 \in \mathbb{N}^n$ satisfy*

$$\Delta A Q_0 + Q_0 \Delta A^T \leqq \Omega_0, \qquad \Delta A \in \mathcal{U}, \tag{9.1}$$

let $\Omega: \mathbb{N}^n \to \mathbb{N}^n$ be such that (3.1) is satisfied, and suppose there exists $\Delta\mathcal{Q} \in \mathbb{N}^n$ satisfying

$$0 = A\Delta\mathcal{Q} + \Delta\mathcal{Q}A^T + \Omega(\Delta\mathcal{Q}) + \Omega_0. \tag{9.2}$$

Then

$$(A + \Delta A, \Omega_0^{1/2}) \text{ is stabilizable}, \qquad \Delta A \in \mathcal{U}, \tag{9.3}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \qquad \Delta A \in \mathcal{U}. \tag{9.4}$$

In this case,

$$Q_0 - \Delta\mathcal{Q} \leqq Q_{\Delta A} \leqq Q_0 + \Delta\mathcal{Q}, \qquad \Delta A \in \mathcal{U}, \tag{9.5}$$

where $Q_{\Delta A}$ is given by (2.7), *and*

$$\operatorname{tr}(Q_0 + \Delta\mathcal{Q})R \leqq J_S(\mathcal{U}) \leqq \operatorname{tr}(Q_0 + \Delta\mathcal{Q})R, \tag{9.6}$$

$$\lambda_{\max}[(Q_0 - \Delta\mathcal{Q})R] \leqq J_D(\mathcal{U}) \leqq \lambda_{\max}[(Q_0 + \Delta\mathcal{Q})R]. \tag{9.7}$$

Proof. Define

$$\Delta Q \triangleq Q_{\Delta A} - Q_0 \tag{9.8}$$

and subtract (3.14) from (2.7) to obtain

$$0 = (A + \Delta A)\Delta Q + \Delta Q(A + \Delta A)^T + \Delta A Q_0 + Q_0 \Delta A^T. \tag{9.9}$$

Now rewrite (9.2) as

$$0 = (A + \Delta A)\Delta\mathcal{Q} + \Delta\mathcal{Q}(A + \Delta A)^T + \Omega(\Delta\mathcal{Q}) - (\Delta A\Delta\mathcal{Q} + \Delta\mathcal{Q}\Delta A^T) + \Omega_0. \tag{9.10}$$

Using (9.10), the equivalence of (9.3) and (9.4) is immediate as in the proof of Theorem 3.1. Next, subtracting (9.9) from (9.10) yields

$$\begin{aligned} 0 = {} & (A + \Delta A)(\Delta\mathcal{Q} - \Delta Q) + (\Delta\mathcal{Q} - \Delta Q)(A + \Delta A)^T + \Omega(\Delta\mathcal{Q}) \\ & - (\Delta A\Delta\mathcal{Q} + \Delta\mathcal{Q}\Delta A^T) + \Omega_0 - (\Delta A Q_0 + Q_0 \Delta A^T). \end{aligned} \tag{9.11}$$

Using (3.1) and (9.1) it follows from (9.11) that

$$\Delta\mathcal{Q} - \Delta Q \geqq 0,$$

or, equivalently,

$$Q_{\Delta A} \leqq Q_0 + \Delta\mathcal{Q}. \tag{9.12}$$

To obtain the lower bound rewrite (9.9) as

$$0 = (A+\Delta A)(-\Delta Q) + (-\Delta Q)(A+\Delta A)^T - (\Delta A Q_0 + Q_0 \Delta A^T). \tag{9.13}$$

Also, note that because of the assumed symmetry of $\mathcal{U}$, (9.1) holds with ΔA appearing in the inequality replaced by $-\Delta A$. Hence it can be shown similarly that

$$\Delta\mathcal{Q} + \Delta Q \geqq 0,$$

or, equivalently,

$$Q_0 - \Delta\mathcal{Q} \leqq Q_{\Delta A}. \tag{9.14}$$

Finally, (9.6) and (9.7) follow from (9.5). □

Remark 9.1. To compare the upper bound in (9.5) with (3.5), rewrite (9.2) as

$$0 = A(Q_0 + \Delta\mathcal{Q}) + (Q_0 + \Delta\mathcal{Q})A^T + \Omega(\Delta\mathcal{Q}) + \Omega_0 + V. \tag{9.15}$$

If $\Omega(\Delta\mathcal{Q}) + \Omega_0 = \Omega(Q_0 + \Delta\mathcal{Q})$ then (9.15) has the same form as (3.2) and thus the two upper bounds are identical. This will be the case, for example, if $\Omega(\cdot) = \Omega_7(\cdot)$ and Ω_0 is chosen to be $\Omega_7(Q_0)$ since $\Omega_7(\cdot)$ is linear. If, however, $\Omega(\Delta\mathcal{Q}) + \Omega_0 < \Omega(Q_0 + \Delta\mathcal{Q})$ then the upper bound in (9.5) will be sharper. In any case it is clear that the individual treatment of $\Delta\mathcal{Q}$ and Q_0 yields potentially new upper bounds.

Remark 9.2. Theorem 9.1 does not guarantee that the lower bound $Q_0 - \Delta\mathcal{Q}$ for $Q_{\Delta A}$ is nonnegative definite. However, $Q_{\Delta A}$ is always nonnegative definite and thus the lower bound in (9.5) may be of limited usefulness. Nevertheless, if $Q_0 - \Delta\mathcal{Q}$ is indefinite then, depending on R, the lower bounds in (9.6) and (9.7) may still be positive and thus be meaningful lower bounds.

10. Analytical examples. In this section we consider simple analytical examples that illustrate the principal results of the paper. These examples also provide insight into the individual characteristics of different bounds as a prelude to numerical examples considered in the following section.

To begin we consider the simplest possible example. Set $n = 1$, $A < 0$, $R > 0$, $V > 0$, $A_1 = 1$, and $\mathcal{U} = \{\Delta A : |\Delta A| \leqq \delta_1\}$. For $\delta_1 < -A$, $Q_{\Delta A} = V/2(|A| - \Delta A)$ and $J_S(\mathcal{U}) = J_D(\mathcal{U}) = RV/2(|A| - \delta_1)$, where this worst-case performance is achieved for $\Delta A = \delta_1$. Solving (MLE1) yields $Q = V/2(|A| - \delta_1)$, which is a nonconservative result for both robust stability and performance. The same result is obtained from (MLE4) by setting $\alpha = \alpha_1 = \delta_1$. To apply (MLE5), set $\delta_1 = \sqrt{MN}$. Choosing $\alpha = 2\delta_1(|A| - \delta_1)NV$ again yields the nonconservative result. Finally, the same result follows from Theorem 8.1.

For the second example we consider nondestabilizing uncertainty in the imaginary component of an uncertain eigenvalue, i.e., frequency uncertainty, in contrast to uncertainty in the real part considered in the previous example. Let $n = 2$,

$$A = \begin{bmatrix} -\nu & \omega \\ -\omega & -\nu \end{bmatrix}, \qquad \nu > 0, \quad \omega \geqq 0,$$

$V = R = I_2$, and $\mathcal{U} = \{\Delta A : \Delta A = \sigma_1 A_1, |\sigma_1| \leqq \delta_1\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Obviously, $A + \Delta A$ remains asymptotically stable for all values of σ_1 since ΔA affects only the imaginary part of the poles of ΔA. The question then is whether the robustness tests are able to guarantee this robustness. Note also that because of the choice of V, $Q_{\Delta A} = Q_0 = (2\nu)^{-1} I_2$ for all $\Delta A \in \mathscr{U}$. For this example we note that (MLE1) is satisfied by $Q = (2\nu)^{-1} I_2$, which is independent of δ_1. Thus (MLE1) possesses a nonnegative-definite solution for all $\delta_1 > 0$, which shows that (MLE1) is nonconservative with respect to robust stability and performance. Since $A(\Delta A) = (\Delta A)A$, it can also be seen that the same result holds for (8.18). The situation is considerably different for (MLE4) and (MLE5). To analyze (MLE4) note that $\mathscr{A}$ has an eigenvalue $-2\nu + \alpha + \delta_1$. (This can be shown by diagonalizing A and A_1 and thus $\mathscr{A}$.) Since, by Proposition 7.1, $\mathscr{A}$ must be asymptotically stable, we require $\delta_1 < 2\nu$. This is, of course, an extremely conservative result, especially when the damping ν is small. For (MLE5) we can factor $A_1 = D_1 E_1$. Thus, let $D_1 = I_2$ and $E_1 = A_1$ and define $M = \delta_1^2 I_2$ and $N = I_2$. Assuming that Q is a multiple of I_2, it follows that Q is nonnegative definite only if $\delta_1 \leq \nu$, which is again an extremely conservative result. The reason for this conservatism becomes clear by noting that M and N as given above will also serve as bounds for perturbations of the form $\sigma_1 I_2$ for which the range of nondestabilizing σ_1 is $|\sigma_1| < \delta_1$. This will also be the case for all factorizations $D_1 E_1$ of A_1 since $D_1 D_1^T$ and $E_1^T E_1$ must be positive definite and thus will also serve as bounds for destabilizing perturbations such as $\sigma_1 I_2$.

Finally, we consider a nondestabilizing uncertainty affecting the interaction of a pair of real poles. Let $n = 2$, $A = -I_2$, $V = R = I_2$, and $\mathscr{U} = \{\Delta A : \Delta A = \sigma_1 A_1, |\sigma_1| \leq \delta_1\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $A + \Delta A$ remains asymptotically stable for all values of σ_1 since ΔA does not affect the nominal poles. Note that

$$Q_{\Delta A} = \begin{bmatrix} \sigma_1^2/4 + \frac{1}{2} & \sigma_1/4 \\ \sigma_1/4 & \frac{1}{2} \end{bmatrix}$$

and $J_S(\mathscr{U}) = \frac{1}{4}\delta_1^2 + 1$, where this worst-case performance is achieved for $\sigma_1 = \delta_1$. In this case (MLE1) has the solution $Q = (2 - \delta_1)^{-1} I_2$, which is valid only for $\delta_1 < 2$, an extremely conservative robust stability result. Furthermore, the corresponding performance bound $\operatorname{tr} QR = 2(2 - \delta_1)^{-1}$ is conservative with respect to the actual worst-case performance $\frac{1}{4}\delta_1^2 + 1$. In contrast, (MLE4) has the solution

$$Q = \begin{bmatrix} (2-\alpha\delta_1)^{-1} + \alpha^{-1}\delta_1(2-\alpha\delta_1)^{-2} & 0 \\ 0 & (2-\alpha\delta_1)^{-1} \end{bmatrix},$$

which is nonnegative definite for all δ_1 so long as $\alpha < 2/\delta_1$. Hence (MLE4) is nonconservative with respect to robust stability. For robust performance,

$$\operatorname{tr} QR = 2(2-\alpha\delta_1)^{-1} + \alpha^{-1}\delta_1(2-\alpha\delta_1)^{-2},$$

which can be shown to be an upper bound for $\frac{1}{4}\delta_1^2 + 1$. Choosing, for example, $\alpha = \delta_1^{-1}$ yields $\operatorname{tr} QR = \delta_1^2 + 2$. The parameter α can also be chosen to minimize $\operatorname{tr} QR$, although this is somewhat tedious to carry out analytically. Finally, (MLE5) has the solution

$$Q = \begin{bmatrix} \frac{1}{2}(1+\alpha^{-1}\delta_1) & 0 \\ 0 & [1-(1-\alpha\delta_1)^{1/2}]/\alpha\delta_1 \end{bmatrix},$$

which exists so long as $\alpha \leqq 1/\delta_1$. Hence (MLE5) is also nonconservative with respect to robust stability. Choosing $\alpha = 1/\delta_1$ yields $\operatorname{tr} QR = \frac{1}{2}\delta_1^2 + \frac{3}{2}$, which lies above the nonconservative bound $\frac{1}{4}\delta_1^2 + 1$. Again, α can be chosen to minimize $\operatorname{tr} QR$.

11. Numerical examples. In this section we consider additional examples illustrating the results developed in earlier sections. In contrast to the analytical examples considered in § 10, however, we consider more complex examples by numerically solving the modified Lyapunov equations. Here we focus on (MLE4) and (MLE5), which are the easiest to solve numerically. Specifically, we solved (MLE4) by using the representation (7.2) (although this may not be practical when n is large), and we solved (MLE5) by means of a standard Riccati package. To simplify matters we consider only uncertainties ΔA of the form $\sigma_1 A_1$. Evaluation and presentation of robust stability and performance results for multiparameter uncertainty can be fairly complex and thus are deferred to a future numerical study.

Since both robustness tests (MLE4) and (MLE5) depend on an arbitrary positive constant α, it is desirable to determine the value of α that yields the tightest (i.e., lowest) performance bound for each robust stability range. To this end we performed a simple one-dimensional search to determine the best such α. Although analytical techniques may assist in determining optimal values of α more efficiently, the search technique proved to be adequate for the examples considered here.

As a first example we consider the control system given in [1] to demonstrate the lack of a guaranteed gain margin for LQG controllers. Hence consider

$$\dot{x}_0(t) = A_0 x_0(t) + B_0 u(t) + w_1(t), \tag{11.1}$$

$$y(t) = C_0 x_0(t) + w_2(t), \tag{11.2}$$

with controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{11.3}$$

$$u(t) = C_c x_c(t), \tag{11.4}$$

and performance

$$J = \lim_{t \to \infty} \mathbb{E}[x_0^T(t) R_1 x_0(t) + u^T(t) R_2 u(t)]. \tag{11.5}$$

The data are

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = [1 \quad 0],$$

$$V_1 = R_1 = \rho \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad V_2 = R_2 = 1,$$

where V_1 and V_2 are the intensities of $w_1(t)$ and $w_2(t)$, respectively. Uncertainty ΔB_0 in B_0 is thus represented by $\sigma_1 B_1$, where $B_1 = [0 \ 1]^T$. Thus the closed-loop system corresponds to

$$A = \begin{bmatrix} A_0 & B_0 C_c \\ B_c C_0 & A_c \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & B_1 C_c \\ 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix},$$

where the zero in the (2, 2) block of R denotes the fact that we are considering the robust performance bound for the state regulation cost only. Choosing $\rho = 60$, it follows that the LQG gains are given by

$$A_c=\begin{bmatrix} -9 & 1 \\ -20 & -9 \end{bmatrix}, \quad B_c=\begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad C_c=[-10 \quad -10].$$

For this controller the actual stability region corresponds to $\sigma_1 \in (-.07, .01)$ so that the largest symmetric region about $\sigma_1 = 0$ is $|\sigma_1| < .01$. The worst-case performance over each stability region $|\sigma_1| < \delta_1$ is denoted by the solid line in Fig. 1, whereas the performance bounds obtained from (MLE4) and (MLE5) are shown for several values of δ_1. For (MLE5) we set $D_1 = [0\ 1\ 0\ 0]^T$ and $E_1 = [0\ 0\ C_c]$. Note that (MLE5) yields considerably tighter estimates of worst-case performance, particularly as δ_1 approaches .01. For (MLE4) optimal values of α were in the range .0012 to .0058, whereas for (MLE5) (with $\Omega_{10'}(\cdot)$, see (5.26)) α was in the range .0143 to .0020.

As a second example we consider a pair of nominally uncoupled oscillators with uncertain coupling. This example was considered in [45] using the majorant Lyapunov technique. Let

$$A=\begin{bmatrix} -\nu & \omega_1 & 0 & 0 \\ -\omega_1 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & \omega_2 \\ 0 & 0 & -\omega_2 & -\nu \end{bmatrix}, \qquad A_1=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\nu=.2, \quad \omega_1=.2, \quad \omega_2=1.8, \quad R=V=I_4,$$

and, for (MLE5), define $D_1 = A_1$ and $E_1 = I_4$. We consider bounds on $J_S(\mathcal{U})$ only.

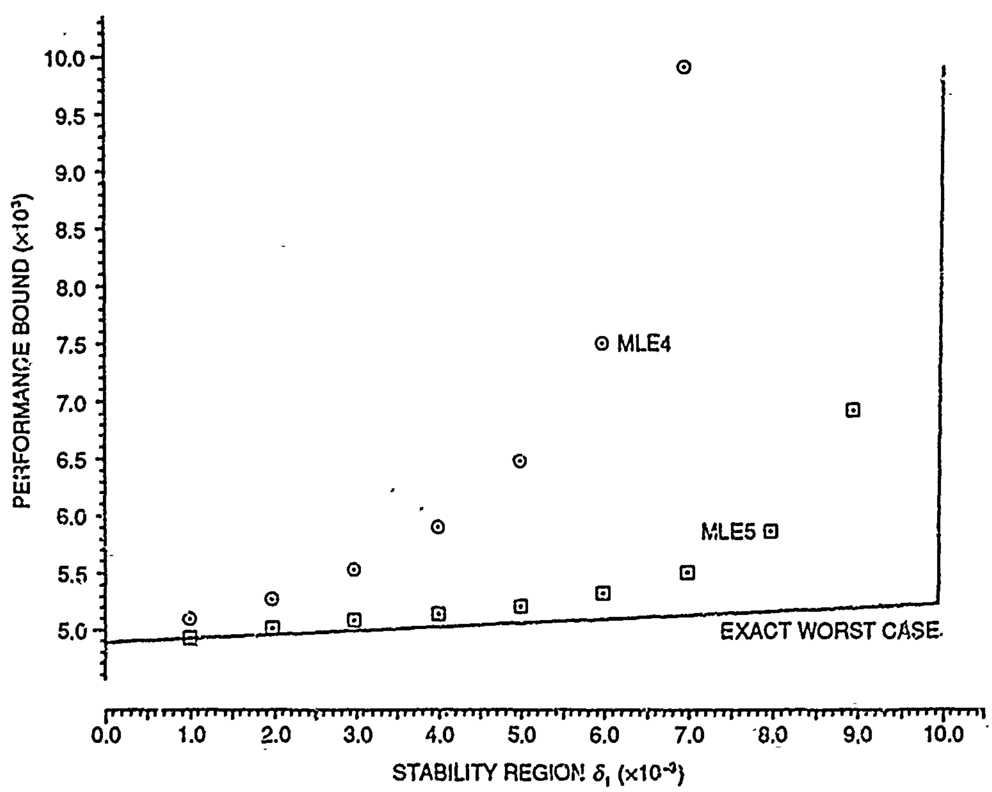

FIG. 1

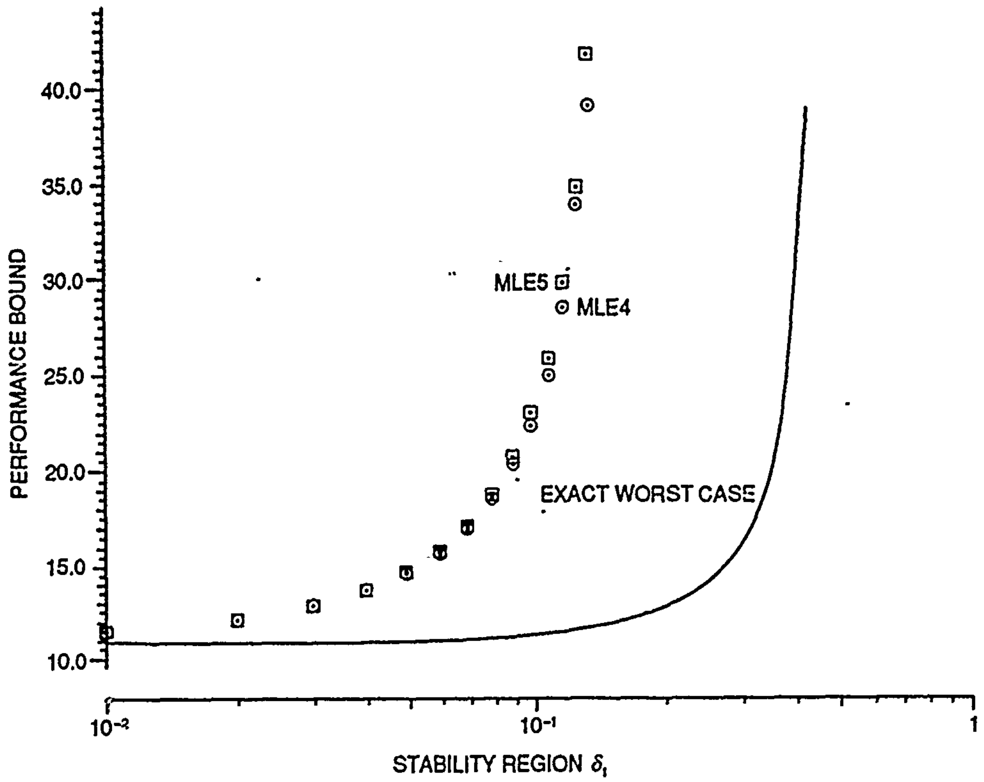

FIG. 2

Figure 2 illustrates the exact worst-case performance along with performance bounds obtained from (MLE4) and (MLE5). For (MLE4) optimal values of α ranged from .036 to .141, whereas for (MLE5) optimal α was between .361 and .096. Although (MLE4) was slightly less conservative than (MLE5), both bounds were able to guarantee robust stability only for $\delta_1 = .15$, whereas the largest stability region is actually $\delta_1 = .54$. It is interesting to contrast this result with [45] where the majorant Lyapunov technique yielded a robust stability range of $\delta_1 = .4$ for a richer class of off-diagonal blocks having maximum singular value less than δ_1.

12. Conclusion. A variety of quadratic Lyapunov bounds have been developed for both robust stability and performance. It seems clear, however, that no single quadratic Lyapunov bound is superior to the others. Although the conservatism of each bound is problem dependent, it is desirable to better understand the nature of the conservatism in order to utilize the bounds in an effective manner. In addition, the issue of *necessity* remains to be addressed. That is, if a system is robustly quadratically stable (i.e., robustly stable with a corresponding Lyapunov function), then is such a Lyapunov function necessarily given by one of the modified Lyapunov equations given in this paper? Furthermore, a better understanding is needed of the gap between robust stability and robust quadratic stability.

Acknowledgment. We thank A. W. Daubendiek for producing the numerical results in § 11.

Note added in proof. (1) The assumption $x(0) = 0$ in (2.2) is stronger than necessary for the treatment of (2.4). If $x(0) \neq 0$, then Lemma 2.1 remains unchanged since the

effect of $x(0)$ vanishes as $t \to \infty$. If, however, $x(0) = 0$, then $Q_{\Delta A}(t)$ is increasing on $[0, \infty)$ and (2.4) is equivalent to

$$J_S(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \sup_{t \in [0,\infty)} \mathbb{E}\{\|y(t)\|_2^2\} \leq \beta_S.$$

For $J_D(\mathscr{U})$, $x(0) = 0$ is essential since $\|y(\cdot)\|_{\infty,2}$ involves the supremum over $[0, \infty)$. If $x(0) \neq 0$, then the analysis can possibly be redone by considering the supremum over $[t, \infty)$ and letting $t \to \infty$ to eliminate the effect of the initial condition.

(2) A relationship between the linear bound $\Omega_7(\cdot)$ and the quadratic bound $\Omega_{10}(\cdot)$ can be seen as follows. If $\Delta A = \sigma_1 A_1$, $|\sigma_1| \leq \delta_1$, then factor $\Delta A = A_L A_R$ as in $\mathscr{U}_3$ according to $A_L = \sigma_1 A_1 Q^{1/2}$ and $A_R = Q^{-1/2}$ with bounds $M = \delta_1^2 A_1 Q A_1^T$ and $N = Q^{-1}$. The unusual feature here is that the "splitting" of ΔA is Q-dependent. Then, by (5.22),

$$\Omega_{10}(Q) = \alpha^{-1}\delta_1^2 A_1 Q A_1^T + \alpha Q,$$

which has the form of $\Omega_5(Q)$.

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Robust Controller Synthesis Using Kharitonov's Theorem

Dennis S. Bernstein
Harris Corporation
Government Aerospace Systems Division
MS 22/4842
Melbourne, FL 32902
(407) 729-2140

Wassim M. Haddad
Department of Mechanical and
Aerospace Engineering
Florida Institute of Technology
Melbourne, FL 32901
(407) 768-8000 Ext. 7241

Abstract

Kharitonov's theorem provides a necessary and sufficient *analysis* test for robust stability of polynomials whose coefficients lie within a hyperrectangle. In this note we present a method based upon Kharitonov's theorem for *synthesizing* robustly stabilizing feedback controllers. Our approach is based upon a multiple plant model formulation with a quadratic cost functional. Sufficient conditions are obtained for characterizing robustly stabilizing static output feedback (proportional) controllers for MIMO plants with denominator polynomial uncertainty.

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1. Introduction

Kharitonov's theorem provides a necessary and sufficient *analysis* test for determining the robust stability of polynomials with perturbed coefficients [1]. Although Kharitonov's original result was limited to uncertain polynomials having independently varying coefficients, considerable progress has been achieved in generalizing this result to more general regions [2–8]. Although these results are useful for analyzing the stability robustness of a given feedback control system, there are relatively few results that exploit Kharitonov's theorem for *synthesizing* robust controllers. Notable exceptions are [9–11] which give necessary and sufficient conditions for robust *stabilizability* of uncertain plants.

The goal of the present paper is to develop sufficient conditions that can be used for synthesizing such robustly stabilizing controllers. Our approach considers a class of MIMO systems in companion state space form with denominator polynomial uncertainty. By limiting the controller to be proportional, i.e., static output feedback, the hyperrectangular structure of the parameter uncertainty is preserved. Hence it suffices to simultaneously stabilize the four "plants" corresponding to Kharitonov's theorem.

Although there exists extensive literature on simultaneous stabilization (see [11] and the references therein), we adopt here a fixed-structure approach involving multiple models with a quadratic performance functional [12]. This approach allows us to develop reasonably general conditions for robust static output feedback (proportional control) synthesis. Although extensions to dynamic compensation are significantly more complex, we also show how SISO systems without zeros can be treated in a similar fashion by means of integral control.

2. Problem Formulation

We begin with a matrix formulation of Kharitonov's theorem. For $i = 0, \ldots, n-1$, let $\underline{\beta}_i$ and $\bar{\beta}_i$ be given uncertainty bounds with $\underline{\beta}_i \leq \bar{\beta}_i$.

Lemma 2.1. Consider the set of matrices

$$\mathcal{A} \triangleq \left\{ \begin{bmatrix} 0_{(n-1)\times 1} & & I_{n-1} \\ -\beta_0 & \cdots & -\beta_{n-1} \end{bmatrix} : \; \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \quad i = 0, \ldots, n-1 \right\}.$$

Then every matrix in $\mathcal{A}$ is stable if and only if the four matrices

$$A_1 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \underline{\beta}_{n-4} - \underline{\beta}_{n-3} & -\bar{\beta}_{n-2} - \bar{\beta}_{n-1} \end{bmatrix}, \quad A_2 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \bar{\beta}_{n-4} - \underline{\beta}_{n-3} & -\underline{\beta}_{n-2} - \bar{\beta}_{n-1} \end{bmatrix},$$

$$A_3 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \bar{\beta}_{n-4} - \bar{\beta}_{n-3} & -\underline{\beta}_{n-2} - \underline{\beta}_{n-1} \end{bmatrix}, \quad A_4 \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \cdots - \underline{\beta}_{n-4} - \bar{\beta}_{n-3} & -\bar{\beta}_{n-2} - \underline{\beta}_{n-1} \end{bmatrix}$$

are stable.

Remark 2.1. As noted in [3], simplification is possible if $n = 2, 3, 4$. If $n = 2$, then it suffices to check A_3. If $n = 3$, then it suffices to check A_3 and $\underline{\beta}_0 > 0$. Hence, for $n = 3$, either of the pairs (A_3, A_1) or (A_3, A_2) suffices. If $n = 4$, then it suffices to check A_2, A_3, and $\underline{\beta}_0 > 0$. Hence either of the triples (A_1, A_2, A_3) or (A_2, A_3, A_4) suffices. Simplification to these cases of the results given in later sections is obvious and thus will not be noted explicitly.

For the statement of the Robust Controller Synthesis Problem, let A, B, C denote $n\times n$, $n\times m$, and $\ell \times n$ matrices, respectively, and let $x = x(t)$, $u = u(t)$, and $y = y(t)$ denote $n-$, $m-$, and $\ell-$dimensional vectors, respectively.

Robust Controller Synthesis Problem. Consider the dynamical system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{2.1}$$

$$y = Cx, \tag{2.2}$$

where $A \in \mathcal{A}$. Then determine an output feedback control law of the form

$$u = Ky \tag{2.3}$$

such that the closed-loop system

$$\dot{x} = (A + BKC)x \tag{2.4}$$

is stable for all $A \in \mathcal{A}$.

The key step in exploiting Lemma 2.1 is to assume that the $n \times m$ matrix B has the form

$$B = \begin{bmatrix} 0_{(n-1)\times m} \\ b \end{bmatrix},$$

where b has dimensions $1 \times m$. No assumptions are needed concerning the structure of C. Then we have the following corollary of Lemma 2.1.

Corollary 2.1. Let K be a given $m \times \ell$ matrix and consider the set of matrices

$$\tilde{\mathcal{A}} \triangleq \{A + BKC\colon\ A \in \mathcal{A}\}.$$

Then every matrix in $\tilde{\mathcal{A}}$ is stable if and only if the four matrices $\tilde{A} \triangleq A_i + BKC,\ i = 1, \ldots, 4$, are stable.

Proof. Every matrix in $\tilde{\mathcal{A}}$ is of the form

$$\begin{bmatrix} 0_{(n-1)\times 1} & & I_{n-1} \\ -\beta_0 + bKC_1 & -\beta_1 + bKC_2 \cdots & -\beta_{n-1} + bKC_n \end{bmatrix},$$

where C_i is the ith column of C. Defining, for $i = 0, \ldots, n-1$,

$$\delta_i \triangleq \beta_i - bKC_{i+1}, \quad \underline{\delta}_i \triangleq \underline{\beta}_i - bKC_{i+1}, \quad \bar{\delta}_i \triangleq \bar{\beta}_i - bKC_{i+1},$$

it follows that $\tilde{\mathcal{A}}$ can be written as

$$\tilde{\mathcal{A}} = \left\{ \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ -\delta_0 & -\delta_1 \cdots - \delta_{n-1} \end{bmatrix} :\ \underline{\delta}_i \le \delta_i \le \bar{\delta}_i, \quad i = 0, \ldots, n-1 \right\}.$$

Now note that the closed-loop matrices in $\tilde{\mathcal{A}}$ have the same structure as the open-loop matrices in $\mathcal{A}$. Furthermore, the matrices $\tilde{A}_1, \ldots, \tilde{A}_4$ now play the same role as $A_1, \ldots, A_4$ with uncertain parameters $\beta_0, \ldots, \beta_{n-1}$, lower bounds $\underline{\beta}_0, \ldots, \underline{\beta}_{n-1}$, and upper bounds $\bar{\beta}_0, \ldots, \bar{\beta}_{n-1}$ replaced by $\delta_0, \ldots, \delta_{n-1}$, lower bounds $\underline{\delta}_0, \ldots, \underline{\delta}_{n-1}$, and upper bounds $\bar{\delta}_0, \ldots, \bar{\delta}_{n-1}$, respectively. □

Next, as in [12], we consider an augmented system of dimension $4n$ that simultaneously includes the dynamics of $\tilde{A}_1, \ldots, \tilde{A}_4$. Specifically, consider

$$\dot{\tilde{x}} = \tilde{A}_a \tilde{x}, \tag{2.5}$$

where

$$\tilde{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \tilde{A}_a = \begin{bmatrix} \tilde{A}_1 & 0 & 0 & 0 \\ 0 & \tilde{A}_2 & 0 & 0 \\ 0 & 0 & \tilde{A}_3 & 0 \\ 0 & 0 & 0 & \tilde{A}_4 \end{bmatrix}.$$

Note that

$$\tilde{A}_a = A_a + \sum_{i=1}^{4} B_{ia} K C_{ia}, \tag{2.6}$$

where

$$A_a \triangleq \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

$$B_{1a} \triangleq \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{2a} \triangleq \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix}, \quad B_{3a} \triangleq \begin{bmatrix} 0 \\ 0 \\ B \\ 0 \end{bmatrix}, \quad B_{4a} \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ B \end{bmatrix},$$

$$C_{1a} \triangleq [C \quad 0 \quad 0 \quad 0], \qquad C_{2a} \triangleq [0 \quad C \quad 0 \quad 0],$$

$$C_{3a} \triangleq [0 \quad 0 \quad C \quad 0], \qquad C_{4a} \triangleq [0 \quad 0 \quad 0 \quad C].$$

(Note our notation scheme: $(\tilde{\ })$ denotes closed loop, $(\ )_a$ denotes augmented system.)

We now turn to the problem of explicitly synthesizing a controller that stabilizes (2.5) and hence the original system (2.1), (2.2) for all $A \in \mathcal{A}$.

3. Controller Synthesis via Quadratically Optimal Control

To synthesize a controller for the system (2.5), we consider a quadratic performance functional of the form

$$J(K) \triangleq \sum_{i=1}^{4} \int_0^\infty \left[x_i^{\mathrm{T}} R_1 x_i + u_i^{\mathrm{T}} R_2 u_i \right] \mathrm{d}t, \tag{4.1}$$

where R_1 and R_2 are $n \times n$ and $m \times m$ positive-definite matrices, and u_i is defined by

$$u_i \triangleq KCx_i.$$

It now follows that $J(K)$ is given by

$$J(K) = \int_0^\infty \tilde{x}^{\mathrm{T}} \tilde{R}_a \tilde{x} \ \mathrm{d}t, \tag{4.2}$$

where

$$\tilde{R}_a \triangleq \begin{bmatrix} R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 & 0 & 0 \\ 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 & 0 \\ 0 & 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC & 0 \\ 0 & 0 & 0 & R_1 + (KC)^{\mathrm{T}} R_2 KC \end{bmatrix}.$$

Note that $\tilde{R}_a = R_a + \sum_{i=1}^{4} C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 K C_{ia}$, where R_a is defined by

$$R_a \triangleq \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.$$

Writing

$$\tilde{x}(t) = e^{\tilde{A}_a t} \tilde{x}_0, \quad \tilde{x}_0 \triangleq \begin{bmatrix} x_0 \\ x_0 \\ x_0 \\ x_0 \end{bmatrix}, \tag{4.3}$$

leads to

$$J(K) = \int_0^\infty \tilde{x}_0^{\mathrm{T}} e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t} \tilde{x}_0 \, \mathrm{d}t, \tag{4.4}$$

where we are now assuming that K is such that $\tilde{A}_a$ is stable. Now, as is common practice [13], we eliminate explicit dependence on the initial condition x_0 by assuming $x_0 x_0^{\mathrm{T}}$ has expected value I_n ($n \times n$ identity). Invoking this step leads to

$$\begin{aligned} J(K) &= \mathbb{E}\left[\operatorname{tr} \int_0^\infty \tilde{x}_0 \tilde{x}_0^{\mathrm{T}} e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t} \, \mathrm{d}t \right] \\ &= \operatorname{tr} I_{4n} \int_0^\infty e^{\tilde{A}_a^{\mathrm{T}} t} \tilde{R}_a e^{\tilde{A}_a t} \, \mathrm{d}t \\ &= \operatorname{tr} \tilde{P}, \end{aligned}$$

where $\tilde{P}$ is the $4n \times 4n$ positive-definite solution to

$$0 = \tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a. \tag{4.5}$$

To minimize $J(K)$ we form the Lagrangian

$$\mathcal{L}(K, \tilde{P}, \tilde{Q}) \triangleq \operatorname{tr}\left[\tilde{P} + \tilde{Q}(\tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a) \right],$$

where $\tilde{Q}$ is a $4n \times 4n$ Lagrange multiplier matrix. Note that

$$\mathcal{L}(K, \tilde{P}, \tilde{Q}) = \operatorname{tr}\left[\tilde{P} + 2A_a \tilde{Q} \tilde{P} + 2\sum_{i=1}^{4} B_{ia} K C_{ia} \tilde{Q} \tilde{P} + \tilde{Q} R_a + \sum_{i=1}^{4} \tilde{Q} C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 K C_{ia} \right].$$

Hence

$$\frac{\partial \mathcal{L}}{\partial K} = 2\sum_{i=1}^{4} \left[C_{ia} \tilde{Q} \tilde{P} B_{ia} + C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} K^{\mathrm{T}} R_2 \right]$$

so that $\partial \mathcal{L} / \partial K = 0$ yields

$$K = -R_2^{-1} \left[\sum_{i=1}^{4} B_{ia}^{\mathrm{T}} \tilde{P} \tilde{Q} C_{ia}^{\mathrm{T}} \right] \left[\sum_{i=1}^{4} C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} \right]^{-1}. \tag{4.6}$$

Similarly, evaluating $\partial K / \partial \tilde{P} = 0$ yields

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^{\mathrm{T}} + I_{4n}. \tag{4.7}$$

We thus have the following result.

Theorem 3.1. Let K be a feedback gain that stabilizes (2.5) and minimizes (4.1). Then K is given by

$$K = -R_2^{-1} \left[\sum_{i=1}^{4} B_{ia}^{\mathrm{T}} \tilde{P} \tilde{Q} C_{ia}^{\mathrm{T}} \right] \left[\sum_{i=1}^{4} C_{ia} \tilde{Q} C_{ia}^{\mathrm{T}} \right]^{-1}, \tag{4.8}$$

where $\tilde{Q}$ and $\tilde{P}$ satisfy

$$0 = \tilde{A}_a \tilde{Q} + \tilde{Q} \tilde{A}_a^{\mathrm{T}} + I_{4n}, \tag{4.9}$$

$$0 = \tilde{A}_a^{\mathrm{T}} \tilde{P} + \tilde{P} \tilde{A}_a + \tilde{R}_a. \tag{4.10}$$

Note that the matrices $\tilde{A}$ and $\tilde{R}_a$ depend upon K so that (4.8)–(4.10) must be solved numerically together. The expression (4.8) for K can be substituted into (4.9), (4.10) to eliminate this dependence. This optimal static output feedback solution is essentially a generalization of the standard theory [13].

4. Sufficiency Result for Robust Stability

Since (4.9) and (4.10) are Lyapunov equations, they guarantee the stability of $\tilde{A}_a$ when they have solutions. Since, furthermore, $\tilde{A}$ is a block-diagonal matrix, each of its (four) diagonal blocks will be stable. Finally, by Corollary 2.1, the stability of these four matrices is sufficient to guarantee the stability of the closed-loop system (2.4) for *all* $A \in \mathcal{A}$, i.e., for all variations in the given uncertainty set.

Theorem 4.1. Suppose that a solution to (4.8)–(4.10) can be computed numerically. Then the resulting gain K solves the Robust Controller Synthesis Problem.

Since (4.8)–(4.10) arise from a parametric LQ problem, there exist a variety of numerical methods that can be used to solve them. Here we note the extensive survey [14] as well as the homotopy-based methods used in [15,16].

5. Integral Control

Static output feedback was considered in previous sections since it preserves the "Kharitonov" structure. Dynamic compensation is, of course, considerably more complex. We now consider, as in [10], the possibility of utilizing an integral controller. Our results are more restrictive than [10] with regard to the admissible plants.

Specifically, we consider the realization

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ -\beta_0 & -\beta_1 \cdots - \beta_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n-1)\times 1} \\ b \end{bmatrix}, \quad C = [1 \quad 0 \cdots 0]$$

corresponding to a SISO plant with no zeros. Letting the control u be given by

$$u = K_P y + K_I \int y$$

leads to closed-loop dynamics of the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x},$$

where

$$\tilde{x} = \begin{bmatrix} x_I \\ x \end{bmatrix} \in \mathbb{R}^{n+1},\ x_I = \text{integrator state},$$

and

$$\tilde{A} = \begin{bmatrix} 0_{n\times 1} & & I_n & \\ bK_I & -\beta_0 + bK_P & \cdots & -\beta_{n-1} + bK_P \end{bmatrix}.$$

Note that $\tilde{A}$ preserves the companion structure needed to apply Lemma 2.1. Optimization can then proceed as in the static controller case. The details are thus omitted.

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Appendix D

herein, although to a lesser extent. For further background see [29], [30]. For extensions to nonstrictly proper controllers see [31], and for extensions to H_∞ control see [32].

To explain the rationale behind the development we briefly describe the main elements of the approach. The following discussion is intended to be descriptive; precise conditions appear in the main body of the paper.

1.1. Robust Stability Problem. For a nominal linear time-invariant (A, B, C) system we consider deterministically modeled bounded but otherwise unknown Lebesgue measurable time-varying parameter variations of the form

$$A+\sum_{i=1}^{p} \hat{\sigma}_i(t)A_i, \quad B+\sum_{i=1}^{p} \hat{\sigma}_i(t)B_i, \quad C+\sum_{i=1}^{p} \hat{\sigma}_i(t)C_i. \tag{1.1}$$

The nominal matrices A, B, C and the perturbation matrices A_i, B_i, C_i denoting the structure of the parametric uncertainty are assumed known, while the time-varying uncertain parameters $\hat{\sigma}_i(t)$ are assumed only to satisfy the bounds

$$|\hat{\sigma}_i(t)| \leqq \delta_i, \quad i=1, \cdots, p, \quad t \in [0, \infty). \tag{1.2}$$

The form of (1.1) permits an arbitrary number of uncertain parameters with arbitrary linear structure. Although we do not require matching conditions as in [21], the linear structure of (1.1) is more restrictive than the functional form $A(q(t))$ used in [21]. It is this structure that we exploit to obtain sufficiency conditions. Note also that the representation (1.1) is independent of state space basis, since replacing A by SAS^{-1} corresponds to replacing A_i by SA_iS^{-1}. As will be seen, our robustness bounds and optimality conditions are also basis independent. Also, scaling techniques [6], [7] will not play a rôle here. Finally, we note that because of the time-varying nature of the uncertain perturbations (1.1) it is virtually impossible to determine the *actual* stability region of a given design by means of empirical methods.

1.2. Quadratic Lyapunov function. As a sufficient condition for characterizing solutions of the Robust Stability Problem we consider a closed-loop quadratic Lyapunov function $V(\tilde{x}) = \tilde{x}^T \mathscr{P} \tilde{x}$, where the matrix $\mathscr{P}$ satisfies

$$0 = \tilde{A}^T \mathscr{P} + \mathscr{P} \tilde{A} + \Omega(\mathscr{P}, B_c, C_c) \tag{1.3}$$

and the function Ω is a bound satisfying

$$\sum_{i=1}^{p} \sigma_i(\tilde{A}_i^T \mathscr{P} + \mathscr{P} \tilde{A}_i) < \Omega(\mathscr{P}, B_c, C_c) \tag{1.4}$$

over the parameter range

$$|\sigma_i| \leqq \delta_i, \qquad i=1, \cdots, p. \tag{1.5}$$

Note that the constant σ_i in (1.4) and (1.5) plays the role of $\hat{\sigma}_i(t)$, i.e., t is "frozen" in (1.4) and (1.5). In (1.3) and (1.4), $\tilde{A}$ and $\tilde{A}_i$ denote the closed-loop dynamics and closed-loop parameter-uncertainty matrices given by

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{A}_i = \begin{bmatrix} A_i & B_iC_c \\ B_cC_i & 0 \end{bmatrix}. \tag{1.6}$$

Since $\tilde{A}_i$ is independent of A_c, Ω depends only on B_c and C_c. As discussed later in this section, (1.4) is automatically satisfied by construction of the function Ω. Furthermore, the existence of a solution $\mathscr{P}$ to (1.3) need not be verified directly, but rather is a result of numerically solving the optimality conditions discussed below.

1.3. Robust Performance Problem. In addition to the *deterministic* parameter uncertainty model (1.1), (1.2), the Robust Performance Problem includes *stochastic* plant disturbances and measurement noise with performance measured by means of the quadratic functional

$$\tilde{J}(t) = x^T(t)R_1x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2u(t). \tag{1.7}$$

To obtain a steady-state design problem we (1) average $\tilde{J}(t)$ over the disturbance and measurement noise statistics; (2) pass to the steady-state limit; and (3) maximize over the class of parameter uncertainties. Hence the performance of a given controller (A_c, B_c, C_c) is given by

$$J(A_c, B_c, C_c) = \sup_{\sigma(\cdot)} \limsup_{t\to\infty} \mathbb{E}[\tilde{J}(t)]. \tag{1.8}$$

The use of "lim sup" is a technicality that accounts for cases in which the steady-state limit may not exist. Note that although (1.8) is an averaging criterion over the disturbances as in LQG theory, it is also a worst-case measure over the uncertain parameters. Thus (1.8) is a *hybrid* criterion in the sense that is *stochastic* in the disturbance space (i.e., external uncertainties) and *deterministic* in the parameter space (i.e., internal uncertainties). By "internal uncertainties" we have in mind quantities such as mass, damping, or stiffness; by "external uncertainties" we are referring to phenomena such as turbulent flow for which only power spectrum statistics may be available. No claim is made, however, with regard to the universal validity of such a mathematical uncertainty model. In particular applications, uncertainty models that are either wholly deterministic or wholly stochastic may be more appropriate. In general, our setting appears to be consistent with the available literature (see [1]-[28]).

1.4. Performance bound. To obtain a tractable design problem, we use the matrix $\mathcal{P}$ to bound the performance of each controller solving the Robust Stability Problem. Specifically, by assuming in addition to (1.4) that

$$\sum_{i=1}^{p} \sigma_i(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i) + \tilde{R} \leqq \Omega(\mathcal{P}, B_c, C_c), \tag{1.9}$$

it follows that

$$J(A_c, B_c, C_c) \leqq \operatorname{tr} \mathcal{P}\tilde{V}. \tag{1.10}$$

In (1.9) and (1.10) $\tilde{R}$ and $\tilde{V}$ denote closed-loop weighting and disturbance intensity matrices. The idea of bounding the performance by means of a Lyapunov function is the basis for guaranteed cost control [14], [17].[1]

1.5. Construction of the Lyapunov function. So far the Lyapunov function has only been abstractly characterized by means of (1.3) and (1.4). To obtain a useful design theory Ω is now given a concrete form. Specifically, to satisfy (1.9) it is assumed that

$$\Omega(\mathcal{P}, B_c, C_c) = \sum_{i=1}^{p} \Lambda_i(\mathcal{P}, B_c, C_c) + \tilde{R}, \tag{1.11}$$

where, for each i, the Λ_i are chosen such that

$$\sigma_i(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i) \leqq \Lambda_i(\mathcal{P}, B_c, C_c), \qquad |\sigma_i| \leqq \delta_i. \tag{1.12}$$

[1] It is also interesting to note that in Hamilton-Jacobi-Bellman sufficiency theory the performance functional is expressed in terms of a value function that also serves as a Lyapunov function for the closed loop system. These connections will be explored in a future paper.

Note that (1.12) implies that (1.4) holds with Ω given by (1.11). Since $\tilde{A}_i$ depends on B_c and C_c, the bound Λ_i will be constructed to be *gain-invariant*, that is, so that (1.12) holds for *all* B_c and C_c. Thus no difficulty will arise from the fact that the controller gains are yet to be determined by optimality considerations.

It should be noted that the bounding in (1.12) is defined in the sense of the cone of nonnegative definite matrices. Since this is only a partial ordering and not a total ordering, a least upper bound (i.e., a "sharpest" bound) does not exist in general and the conservatism of the inequality in (1.12) cannot be quantified by a scalar measure. Hence, Λ_i satisfying (1.12) is not necessarily unique and two particular choices of Λ_i are developed in this paper. Since we shall utilize first-order necessary conditions for optimality, we confine our consideration to bounds that are differentiable. The first choice of Λ_i satisfying (1.12) is given by the linear (in $\mathscr{P}$) function

$$\Lambda_i(\mathscr{P}, B_c, C_c) = \delta_i(\alpha_i \mathscr{P} + \alpha_i^{-1} \tilde{A}_i^T \mathscr{P} \tilde{A}_i), \tag{1.13}$$

where α_i is an arbitrary positive number. As shown in [33], the bound (1.13) can be viewed as arising from a stochastic optimal control problem with exponentially weighted cost and state-, control- and measurement-dependent white noise. The stochastic multiplicative white noise model serves only as an *interpretation*, however, and need not be viewed as having physical significance. A similar bound is used in [28].

The second choice for Λ_i satisfying (1.12) is given by the quadratic (in $\mathscr{P}$) function

$$\Lambda_i(\mathscr{P}, B_c, C_c) = \delta_i(\tilde{E}_i^T \tilde{E}_i + \mathscr{P} \tilde{D}_i \tilde{D}_i^T \mathscr{P}), \tag{1.14}$$

where $\tilde{D}_i$, $\tilde{E}_i$ denote an arbitrary factorization of $\tilde{A}_i$ of the form

$$\tilde{A}_i = \tilde{D}_i \tilde{E}_i. \tag{1.15}$$

The bound (1.14) was used in [26] for full-state feedback with rank 1 uncertainties. Note that using congruence transformations shows that both bounds (1.13) and (1.14) are basis independent; that is, replacing $\tilde{A}_i$ by $\tilde{S}\tilde{A}_i\tilde{S}^{-1}$ leads to replacing $\mathscr{P}$ by $\tilde{S}^{-T}\mathscr{P}\tilde{S}^{-1}$.

1.6. Auxiliary Minimization Problem. The next step in our development for robust performance is the following. Inasmuch as the performance of a robustly stabilizing controller is bounded via (1.10) over the given range of parameter variations, it is desirable to minimize the upper bound

$$\mathscr{J}(\mathscr{P}, A_c, B_c, C_c) \triangleq \operatorname{tr} \mathscr{P} \tilde{V} \tag{1.16}$$

subject to the constraint (1.3). This is referred to as the Auxiliary Minimization Problem. For a given choice (1.13) or (1.14) of Λ_i for each i, a solution of the Auxiliary Minimization Problem provides a controller whose steady-state performance is guaranteed to remain below the bound (1.16) over the range of parameter variations, hence guaranteeing robust performance. Since the Auxiliary Minimization Problem is a smooth mathematical programming problem, a minimum always exists on compact sets. To actually characterize extremals of the Auxiliary Minimization Problem we proceed by deriving first-order necessary conditions. Because these necessary conditions are derived for the Auxiliary Minimization Problem, they effectively serve as sufficient conditions for robustness in the original problem.

It should be noted that the guaranteed cost control approach developed in [14] does not permit this line of development since Λ_i is given by

$$\Lambda_i(\mathscr{P}, B_c, C_c) = \delta_i |\tilde{A}_i^T \mathscr{P} + \mathscr{P} \tilde{A}_i|, \tag{1.17}$$

where $|\cdot|$ denotes the matrix obtained by replacing each eigenvalue by its absolute value. Since this bound is not differentiable with respect to the controller gains, first-order necessary conditions cannot be used.

1.7. The optimality conditions: full-order case. For the full-order case, i.e., when the order of the controller is equal to the order of the plant, the first-order necessary conditions can be derived in a form that is a direct generalization of the pair of separated Riccati equations of LQG theory. Specifically, the necessary conditions comprise a coupled system of four algebraic matrix equations including a pair of modified Riccati equations and a pair of Lyapunov equations. For plant models involving multiplicative white noise these equations have been studied in [34]–[36]. This form of the equations thus essentially corresponds to choosing bound (1.13).

1.8. The optimality conditions: reduced-order case. For design flexibility we also consider controllers of arbitrary reduced dimension. For the linear-quadratic problem without parameter uncertainty, the formulation of the necessary conditions given in [29] provides a generalization of LQG theory. Specifically, the optimal gains are characterized by a system of algebraic matrix equations consisting of a pair of modified Riccati equations and a pair of modified Lyapunov equations coupled by an oblique projection. When the order of the controller is equal to the order of the plant, the projection becomes the identity and the standard LQG result is recovered.

The outcome of the development above is a set of algebraic matrix equations that correspond to the necessary conditions for the Auxiliary Minimization Problem and hence to sufficient conditions for robust stability and performance. These necessary conditions characterize full- or reduced-order controllers with either choice of bounds (1.13) and (1.14) for each uncertain parameter. For control-system design, these equations can be used as follows. If a solution to the necessary conditions is obtained computationally and if certain definiteness conditions hold, *then* the explicitly synthesized controller (1) solves the Robust Stability Problem and (2) is guaranteed to provide robust performance bounded by $\operatorname{tr} \mathcal{P}\tilde{V}$ over the stipulated uncertainty range.

The applicability of these results is, of course, limited to plants that are nominally stabilizable via controllers of the given order. Indeed, in this case it has been shown [37] via topological degree theory that the optimality conditions for the case $\delta_i = 0$, $i = 1, \cdots, p$, possess at least one stabilizing solution. For the parameter uncertainty problem, i.e., $\delta_i > 0$, it follows from continuity properties that a solution also exists for sufficiently small δ_i. The *actual* range of uncertainty that can be stabilized and the tightness of the performance bound depend on the *conservatism* of our bounds. As will be seen from a numerical example, our bounds are not generally sharp. This is not unexpected, however, due to both the sense of the partial ordering employed in (1.12) and the fact that our choice of gain-invariant bounds permits a one-step, *noniterative* synthesis (rather than analysis) procedure. It should be noted that necessary and sufficient conditions for robust analysis of a block-structured class of uncertainties are obtainable using the μ-function [6]. This block structure, however, does not appear to include either the linear uncertainty model (1.1) or the matched uncertainty model of [21] as special cases.

In the present paper we present results of an illustrative numerical study for a well-known example used in [2] to demonstrate the lack of gain margin for LQG controllers. This type of uncertainty is a special case of (1.1) obtained by taking $p = m$ and defining B_i to be the matrix whose ith column is the same as the ith column of B, and zero otherwise. To obtain full-order, robustified controllers exhibiting performance/robustness tradeoffs, we use bound (1.13) for several values of δ_i. To obtain

these numerical results we used a straightforward iterative algorithm that requires only an LQG-type software package. The homotopy algorithm of [37] with appropriate extensions can also be used. Further descriptions of related algorithms and numerical results can be found in [38]-[40].

The development herein is self-contained, with the exception that the detailed derivation of the optimality conditions has been omitted. In specialized cases the derivation has been given previously. For the case of bound (1.13) only, a derivation using Kronecker products appears in [36]. Also, a derivation without parameter uncertainties has been given in [29] using Lagrange multipliers. Overall, the derivation involves considerable matrix manipulation. Since the detailed derivation does not appear to warrant the required space, we give an outline of the proof to assist the sufficiently motivated reader in reconstructing the details.

2. Notation and definitions. (Note that all matrices have real entries.)

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation
$\|\cdot\|$	Euclidean vector norm
$I_r, 0_{r\times s}, 0_r$	$r\times r$ identity matrix, $r\times s$ zero matrix, $0_{r\times r}$
$(\ )^T, (\ )^{-1}, (\ )^{-T}$	transpose, inverse, inverse transpose
tr	trace
$\oplus, \otimes$	Kronecker sum, Kronecker product [41]
$\mathbb{S}^r$	$r\times r$ symmetric matrices
$\mathbb{N}^r$	$r\times r$ symmetric nonnegative-definite matrices
$\mathbb{P}^r$	$r\times r$ symmetric positive-definite matrices
$Z_1 \geqq Z_2$	$Z_1 - Z_2 \in \mathbb{N}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
$Z_1 > Z_2$	$Z_1 - Z_2 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
asymptotically stable matrix	matrix with eigenvalues in open left half-plane
$n, m, l, p, n_c, n_i, m_i$	positive integers, $i \in \{1, \cdots, p\}$
$\tilde{n}, \tilde{n}_i$	$n + n_c$, $n_i + m_i$, $i \in \{1, \cdots, p\}$
x, u, y, x_c	n, m, l, n_c-dimensional vectors
$A, A_i; B, B_i; C, C_i$	$n\times n$ matrices, $n\times m$ matrices, $l\times n$ matrices, $i \in \{1, \cdots, p\}$
A_c, B_c, C_c	$n_c\times n_c$, $n_c \times l$, $m\times n_c$ matrices
$\tilde{A}, \tilde{A}_i$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \begin{bmatrix} A_i & B_iC_c \\ B_cC_i & 0 \end{bmatrix}, i \in \{1, \cdots, p\}$
δ_i	positive number, $i \in \{1, \cdots, p\}$
Δ	$[-\delta_1, \delta_1] \times \cdots \times [-\delta_p, \delta_p]$
σ_i	real number, $i \in \{1, \cdots, p\}$
σ	$(\sigma_1, \cdots, \sigma_p)$
$\hat{\sigma}_i(\cdot)$	Lebesgue measurable function on $[0, \infty)$, $i \in \{1, \cdots, p\}$
$\hat{\sigma}(\cdot)$	$(\hat{\sigma}_1(\cdot), \cdots, \hat{\sigma}_p(\cdot))$
$L_\infty([0, \infty), \Delta)$	Lebesgue measurable functions on $[0, \infty)$ with values in Δ
α_i	positive number, $i \in \{1, \cdots, p\}$
D_i, E_i, H_i, K_i	$n\times n_i$, $n_i \times n$, $n\times m_i$, $m_i\times m$ matrices, $i \in \{1, \cdots, p\}$
$\tilde{D}_i, \tilde{E}_i$	$\tilde{n}\times \tilde{n}_i$, $\tilde{n}_i \times \tilde{n}$ matrices, $i \in \{1, \cdots, p\}$
Σ', Σ''	see § 6
R_1	state weighting matrix in $\mathbb{N}^n$

R_2	control weighting matrix in $\mathbb{P}^m$
R_{12}	$n\times m$ cross weighting matrix such that $R_1 - R_{12}R_2^{-1}R_{12}^T \geqq 0$
$\tilde{R}$	$\begin{bmatrix} R_1 & R_{12}C_c \\ C_c^TR_{12}^T & C_c^TR_2C_c \end{bmatrix}$
$w_1(\cdot)$	n-dimensional white noise
$w_2(\cdot)$	l-dimensional white noise
V_1	intensity of $w_1(\cdot)$ in $\mathbb{N}^n$
V_2	intensity of $w_2(\cdot)$ in $\mathbb{P}^l$
V_{12}	$n\times l$ cross-intensity of $w_1(\cdot)$, $w_2(\cdot)$
$\tilde{V}$	$\begin{bmatrix} V_1 & V_{12}B_c^T \\ B_cV_{12}^T & B_cV_2B_c^T \end{bmatrix}$

3. Robust Stability and Robust Performance Problems. In this section we state the Robust Stability Problem and Robust Performance Problem along with related notation for later use.

3.1. Robust Stability Problem. For fixed $n_c \leqq n$, determine $(A_c, B_c, C_c) \in \mathbb{R}^{n_c\times n_c}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}$ such that the closed-loop system consisting of the nth-order controlled plant

$$(3.1)\qquad \dot{x}(t) = \left(A + \sum_{i=1}^{p} \hat{\sigma}_i(t)A_i\right)x(t) + \left(B + \sum_{i=1}^{p} \hat{\sigma}_i(t)B_i\right)u(t) \quad \text{a.a. } t\in[0,\infty),$$

measurements

$$(3.2)\qquad y(t) = \left(C + \sum_{i=1}^{p} \hat{\sigma}_i(t)C_i\right)x(t),$$

and n_cth-order dynamic compensator

$$(3.3)\qquad \dot{x}_c(t) = A_cx_c(t) + B_cy(t),$$

$$(3.4)\qquad u(t) = C_cx_c(t)$$

are asymptotically stable[2] for all $\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta)$.

3.2. Robust Performance Problem. For fixed $n_c \leqq n$, determine $(A_c, B_c, C_c) \in \mathbb{R}^{n_c\times n_c}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}$ such that, for the closed-loop system consisting of the nth-order controlled and disturbed plant

$$(3.5)\qquad \dot{x}(t) = \left(A + \sum_{i=1}^{p} \hat{\sigma}_i(t)A_i\right)x(t) + \left(B + \sum_{i=1}^{p} \hat{\sigma}_i(t)B_i\right)u(t) + w_1(t) \quad \text{a.a. } t\in[0,\infty),$$

noisy measurements

$$(3.6)\qquad y(t) = \left(C + \sum_{i=1}^{p} \hat{\sigma}_i(t)C_i\right)x(t) + w_2(t),$$

and n_cth-order dynamic compensator (3.3), (3.4), the performance criterion

$$(3.7)\qquad \begin{aligned} &J(A_c, B_c, C_c) \\ &\triangleq \sup_{\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta)} \limsup_{t\to\infty} \mathbb{E}[x^T(t)R_1x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2u(t)] \end{aligned}$$

is minimized.

[2] Asymptotic stability for a nonautonomous system is defined in the standard way (see, e.g., [42]).

For each controller (A_c, B_c, C_c) and parameter variation $\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta)$ the undisturbed closed-loop system (3.1)-(3.4) is given by

$$\dot{\tilde{x}}(t)=\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\tilde{x}(t)\quad \text{a.a. } t\in[0,\infty), \tag{3.8}$$

while the disturbed closed-loop system (3.3)-(3.6) is

$$\dot{\tilde{x}}(t)=\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\tilde{x}(t)+\tilde{w}(t)\quad \text{a.a. } t\in[0,\infty). \tag{3.9}$$

Also (see, e.g., [43, p. 194]), let $\tilde{\Phi}:[0,\infty)\to\mathbb{R}^{\tilde{n}\times\tilde{n}}$ be the unique absolutely continuous solution to

$$\dot{\tilde{\Phi}}(t)=\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\tilde{\Phi}(t)\quad \text{a.a. } t\in[0,\infty), \tag{3.10}$$

$$\tilde{\Phi}(0)=I_{\tilde{n}}, \tag{3.11}$$

and recall that $\tilde{\Phi}^{-1}(\cdot)$ satisfies

$$\frac{d}{dt}\tilde{\Phi}^{-1}(t)=-\tilde{\Phi}^{-1}(t)\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\quad \text{a.a. } t\in[0,\infty). \tag{3.12}$$

4. Sufficient conditions for robust stability and performance. For robust stability we characterize quadratic Lyapunov functions for the closed-loop system.

THEOREM 4.1. *Let* $\Omega:\mathbb{P}^{\tilde{n}}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}\to\mathbb{S}^{\tilde{n}}$ *satisfy*

$$\sum_{i=1}^{p}\sigma_i(\tilde{A}_i^T\mathcal{P}+\mathcal{P}\tilde{A}_i)<\Omega(\mathcal{P},B_c,C_c),\qquad \sigma\in\Delta,$$
$$(\mathcal{P},B_c,C_c)\in\mathbb{P}^{\tilde{n}}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}. \tag{4.1}$$

If, for some $(A_c,B_c,C_c)\in\mathbb{R}^{n_c\times n_c}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}$, *there exists* $\mathcal{P}\in\mathbb{P}^{\tilde{n}}$ *satisfying*

$$0=\tilde{A}^T\mathcal{P}+\mathcal{P}\tilde{A}+\Omega(\mathcal{P},B_c,C_c), \tag{4.2}$$

then (A_c,B_c,C_c) *solves the Robust Stability Problem.*

Proof. Define the Lyapunov function

$$V(\tilde{x})\triangleq\tilde{x}^T\mathcal{P}\tilde{x},\qquad \tilde{x}\in\mathbb{R}^{\tilde{n}}.$$

For almost all $t\in[0,\infty)$ and $\tilde{x}(t)$ satisfying (3.8), it follows from (4.2) that

$$\begin{aligned}\dot{V}(\tilde{x}(t))&=\dot{\tilde{x}}^T(t)\mathcal{P}\tilde{x}(t)+\tilde{x}^T(t)\mathcal{P}\dot{\tilde{x}}(t)\\&=\tilde{x}^T(t)\left[\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)^T\mathcal{P}+\mathcal{P}\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\right]\tilde{x}(t)\\&=\tilde{x}^T(t)\left[\sum_{i=1}^{p}\hat{\sigma}_i(t)(\tilde{A}_i^T\mathcal{P}+\mathcal{P}\tilde{A}_i)-\Omega(\mathcal{P},B_c,C_c)\right]\tilde{x}(t).\end{aligned}$$

Since $\hat{\sigma}(t)\in\Delta$, almost all $t\in[0,\infty)$, it follows from (4.1) that there exists $\gamma>0$ such that $\dot{V}(\tilde{x}(t))\leqq-\gamma\|\tilde{x}(t)\|^2$, almost all $t\in[0,\infty)$. □

Remark 4.1. If (A_c,B_c,C_c) solves the Robust Stability Problem, then

$$\lim_{t\to\infty}\tilde{\Phi}(t)=0,\qquad \hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta). \tag{4.3}$$

Remark 4.2. As will be seen, the bound (4.1) will be guaranteed for all $\mathcal{P}, B_c, C_c$ by suitable construction of the function Ω. In addition, the existence of a solution $\mathcal{P}$ to (4.2) need not be verified in practice. Rather, (4.2) is a result of numerically solving the necessary conditions for the Auxiliary Minimization Problem given in Theorem 6.1.

For the Robust Performance Problem the cost can be expressed in terms of the closed-loop second-moment matrix.

PROPOSITION 4.1. *For $(A_c, B_c, C_c) \in \mathbb{R}^{n_c \times n_c} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}$ and $\hat{\sigma}(\cdot) \in L_\infty([0,\infty), \Delta)$ the second-moment matrix*

$$\tilde{Q}(t) \triangleq \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)], \qquad t \in [0,\infty), \tag{4.4}$$

satisfies

$$\dot{\tilde{Q}}(t) = \left(\tilde{A} + \sum_{i=1}^{p} \hat{\sigma}_i(t)\tilde{A}_i\right)\tilde{Q}(t) + \tilde{Q}(t)\left(\tilde{A} + \sum_{i=1}^{p} \hat{\sigma}_i(t)\tilde{A}_i\right)^T + \tilde{V} \quad a.a.\ t \in [0,\infty), \tag{4.5}$$

or, equivalently,

$$\tilde{Q}(t) = \tilde{\Phi}(t)\tilde{Q}(0)\tilde{\Phi}^T(t) + \int_0^t \tilde{\Phi}(t)\tilde{\Phi}^{-1}(s)\tilde{V}\tilde{\Phi}^{-T}(s)\tilde{\Phi}^T(t)\,ds, \qquad t \in [0,\infty). \tag{4.6}$$

Furthermore,

$$J(A_c, B_c, C_c) = \sup_{\hat{\sigma}(\cdot) \in L_\infty([0,\infty),\Delta)} \limsup_{t\to\infty} \operatorname{tr} \tilde{Q}(t)\tilde{R}, \tag{4.7}$$

or, equivalently,

$$\begin{aligned} J(A_c, B_c, C_c) \triangleq \sup_{\hat{\sigma}(\cdot) \in L_\infty([0,\infty),\Delta)} \limsup_{t\to\infty} \operatorname{tr} \Bigg[&\tilde{\Phi}(t)\tilde{Q}(0)\tilde{\Phi}^T(t)\tilde{R} \\ &+ \int_0^t \tilde{\Phi}(t)\tilde{\Phi}^{-1}(s)\tilde{V}\tilde{\Phi}^{-T}(s)\tilde{\Phi}^T(t)\,ds\,\tilde{R}\Bigg]. \end{aligned} \tag{4.8}$$

Proof. The second-moment equation (4.5) is a direct consequence of the Itô differential rule (see [44, p. 142]), while (4.6) follows by direct verification. Finally, (4.7) is immediate. $\square$

We now obtain an upper bound for J in terms of the matrix $\mathcal{P}$. The following lemma is required.

LEMMA 4.1. *Let $\Omega: \mathbb{P}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \to \mathbb{S}^{\tilde{n}}$ and $(A_c, B_c, C_c) \in \mathbb{R}^{n_c \times n_c} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}$ be given. Then $\mathcal{P} \in \mathbb{P}^{\tilde{n}}$ satisfies* (4.2) *if and only if $\mathcal{P}$ satisfies*

$$\begin{aligned} \mathcal{P} = \tilde{\Phi}^T(t)\mathcal{P}\tilde{\Phi}(t) + \int_0^t &\tilde{\Phi}^T(t)\tilde{\Phi}^{-T}(s) \\ &\cdot \left[\Omega(\mathcal{P}, B_c, C_c) - \sum_{i=1}^{p} \hat{\sigma}_i(t)(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(s)\tilde{\Phi}(t)\,ds, \\ &\hat{\sigma}(\cdot) \in L_\infty([0,\infty),\Delta), \qquad t \in [0,\infty). \end{aligned} \tag{4.9}$$

Proof. Suppose $\mathcal{P}$ satisfies (4.2). Then for $t \in [0,\infty)$,

$$\begin{aligned} 0 &= \tilde{\Phi}^{-T}(t)\left(\tilde{A} + \sum_{i=1}^{p} \hat{\sigma}_i(t)\tilde{A}_i\right)^T \mathcal{P}\tilde{\Phi}^{-1}(t) + \tilde{\Phi}^{-T}(t)\mathcal{P}\left(\tilde{A} + \sum_{i=1}^{p} \hat{\sigma}_i(t)\tilde{A}_i\right)\tilde{\Phi}^{-1}(t) \\ &\quad + \tilde{\Phi}^{-T}(t)\left[\Omega(\mathcal{P}, B_c, C_c) - \sum_{i=1}^{p} \hat{\sigma}_i(t)(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(t) \\ &= -\frac{d}{dt}\left[\tilde{\Phi}^{-T}(t)\mathcal{P}\tilde{\Phi}^{-1}(t)\right] + \tilde{\Phi}^{-T}(t)\left[\Omega(\mathcal{P}, B_c, C_c) - \sum_{i=1}^{p} \hat{\sigma}_i(t)(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(t), \end{aligned}$$

which yields

$$\begin{aligned} 0 &= -\tilde{\Phi}^{-T}(t)\mathcal{P}\tilde{\Phi}^{-1}(t) + \mathcal{P} \\ &\quad + \int_0^t \tilde{\Phi}^{-T}(s)\left[\Omega(\mathcal{P}, B_c, C_c) - \sum_{i=1}^{p} \hat{\sigma}_i(s)(\tilde{A}_i^T\mathcal{P} + \mathcal{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(s)\,ds. \end{aligned}$$

Thus (4.9) is satisfied. Conversely, suppose $\mathscr{P}$ satisfies (4.9). Differentiating with respect to t using Leibniz's rule yields

$$
\begin{aligned}
0=&\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)^T\tilde{\Phi}^T(t)\mathscr{P}\tilde{\Phi}(t)+\tilde{\Phi}^T(t)\mathscr{P}\tilde{\Phi}(t)\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\\
&+\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)^T\int_0^t\tilde{\Phi}^T(t)\tilde{\Phi}^{-T}(s)\\
&\cdot\left[\Omega(\mathscr{P},B_c,C_c)-\sum_{i=1}^{p}\hat{\sigma}_i(s)(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(s)\tilde{\Phi}(t)\,ds\\
&+\int_0^t\tilde{\Phi}^T(t)\tilde{\Phi}^{-T}(s)\left[\Omega(\mathscr{P},B_c,C_c)-\sum_{i=1}^{p}\hat{\sigma}_i(s)(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)\right]\\
&\cdot\tilde{\Phi}^{-1}(s)\tilde{\Phi}(t)\,ds\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)\\
&+\Omega(\mathscr{P},B_c,C_c)-\sum_{i=1}^{p}\hat{\sigma}_i(t)(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)\\
=&\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)^T\mathscr{P}+\mathscr{P}\left(\tilde{A}+\sum_{i=1}^{p}\hat{\sigma}_i(t)\tilde{A}_i\right)+\Omega(\mathscr{P},B_c,C_c)-\sum_{i=1}^{p}\hat{\sigma}_i(t)(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)\\
=&\tilde{A}^T\mathscr{P}+\mathscr{P}\tilde{A}+\Omega(\mathscr{P},B_c,C_c).
\end{aligned}
$$

Hence (4.2) is satisfied. □

Remark 4.3. Note the identity

$$
\begin{gathered}
\operatorname{tr}\int_0^t\tilde{\Phi}(t)\tilde{\Phi}^{-1}(s)\tilde{V}\tilde{\Phi}^{-T}(s)\tilde{\Phi}^T(t)\,ds\,\tilde{R}=\operatorname{tr}\int_0^t\tilde{\Phi}^T(t)\tilde{\Phi}^{-T}(s)\tilde{R}\tilde{\Phi}^{-1}(s)\tilde{\Phi}(t)\,ds\,\tilde{V},\\
(A_c,B_c,C_c)\in\mathbb{R}^{n_c\times n_c}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c},\quad\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta),\quad t\in[0,\infty).
\end{gathered}\tag{4.10}
$$

We are now in a position to bound the cost J by means of the matrix $\mathscr{P}$.

THEOREM 4.2. *Let* $\Omega:\mathbb{P}^{\tilde{n}}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}\to\mathbb{S}^{\tilde{n}}$ *satisfy* (4.1) *and*

$$
\begin{gathered}
\sum_{i=1}^{p}\sigma_i(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)+\tilde{R}\leqq\Omega(\mathscr{P},B_c,C_c),\qquad\sigma\in\Delta,\\
(\mathscr{P},B_c,C_c)\in\mathbb{P}^{\tilde{n}}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}.
\end{gathered}\tag{4.11}
$$

If, for some $(A_c,B_c,C_c)\in\mathbb{R}^{n_c\times n_c}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}$, *there exists* $\mathscr{P}\in\mathbb{P}^{\tilde{n}}$ *satisfying* (4.2), *then*

$$
J(A_c,B_c,C_c)\leqq\operatorname{tr}\mathscr{P}\tilde{V}.\tag{4.12}
$$

Proof. From (4.8)-(4.10) and (4.3) it follows that

$$
\begin{aligned}
&J(A_c,B_c,C_c)\\
&=\sup_{\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta)}\limsup_{t\to\infty}\operatorname{tr}\Big\{\tilde{\Phi}(t)\tilde{Q}(0)\tilde{\Phi}^T(t)\tilde{R}+\mathscr{P}\tilde{V}-\tilde{\Phi}^T(t)\mathscr{P}\tilde{\Phi}(t)\tilde{V}\\
&\quad-\int_0^t\tilde{\Phi}^T(t)\tilde{\Phi}^{-T}(s)\left[\Omega(\mathscr{P},B_c,C_c)-\tilde{R}-\sum_{i=1}^{p}\hat{\sigma}_i(s)(\tilde{A}_i^T\mathscr{P}+\mathscr{P}\tilde{A}_i)\right]\tilde{\Phi}^{-1}(s)\tilde{\Phi}(t)\,ds\,\tilde{V}\Big\}\\
&\leqq\sup_{\hat{\sigma}(\cdot)\in L_\infty([0,\infty),\Delta)}\limsup_{t\to\infty}\operatorname{tr}[\tilde{\Phi}(t)\tilde{Q}(0)\tilde{\Phi}^T(t)\tilde{R}+\mathscr{P}\tilde{V}]\\
&=\operatorname{tr}\mathscr{P}\tilde{V}.
\end{aligned}
$$

□

Remark 4.4. Note that since $\tilde{R} \geqq 0$, (4.11) implies

$$\sum_{i=1}^{p} \sigma_i(\tilde{A}_i^T \mathscr{P} + \mathscr{P}\tilde{A}_i) \leqq \Omega(\mathscr{P}, B_c, C_c), \qquad \sigma \in \Delta, \tag{4.13}$$

which is a weak form of (4.1). If $\tilde{R} > 0$ then (4.11) implies (4.1). This implication is not surprising since (4.11) implies robust performance while (4.1) implies robust stability.

5. Choice of bounds. To satisfy (4.11), $\Omega(\cdot, \cdot, \cdot)$ is chosen to be of the form

$$\Omega(\mathscr{P}, B_c, C_c) = \sum_{i=1}^{p} \Lambda_i(\mathscr{P}, B_c, C_c) + \tilde{R}, \tag{5.1}$$

where, for each $i = 1, \cdots, p$, $\Lambda_i : \mathbb{P}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \to \mathbb{S}^{\tilde{n}}$ satisfies

$$\begin{aligned} &\sigma_i(\tilde{A}_i^T \mathscr{P} + \mathscr{P}\tilde{A}_i) \leqq \Lambda_i(\mathscr{P}, B_c, C_c), \qquad \sigma_i \in [-\delta_i, \delta_i], \\ &(\mathscr{P}, B_c, C_c) \in \mathbb{P}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}. \end{aligned} \tag{5.2}$$

Two distinct choices for the bound Λ_i are considered. As we pointed out in § 1, the first choice corresponds to a right shift/multiplicative white noise model [33], while the second bound generalizes results found in [26].

PROPOSITION 5.1. *For all* $\alpha_i > 0$ *the function*

$$\Lambda_i(\mathscr{P}, B_c, C_c) = \delta_i(\alpha_i \mathscr{P} + \alpha_i^{-1} \tilde{A}_i^T \mathscr{P} \tilde{A}_i) \tag{5.3}$$

satisfies (5.2).

Proof. Note that

$$\begin{aligned} 0 &\leqq [\sigma_i(\alpha_i/\delta_i)^{1/2} I_{\tilde{n}} - (\delta_i/\alpha_i)^{1/2} \tilde{A}_i]^T \mathscr{P} [\sigma_i(\alpha_i/\delta_i)^{1/2} I_{\tilde{n}} - (\delta_i/\alpha_i)^{1/2} \tilde{A}_i] \\ &= \sigma_i^2(\alpha_i/\delta_i)\mathscr{P} + (\delta_i/\alpha_i)\tilde{A}_i^T \mathscr{P} \tilde{A}_i - \sigma_i(\tilde{A}_i^T \mathscr{P} + \mathscr{P}\tilde{A}_i), \end{aligned}$$

which, since $\sigma_i^2 \leqq \delta_i^2$, implies (5.2). □

PROPOSITION 5.2. *For all* $\tilde{D}_i \in \mathbb{R}^{\tilde{n} \times \tilde{n}_i}$ *and* $\tilde{E}_i \in \mathbb{R}^{\tilde{n}_i \times \tilde{n}}$ *satisfying*

$$\tilde{A}_i = \tilde{D}_i \tilde{E}_i, \tag{5.4}$$

the function

$$\Lambda_i(\mathscr{P}, B_c, C_c) = \delta_i(\tilde{E}_i^T \tilde{E}_i + \mathscr{P}\tilde{D}_i \tilde{D}_i^T \mathscr{P}) \tag{5.5}$$

satisfies (5.2).

Proof. Note that

$$\begin{aligned} 0 &\leqq [\delta_i^{1/2} \tilde{E}_i - \sigma_i \delta_i^{-1/2} \tilde{D}_i^T \mathscr{P}]^T [\delta_i^{1/2} \tilde{E}_i - \sigma_i \delta_i^{-1/2} \tilde{D}_i^T \mathscr{P}] \\ &= \delta_i \tilde{E}_i^T \tilde{E}_i + (\sigma_i^2/\delta_i)\mathscr{P}\tilde{D}_i \tilde{D}_i^T \mathscr{P} - \sigma_i(\tilde{A}_i^T \mathscr{P} + \mathscr{P}\tilde{A}_i), \end{aligned}$$

which implies (5.2). □

6. The auxiliary minimization problem and necessary conditions for optimality. To optimize robust performance while retaining robust stability, we consider the following problem for which the cost functional is given by the bound (4.12).

6.1. Auxiliary Minimization Problem. For $i = 1, \cdots, p$, let Λ_i be given by either (5.3) or (5.5). Determine $(\mathscr{P}, A_c, B_c, C_c) \in \mathbb{P}^{\tilde{n}} \times \mathbb{R}^{n_c \times n_c} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}$, which minimizes

$$\mathscr{J}(\mathscr{P}, A_c, B_c, C_c) \triangleq \operatorname{tr} \mathscr{P}\tilde{V} \tag{6.1}$$

subject to

$$0=\tilde{A}^T\mathcal{P}+\mathcal{P}\tilde{A}+\sum_{i=1}^{p}\Lambda_i(\mathcal{P}, B_c, C_c)+\tilde{R}, \tag{6.2}$$

$$\sum_{i=1}^{p}\sigma_i(\tilde{A}_i^T\mathcal{P}+\mathcal{P}\tilde{A}_i)<\sum_{i=1}^{p}\Lambda_i(\mathcal{P}, B_c, C_c)+\tilde{R}, \qquad \sigma\in\Delta. \tag{6.3}$$

Remark 6.1. Note that (6.3) enforces both (4.1) and (4.11) to guarantee robust stability and performance.

To derive first-order necessary conditions for the Auxiliary Minimization Problem, note that the constraint (6.3) defines an open set.

PROPOSITION 6.1. *The set of* $(\mathcal{P}, B_c, C_c)\in\mathbb{P}^{\tilde{n}}\times\mathbb{R}^{n_c\times l}\times\mathbb{R}^{m\times n_c}$ *satisfying* (6.3) *is open.*

Proof. Since $\Lambda_i(\cdot,\cdot,\cdot)$ is continuous it can be shown that the function

$$f(\mathcal{P}, B_c, C_c)\triangleq\min_{\sigma\in\Delta}\lambda_{\min}\left\{\sum_{i=1}^{p}\Lambda_i(\mathcal{P}, B_c, C_c)+\tilde{R}-\sum_{i=1}^{p}\sigma_i(\tilde{A}_i^T\mathcal{P}+\mathcal{P}\tilde{A}_i)\right\}$$

is also continuous. Since (6.3) is equivalent to $0<f(\mathcal{P}, B_c, C_c)$, the result is immediate. □

To obtain explicit feedback gain expressions we shall require two additional technical assumptions. If bound (5.3) is chosen for a given $i\in\{1,\cdots,p\}$ we require

$$B_i\neq 0\Rightarrow C_i=0, \tag{6.4}$$

i.e., B_i and C_i are not simultaneously nonzero. Of course, both B_i and C_i may be zero. Assumption (6.4) implies that parameter uncertainties in B and C must be modeled as uncorrelated. Correlation between uncertainties in A and B or A and C is, of course, permitted. Furthermore, if bound (5.5) is chosen for a given $i\in\{1,\cdots,p\}$ we require

$$C_i=0. \tag{6.5}$$

We stress that (6.4) and (6.6) can be removed, but at the expense of explicit gain expressions.

When we use bound (5.3) the positive constant α_i will be considered fixed but arbitrary. Furthermore, for bound (5.5), let $D_i\in\mathbb{R}^{n\times n_i}$, $E_i\in\mathbb{R}^{n_i\times n}$, $H_i\in\mathbb{R}^{n\times m_i}$, and $K_i\in\mathbb{R}^{m_i\times m}$ satisfy

$$A_i=D_iE_i, \qquad B_i=H_iK_i, \tag{6.6}$$

and define $\tilde{D}_i$, $\tilde{E}_i$ satisfying (5.4) by

$$\tilde{D}_i\triangleq\begin{bmatrix} D_i & H_i \\ 0_{n_c\times n_i} & 0_{n_c\times m_i}\end{bmatrix}, \qquad \tilde{E}_i\triangleq\begin{bmatrix} E_i & 0_{n_i\times n_c} \\ 0_{m_i\times n} & K_iC_c\end{bmatrix}. \tag{6.7}$$

In addition to the open set defined by (6.3), the derivation of the necessary conditions requires that $(\mathcal{P}, A_c, B_c, C_c)$ be further restricted so that

$$\begin{aligned}&\left(\tilde{A}+\frac{1}{2}\sum{}'\,\delta_i\alpha_iI_{\tilde{n}}+\sum{}''\,\delta_i\tilde{D}_i\tilde{D}_i^T\mathcal{P}\right)\oplus\left(\tilde{A}+\frac{1}{2}\sum{}'\,\delta_i\alpha_iI_{\tilde{n}}+\sum{}''\,\delta_i\tilde{D}_i\tilde{D}_i^T\mathcal{P}\right)\\ &\qquad +\sum{}'\,(\delta_i\alpha_i^{-1})\tilde{A}_i\otimes\tilde{A}_i \quad \text{is asymptotically stable,}\end{aligned} \tag{6.8}$$

$$(A_c, B_c, C_c)\ \text{is controllable and observable.} \tag{6.9}$$

In (6.8) the notation $\sum'$ and $\sum''$ denotes summation over indices for which bounds (5.3) and (5.5), respectively, have been chosen. Note that (6.8) and (6.9) play no role in the Auxiliary Minimization Problem and thus need not be verified for robust stability or robust performance.

For arbitrary $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n\times n}$ define the following notation:

$$R_{2a} \triangleq R_2 + \sum{}' (\delta_i\alpha_i^{-1})B_i^T(P+\hat{P})B_i + \sum{}'' \delta_i K_i^T K_i,$$

$$V_{2a} \triangleq V_2 + \sum{}' (\delta_i\alpha_i^{-1})C_i(Q+\hat{Q})C_i^T,$$

$$P_a \triangleq B^T P + R_{12}^T + \sum{}' (\delta_i\alpha_i^{-1})B_i^T(P+\hat{P})A_i,$$

$$Q_a \triangleq QC^T + V_{12} + \sum{}' (\delta_i\alpha_i^{-1})A_i(Q+\hat{Q})C_i^T,$$

$$D \triangleq \sum{}'' \delta_i(D_iD_i^T + H_iH_i^T), \qquad E \triangleq \sum{}'' \delta_i E_i^T E_i,$$

$$\hat{A} \triangleq A + \frac{1}{2}\sum{}' \delta_i\alpha_i I_n, \quad \hat{A}_P \triangleq \hat{A} - BR_{2a}^{-1}P_a, \quad \hat{A}_Q \triangleq \hat{A} - Q_aV_{2a}^{-1}C.$$

The following lemma will be needed.

LEMMA 6.1. *If $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and* rank $\hat{Q}\hat{P} = n_c$, *then there exist $G, \Gamma \in \mathbb{R}^{n_c\times n}$ and invertible $M \in \mathbb{R}^{n_c\times n_c}$ such that*

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{6.10}$$

$$\Gamma G^T = I_{n_c}. \tag{6.11}$$

Furthermore, G, M, and Γ are unique except for a change of basis in $\mathbb{R}^{n_c}$.

Proof. The result is an immediate consequence of [45, Thm. 6.2.5, p. 123]. □

Note that because of (6.11), the $n\times n$ matrix $\tau \triangleq G^T\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Since τ is not necessarily symmetric, it is an oblique projection. Also, define $\tau_\perp \triangleq I_n - \tau$.

THEOREM 6.1. *Suppose $(\mathcal{P}, A_c, B_c, C_c)$ solves the Auxiliary Minimization Problem subject to* (6.8) *and* (6.9). *Then there exist $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ such that $\mathcal{P}, A_c, B_c, C_c$ are given by*

$$\mathcal{P} = \begin{bmatrix} P+\hat{P} & -\hat{P}G^T \\ -G\hat{P} & G\hat{P}G^T \end{bmatrix}, \tag{6.12}$$

$$A_c = \Gamma(A - Q_aV_{2a}^{-1}C - BR_{2a}^{-1}P_a + DP)G^T, \tag{6.13}$$

$$B_c = \Gamma Q_a V_{2a}^{-1}, \tag{6.14}$$

$$C_c = -R_{2a}^{-1}P_aG^T, \tag{6.15}$$

and such that $P, Q, \hat{P}, \hat{Q}$ satisfy

$$\begin{aligned} 0 = \hat{A}^TP + P\hat{A} + R_1 + \sum{}' (\delta_i\alpha_i^{-1})[A_i^TPA_i + (A_i - Q_aV_{2a}^{-1}C_i)^T\hat{P}(A_i - Q_aV_{2a}^{-1}C_i)] \\ + E + PDP - P_a^TR_{2a}^{-1}P_a + \tau_\perp^TP_a^TR_{2a}^{-1}P_a\tau_\perp, \end{aligned} \tag{6.16}$$

$$\begin{aligned} 0 = [\hat{A} + D(P+\hat{P})]Q + Q[\hat{A} + D(P+\hat{P})]^T + V_1 \\ + \sum{}' (\delta_i\alpha_i^{-1})[A_iQA_i^T + (A_i - B_iR_{2a}^{-1}P_a)\hat{Q}(A_i - B_iR_{2a}^{-1}P_a)^T] \\ - Q_aV_{2a}^{-1}Q_a^T + \tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \end{aligned} \tag{6.17}$$

$$0 = (\hat{A}_Q + DP)^T\hat{P} + \hat{P}(\hat{A}_Q + DP) + \hat{P}D\hat{P} + P_a^TR_{2a}^{-1}P_a - \tau_\perp^TP_a^TR_{2a}^{-1}P_a\tau_\perp, \tag{6.18}$$

$$0 = (\hat{A}_P + DP)\hat{Q} + \hat{Q}(\hat{A}_P + DP)^T + Q_aV_{2a}^{-1}Q_a^T - \tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \tag{6.19}$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c. \tag{6.20}$$

Conversely, if there exist $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ *satisfying* (6.16)-(6.20), *then* $\mathcal{P}$ *given by* (6.12) *satisfies* (6.2) *or, equivalently,* (4.2) *with* (A_c, B_c, C_c) *given by* (6.13)-(6.15).

Outline of proof. As discussed in § 1, we limit the presentation of the proof to the salient details. First note that with the choice of bounds Λ_i, (6.2) becomes

$$0=\left(\tilde{A}+\frac{1}{2}\sum{}' \delta_i\alpha_i I_{\tilde{n}}\right)^T \mathcal{P}+\mathcal{P}\left(\tilde{A}+\frac{1}{2}\sum{}' \delta_i\alpha_i I_{\tilde{n}}\right)+\tilde{R}$$
$$+\sum{}' (\delta_i\alpha_i^{-1})\tilde{A}_i^T\mathcal{P}\tilde{A}_i+\sum{}'' \delta_i(\tilde{E}_i^T\tilde{E}_i+\mathcal{P}\tilde{D}_i\tilde{D}_i^T\mathcal{P}). \tag{6.21}$$

By introducing multipliers $\lambda \in \mathbb{R}$, $\lambda \geqq 0$, and $\mathcal{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, a Lagrangian can be defined as

$$\mathcal{L}(\mathcal{P}, A_c, B_c, C_c) \triangleq \operatorname{tr}[\lambda\mathcal{P}\tilde{V}+\mathcal{Q}(\text{RHS of (6.21)})]. \tag{6.22}$$

Setting $\partial\mathcal{L}/\partial\mathcal{P}=0$ and using (6.8) implies that $\lambda=1$ without loss of generality, $\mathcal{Q}\geqq 0$, and $\mathcal{Q}$ satisfies

$$0=\left(\tilde{A}+\frac{1}{2}\sum{}' \delta_i\alpha_i I_{\tilde{n}}+\sum{}'' \tilde{D}_i\tilde{D}_i^T\mathcal{P}\right)\mathcal{Q}+\mathcal{Q}\left(\tilde{A}+\frac{1}{2}\sum{}' \delta_i\alpha_i I_{\tilde{n}}+\sum{}'' \tilde{D}_i\tilde{D}_i^T\mathcal{P}\right)^T$$
$$+\sum{}' (\delta_i\alpha_i^{-1})\tilde{A}_i\mathcal{Q}\tilde{A}_i^T+\tilde{V}. \tag{6.23}$$

The remainder of the derivation is exactly parallel to the techniques utilized in [29] and [36]. Briefly, the principal steps are as follows:

Step 1. Compute $\partial\mathcal{L}/\partial A_c$, $\partial\mathcal{L}/\partial B_c$, and $\partial\mathcal{L}/\partial C_c$.
Step 2. Use (6.9) to show that the lower right $n_c\times n_c$ blocks of $\mathcal{Q}$ and $\mathcal{P}$ are positive definite.
Step 3. Use $\partial\mathcal{L}/\partial A_c=0$ to define a projection τ and new variables $P, Q, \hat{P}, \hat{Q}, G, \Gamma$.
Step 4. Partition (6.21) and (6.23) into six equations (1)-(6) corresponding to the $n\times n$, $n\times n_c$ and $n_c\times n_c$ blocks of $\mathcal{P}$ and $\mathcal{Q}$, respectively.
Step 5. Use (2) and (3) to solve for A_c; show that (5) and (6) also yield A_c; note that with A_c now given, (3) and (6) are superfluous and can be eliminated.
Step 6. Manipulate (1), (2), (4), and (5) to yield (6.16)-(6.19).
Step 7. Show that Steps 4-6 are reversible so that (6.16)-(6.20) are equivalent to (6.2) or, equivalently, (4.2).

By enforcing the strict inequalities $\mathcal{P}>0$ and (6.3), solutions of (6.16)-(6.20) guarantee robust stability with a robust performance bound. The following result follows from Theorem 4.1, Theorem 4.2, and the converse of Theorem 6.1.

THEOREM 6.2. *Suppose there exist* $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ *satisfying* (6.16)-(6.20), *and suppose that* (6.3) *and* $\mathcal{P}>0$ *are satisfied with* $(\mathcal{P}, A_c, B_c, C_c)$ *given by* (6.12)-(6.15). *Then the compensator* A_c, B_c, C_c *given by* (6.13)-(6.15) *solves the Robust Stability Problem and the closed-loop performance* (3.7) *satisfies the bound*

$$J(A_c, B_c, C_c) \leqq \operatorname{tr}\mathcal{P}\tilde{V}. \tag{6.24}$$

The following existence result concerns the solvability of (6.16)-(6.20). Let n_u denote the dimension of the unstable subspace of the plant dynamics matrix A.

THEOREM 6.3. *Assuming* $n_c \geqq n_u$; $R_1>0$, $V_1>0$, *suppose the nominal plant, i.e.,* (3.1), (3.2) *with* $\delta_i=0$, $i=1,\cdots,p$, *is stabilizable and detectable and, in addition, is stabilizable by means of an* n_c*th-order strictly proper dynamic compensator* (3.3), (3.4). *Then there exist* $\bar{\delta}_1,\cdots,\bar{\delta}_p>0$ *such that if* $\delta_i\in[0,\bar{\delta}_i)$, $i=1,\cdots,p$, *then* (6.16)-(6.20) *have a solution* $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ *for which* (A_c, B_c, C_c) *given by* (6.13)-(6.15) *solve the robust stability problem with robust performance bound* (6.24).

Proof. From Theorem 3.1 of [37] it follows that there exists a solution to (6.16)–(6.20) that stabilizes the nominal plant. By continuity there exists a neighborhood over which robust stability with performance bound (6.24) holds. □

Theorem 6.3 is an existence result that guarantees solvability of the sufficiency conditions over a range of parameter uncertainties. The actual range of uncertainty that can be bounded and the conservatism of the performance bound are problem dependent. To this end we now consider a numerical example.

7. Illustrative numerical example. To demonstrate the theory above we present an illustrative numerical example. The example chosen was originally used in [2] to illustrate the lack of a guaranteed gain margin for LQG controllers. This example was also considered in [35] for a preliminary robustness study and reconsidered in [46] using μ-analysis. Define the following:

$$n = n_u = 2, \qquad m = l = p = 1,$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0],$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [0 \quad 0],$$

$$R_1 = V_1 = \begin{bmatrix} 60 & 60 \\ 60 & 60 \end{bmatrix}, \quad R_{12} = V_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 = V_2 = 1.$$

Note that the system is open-loop unstable and becomes uncontrollable at $\sigma_1 = -1$. As can be seen using root locus, a strictly proper stabilizing controller must be of at least second order. Hence we consider (6.16)–(6.20) with $n_c = n$ and $\tau_\perp = 0$. Furthermore, we use bound (5.3) and thus set $D = E = 0$. Using algorithms described in

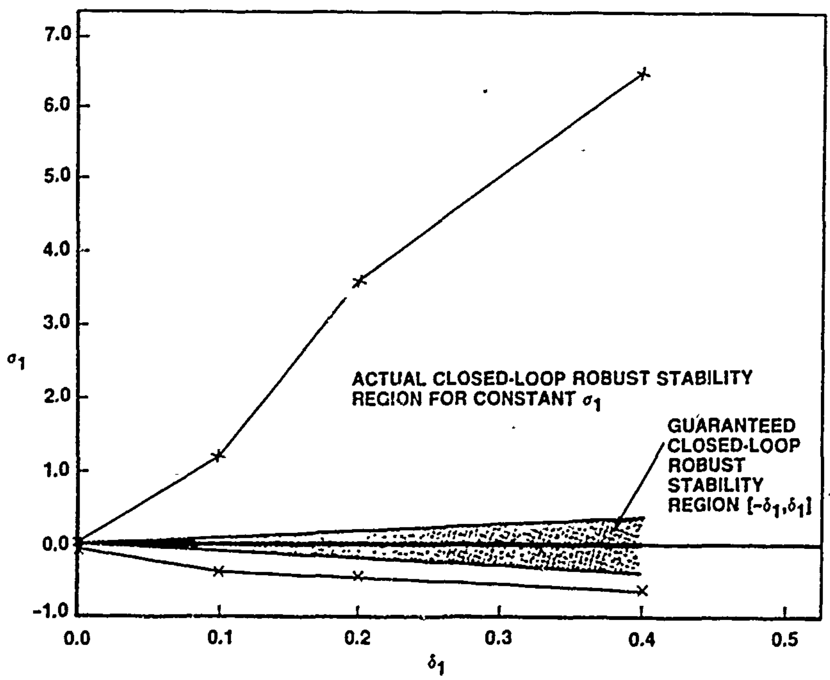

FIG. 1

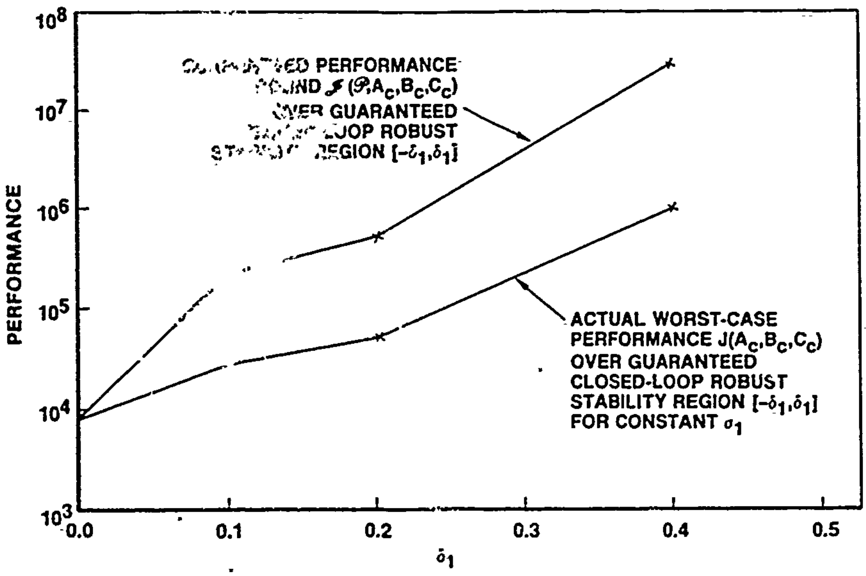

FIG. 2

TABLE 1

(δ_1, α_1)	A_c	B_c	C_c
(.1, 1)	$\begin{bmatrix} -14.917 & 1.0 \\ -85.177 & 3.9657 \end{bmatrix}$	$\begin{bmatrix} 15.917 \\ 79.959 \end{bmatrix}$	$[-5.2182 \quad -4.9657]$
(.2, 2)	$\begin{bmatrix} -17.963 & 1.0 \\ -133.65 & -4.4614 \end{bmatrix}$	$\begin{bmatrix} 18.963 \\ 127.05 \end{bmatrix}$	$[-6.6011 \quad -5.4614]$
(.4, 4)	$\begin{bmatrix} -47.813 & 1.0 \\ -1087.3 & -6.5463 \end{bmatrix}$	$\begin{bmatrix} 48.813 \\ 1073.5 \end{bmatrix}$	$[-13.766 \quad -7.5463]$

[38]–[40], controllers were obtained by solving (6.16)–(6.20) for $(\delta_1, \alpha_1) = (.1, 1), (.2, 2)$, and $(.4, 4)$. As stated previously, these numerical solutions also verify (4.2) with $\mathcal{P}$ given by (6.12). Figure 1 compares the guaranteed robust stability region to the "actual" robust stability region. This robust stability region was evaluated assuming constant $\hat{\sigma}_1(\cdot)$, although the theory actually guarantees robustness with respect to time-varying uncertainties. Thus, the gap between these regions may not be a reliable measure of the conservatism of the results. Note, however, that the design approach appears to provide more stability than is guaranteed a priori. This feature may be attributable to the desire for a symmetric stability interval so close to an unstabilizable plant perturbation, i.e., $\sigma_1 = -1$. Nevertheless, the stability design objectives have been met in accordance with Theorem 6.2. Interestingly, the form of the actual stability region mimics the classical 6-dB-downward/infinite-dB-upward gain margin of full-state-feedback LQR controllers [1]. Thus, this approach appears to provide an alternative to gain-margin recovery techniques [9], which address this specialized form of plant uncertainty. Finally, Fig. 2 compares guaranteed closed loop performance to "actual" closed-loop performance over the guaranteed closed-loop robust stability region. Again the "actual" region was determined for constant $\hat{\sigma}_1(\cdot)$. Controller gains are given in

Table 1. Finally, we note that higher-order robust controllers were obtained for this example in [46] using the μ-function approach.

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The Optimal Projection Equations with Petersen–Hollot Bounds: Robust Stability and Performance Via Fixed-Order Dynamic Compensation for Systems with Structured Real-Valued Parameter Uncertainty

DENNIS S. BERNSTEIN AND WASSIM M. HADDAD

Dedicated to the memory of Vicki Ringer Jones
October 26, 1951–December 12, 1987

Abstract—A feedback control-design problem involving structured real-valued plant parameter uncertainties is considered. A quadratic Lyapunov bound suggested by recent work of Petersen and Hollot is utilized in conjunction with the guaranteed cost approach of Chang and Peng to guarantee robust stability with robust performance bound. Necessary conditions which generalize the optimal projection equations for fixed-order dynamic compensation are used to characterize the controller which minimizes the performance bound. The design equations thus effectively serve as sufficient conditions for synthesizing dynamic output-feedback controllers which provide robust stability and performance.

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D. S. Bernstein is with the Government Aerospace Systems Division, Harris Corporation, Melbourne, FL 32902.

W. M. Haddad is with the Department of Mechanical Engineering, Florida Institute of Technology, Melbourne, FL 32901.

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I. INTRODUCTION

As is well known, LQR and LQG controllers lack guaranteed robustness with respect to arbitrary parameter variations [1], [2]. Thus, it is not surprising that there is considerable interest in the analysis and synthesis of feedback controllers which are robust with respect to structured real-valued plant parameter uncertainty. The present paper was motivated in particular by the guaranteed cost control approach of Chang and Peng [3], [4] and the robust stability technique of Petersen and Hollot [5]–[7]. In [3], Chang and Peng consider a modified Riccati equation whose solutions are guaranteed to provide both robust stability and performance over a specified range of parameter variations. On the other hand, Petersen and Hollot in [5]–[7] consider a different modified Riccati equation which utilizes a quadratic Lyapunov bound to provide robust stability over a range of structured plant variations. In the present paper, we combine aspects of both of these approaches to obtain both robust stability and performance.

Our preference for the Petersen–Hollot bound over the bound originally proposed by Chang and Peng is based upon the fact that the former is differentiable with respect to the Riccati solution, while the latter is not. We exploit this smoothness by utilizing the optimal projection approach for fixed-order dynamic compensation [8] in place of full-state feedback considered in [3], [4], [6], [7]. A systematic, in-depth treatment of the Chang–Peng, Petersen–Hollot, and other bounds (such as the right shift/multiplicative white noise bound considered in [9]–[11]) will be the subject of a future paper [12].

As discussed in [8], the optimal projection approach to fixed-order dynamic compensation is based upon a system of two modified algebraic Riccati equations and two modified algebraic Lyapunov equations which directly generalize LQG theory to the case of reduced-order controllers. To ensure robust stability and performance for reduced-order controllers, the present paper utilizes the Petersen–Hollot quadratic Lyapunov technique to bound the performance of controllers of fixed dimension. The performance bound is then interpreted as the cost functional for an *auxiliary minimization problem* whose optimality conditions directly generalize the results of [8]. Specifically, we again obtain a coupled system of algebraic Riccati and Lyapunov equations with additional terms arising from the Petersen–Hollot bound. When uncertainty is absent, these equations specialize immediately to the result of [8] which, in turn, specializes to LQG when the compensator order is equal to the plant dimension.

Although the optimal projection equations are necessary conditions for optimality, it is important to stress that in the present paper they are obtained not for the original cost function, but rather for a *bound* on the cost. The necessary conditions for the auxiliary minimization problem thus effectively serve as sufficient conditions for the original problem. Hence, even if a numerical solution of the extended optimal projection equations fails to produce the globally optimal controller, robust stability and performance are still guaranteed for all local extremals. Our approach thus seeks to rectify one of the main drawbacks of necessity theory by guaranteeing both robust stability and performance. Nevertheless, a numerical algorithm for computing the global optimum is given in [15].

In summary, *the main contribution of the present paper is the generalization of the optimal projection equations by means of the Petersen–Hollot quadratic Lyapunov bound to synthesize robustly stabilizing fixed-order dynamic compensators with guaranteed performance bound.* It is interesting to note that even in the full-order case, our results, which specialize to a coupled system of *three* matrix equations, are distinct from the results of [5] which involve a pair of modified Riccati equations and an auxiliary inequality. Furthermore, the

present paper provides a robust performance bound not obtained in [5]–[7]. An additional, conceptual benefit of our approach is a rigorous optimization interpretation for the Petersen–Hollot Riccati equation approach. Finally, as shown in [20] for full-state feedback, the results given herein can be directly applied to the H_∞ design problem. For details, see [21].

Due to space constraints, the contents of the paper will not be reviewed here. We note only that the proof of Theorem 8.1, which has been omitted for this reason, can be found in [13], [14]. Finally, although numerical algorithms are outside the scope of this note, related results can be found in [15].

II. Notation and Definitions

Note: All matrices have real entries.

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathcal{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.
$I_r, ()^T$	$r \times r$ identity matrix, transpose.
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$.
$n, m, l, n_c; \tilde{n}$	Positive integers; $n + n_c$.
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors.
$A, \Delta A; B, \Delta B; C, \Delta C$	$n \times n$ matrices; $n \times m$ matrices; $l \times n$ matrices.
A_c, B_c, C_c	$n_c \times n_c$; $n_c \times l$; $m \times n_c$ matrices.
$\tilde{A}, \Delta\tilde{A}$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & 0 \end{bmatrix}$.
R_1, R_2	$n \times n$, $m \times m$ state, control weighting matrices; $R_1 \ge 0$, $R_2 > 0$.
R_{12}	$n \times m$ cross weighting matrix: $R_1 - R_{12}R_2^{-1}R_{12}^T \ge 0$.
$w_1(\cdot), w_2(\cdot)$	n, l-dimensional white noise.
V_1, V_2	Intensity of $w_1(\cdot), w_2(\cdot)$; $V_1 \ge 0$, $V_2 > 0$.
V_{12}	$n \times l$ cross intensity of $w_1(\cdot), w_2(\cdot)$.
$\tilde{w}(\cdot), \tilde{V}$	$\begin{bmatrix} w_1(\cdot) \\ B_cw_2(\cdot) \end{bmatrix}, \begin{bmatrix} V_1 & V_{12}B_c^T \\ B_cV_{12}^T & B_cV_2B_c^T \end{bmatrix}$
$\tilde{R}$	$\begin{bmatrix} R_1 & R_{12}C_c \\ C_c^TR_{12}^T & C_c^TR_2C_c \end{bmatrix}$.

III. Robust Stability and Robust Performance Problems

Let $\mathcal{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times n}$ denote the set of uncertain perturbations $(\Delta A, \Delta B, \Delta C)$ of the nominal plant matrices A, B, and C.

Robust Stability Problem. For fixed $n_c \le n$, determine (A_c, B_c, C_c) such that the closed-loop system consisting of the nth-order controlled plant

$$\dot{x}(t) = (A+\Delta A)x(t) + (B+\Delta B)u(t), \quad t \in [0, \infty), \tag{3.1}$$

measurements

$$y(t) = (C+\Delta C)x(t), \tag{3.2}$$

and n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t), \tag{3.3}$$

$$u(t) = C_cx_c(t) \tag{3.4}$$

is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$.

Robust Performance Problem. For fixed $n_c \le n$, determine (A_c, B_c, C_c) such that, for the closed-loop system consisting of the nth-order disturbed plant

$$\dot{x}(t) = (A+\Delta A)x(t) + (B+\Delta B)u(t) + w_1(t), \quad t \in [0, \infty), \tag{3.5}$$

noisy measurements

$$y(t) = (C+\Delta C)x(t) + w_2(t), \tag{3.6}$$

and n_cth-order dynamic compensator (3.3), (3.4), the performance criterion

$$J(A_c, B_c, C_c) \triangleq \sup_{(\Delta A, \Delta B, \Delta C)\in\mathcal{U}} \limsup_{t\to\infty} \mathcal{E}[x^T(t)R_1x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2u(t)] \tag{3.7}$$

is minimized.

Remark 3.1: Note that (3.7) is precisely the LQG criterion except for the supremum over $\mathcal{U}$ for worst-case performance.

For each controller (A_c, B_c, C_c) and plant variation $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, the undisturbed closed-loop system (3.1)–(3.4) is given by

$$\dot{\tilde{x}}(t) = (\tilde{A}+\Delta\tilde{A})\tilde{x}(t), \quad t \in [0, \infty), \tag{3.8}$$

while the disturbed closed-loop system (3.3)–(3.6) is

$$\dot{\tilde{x}}(t) = (\tilde{A}+\Delta\tilde{A})\tilde{x}(t) + \tilde{w}(t), \quad t \in [0, \infty), \tag{3.9}$$

where $\tilde{x}(t) \triangleq [x^T(t), x_c^T(t)]^T$ and $\tilde{w}(\cdot)$ is white noise with intensity $\tilde{V} \in \mathbb{N}^{\tilde{n}}$.

IV. Sufficient Conditions for Robust Stability and Performance

In practice, steady-state performance is only of interest when the closed-loop system (3.8) is stable over $\mathcal{U}$. The following result expresses the performance in terms of the steady-state closed-loop second-moment matrix.

Lemma 4.1: Suppose (3.8) is stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Then

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C)\in\mathcal{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}\tilde{R}, \tag{4.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty} \mathcal{E}[\tilde{x}(t)\tilde{x}^T(t)] \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A}+\Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A}+\Delta\tilde{A})^T + \tilde{V}. \tag{4.2}$$

We now seek upper bounds for $J(A_c, B_c, C_c)$.

Theorem 4.1: Let $\Omega: \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c\times l} \times \mathbb{R}^{m\times n_c} \to \mathbb{S}^{\tilde{n}}$ be such that

$$\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T \le \Omega(\mathcal{Q}, B_c, C_c),$$
$$(\Delta A, \Delta B, \Delta C) \in \mathcal{U}, \ (\mathcal{Q}, B_c, C_c) \in \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c\times l} \times \mathbb{R}^{m\times n_c}, \tag{4.3}$$

and, for given (A_c, B_c, C_c), suppose there exists $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \Omega(\mathcal{Q}, B_c, C_c) + \tilde{V}, \tag{4.4}$$

and suppose the pair $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$,

$$\tilde{Q}_{\Delta\tilde{A}} \le \mathcal{Q}, \quad (\Delta A, \Delta B, \Delta C) \in \mathcal{U}, \tag{4.5}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ satisfies (4.2), and

$$J(A_c, B_c, C_c) \le \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{4.6}$$

Proof: For all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, (4.4) is equivalent to

$$0 = (\tilde{A}+\Delta\tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A}+\Delta\tilde{A})^T + \Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) + \tilde{V}, \tag{4.7}$$

where

$$\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) \triangleq \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T).$$

Note that by (4.3), $\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) \ge 0$ for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, it follows from [16, Theorem 3.6] that $((\tilde{V} + \Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}))^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Hence, [16, Lemma 12.2] implies $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in$

$\mathfrak{U}$. Next, subtracting (4.2) from (4.7) yields

$$0=(\tilde{A}+\Delta\tilde{A})(\mathcal{Q}-\tilde{\mathcal{Q}}_{\Delta\tilde{A}})+(\mathcal{Q}-\tilde{\mathcal{Q}}_{\Delta\tilde{A}})(\tilde{A}+\Delta\tilde{A})^T+\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A})$$

or, equivalently (since $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable),

$$\mathcal{Q}-\tilde{\mathcal{Q}}_{\Delta\tilde{A}}=\int_0^\infty e^{(\tilde{A}+\Delta\tilde{A})t}\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A})e^{(\tilde{A}+\Delta\tilde{A})^Tt}\,dt\geq 0,$$

which implies (4.5). Finally, (4.5) and (4.1) yield (4.6). □

V. Uncertainty Structure

To obtain explicit expressions for (A_c, B_c, C_c), we require that $\Delta B = 0$, $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$. Hence, for simplicity, we write $(\Delta A, \Delta C) \in \mathfrak{U}$. The dual case $\Delta B \neq 0$ and $\Delta C = 0$ is treated in Section X. Thus, $\mathfrak{U}$ is assumed to be of the form

$$\mathfrak{U}=\Big\{(\Delta A, \Delta C)\in \mathbb{R}^{n\times n}\times\mathbb{R}^{l\times n}: \Delta A=\sum_{i=1}^p D_iM_iN_iE_i,$$
$$\Delta C=\sum_{i=1}^p F_iM_iN_iE_i,\quad M_iM_i^T\leq \hat{M}_i,\ N_i^TN_i\leq\hat{N}_i,\ i=1,\cdots,p\Big\}, \quad (5.1)$$

where, for $i=1,\cdots,p$: $D_i\in\mathbb{R}^{n\times r_i}$, $E_i\in\mathbb{R}^{s_i\times n}$, and $F_i\in\mathbb{R}^{l\times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\hat{M}_i\in\mathbb{N}^{r_i}$ and $\hat{N}_i\in\mathbb{N}^{s_i}$ are given uncertainty bounds; and $M_i\in\mathbb{R}^{r_i\times s_i}$ and $N_i\in\mathbb{R}^{s_i\times s_i}$ are uncertain matrices. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A}=\sum_{i=1}^p\tilde{D}_iM_iN_i\tilde{E}_i,$$

where

$$\tilde{D}_i\triangleq\begin{bmatrix}D_i\\ B_cF_i\end{bmatrix},\ \tilde{E}_i\triangleq[E_i\quad 0].$$

The special case $\hat{M}_i=\mu_i^2I_{r_i}$, $\hat{N}_i=\nu_i^2I_{s_i}$ is worth noting.

Proposition 5.1: Let $\mu_i, \nu_i\geq 0$, $i=1,\cdots,p$. Then $M_iM_i^T\leq\mu_i^2I_{r_i}$ and $N_i^TN_i\leq\nu_i^2I_{s_i}$ if and only if $\sigma_{\max}(M_i)\leq\mu_i$ and $\sigma_{\max}(N_i)\leq\nu_i$.

Remark 5.1: The form of $\mathfrak{U}$ given by (5.1) is directly related to the structured stability radius introduced by Hinrichsen and Pritchard [17], [18]. Specifically, let $p=1$, $\hat{M}_1=\mu_1I_{r_1}$, $r_1=s_1$, and $N_1\equiv\hat{N}_1=I_{r_1}$.

VI. The Petersen-Hollot Bound

Given $\mathfrak{U}$, we now specify the bound Ω satisfying (4.3). Note that because of $\Delta B=0$, Ω is independent of C_c. Hence, we write $\Omega(\mathcal{Q}, B_c)$ for $\Omega(\mathcal{Q}, B_c, C_c)$.

Proposition 6.1: The function

$$\Omega(\mathcal{Q}, B_c)\triangleq\sum_{i=1}^p\tilde{D}_i\hat{M}_i\tilde{D}_i^T+\mathcal{Q}\tilde{E}_i^T\hat{N}_i\tilde{E}_i\mathcal{Q} \qquad (6.1)$$

satisfies (4.3) with $\mathfrak{U}$ given by (5.1).

Proof: For $i=1,\cdots,p$,

$$\begin{aligned}0&\leq[\tilde{D}_iM_i-\mathcal{Q}\tilde{E}_i^TN_i^T][\tilde{D}_iM_i-\mathcal{Q}\tilde{E}_i^TN_i^T]^T\\&=\tilde{D}_iM_iM_i^T\tilde{D}_i^T+\mathcal{Q}\tilde{E}_i^TN_i^TN_i\tilde{E}_i\mathcal{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathcal{Q}+\mathcal{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T)\\&\leq\tilde{D}_i\hat{M}_i\tilde{D}_i^T+\mathcal{Q}\tilde{E}_i^T\hat{N}_i\tilde{E}_i\mathcal{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathcal{Q}+\mathcal{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T).\end{aligned}$$

Summing over i yields (4.3). □

Remark 6.1: The bound (6.1) was originally proposed by Petersen in [5] for unit-rank perturbations with scalar uncertain parameters. A more general treatment appears in [7]. Note that we absorb the epsilon used in [7] into D_i and E_i.

VII. The Auxiliary Minimization Problem

To optimize robust performance while guaranteeing robust stability, we consider the following problem.

Auxiliary Minimization Problem: Determine $(\mathcal{Q}, A_c, B_c, C_c)$ which minimizes

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)\triangleq\operatorname{tr}\mathcal{Q}\tilde{R} \qquad (7.1)$$

subject to

$$\mathcal{Q}\in\mathbb{N}^{\tilde{n}}, \qquad (7.2)$$

$$0=\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^T+\sum_{i=1}^p[\tilde{D}_i\hat{M}_i\tilde{D}_i^T+\mathcal{Q}\tilde{E}_i^T\hat{N}_i\tilde{E}_i\mathcal{Q}]+\tilde{V}, \qquad (7.3)$$

$$(\tilde{V}^{1/2}, \tilde{A}+\Delta\tilde{A})\text{ is stabilizable, }(\Delta A, \Delta C)\in\mathfrak{U}. \qquad (7.4)$$

Proposition 7.1: If $(\mathcal{Q}, A_c, B_c, C_c)$ satisfies (7.2)–(7.4), then $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C)\in\mathfrak{U}$ and

$$J(A_c, B_c, C_c)\leq\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c). \qquad (7.5)$$

Proof: With Ω given by (6.1), the hypotheses of Theorem 4.1 are satisfied so that robust stability is guaranteed with performance bound (4.6). □

VIII. Necessary Conditions for the Auxiliary Minimization Problem

Rigorous derivation of the necessary conditions for the Auxiliary Minimization Problem requires additional technical assumptions. Specifically, in addition to (7.2), we restrict $(\mathcal{Q}, A_c, B_c, C_c)$ to the open set

$$\mathcal{S}\triangleq\{(\mathcal{Q}, A_c, B_c, C_c): \mathcal{Q}\in\mathbb{P}^{\tilde{n}},\ \tilde{\mathcal{A}}\text{ is asymptotically stable,}$$
$$\text{and }(A_c, B_c, C_c)\text{ is controllable and observable}\},$$

where (see [19] for the definition of the Kronecker sum)

$$\tilde{\mathcal{A}}\triangleq\left(\tilde{A}+\sum_{i=1}^p\mathcal{Q}\tilde{E}_i^T\hat{N}_i\tilde{E}_i\right)\oplus\left(\tilde{A}+\sum_{i=1}^p\mathcal{Q}\tilde{E}_i^T\hat{N}_i\tilde{E}_i\right).$$

Furthermore, the constraint (7.4) will not be accounted for explicitly since it can be shown that the compactness of $\mathfrak{U}$ implies that the set of (A_c, B_c, C_c) satisfying (7.4) is open.

Remark 8.1: The constraint $(\mathcal{Q}, A_c, B_c, C_c)\in\mathcal{S}$ is not required for either robust stability or robust performance since Proposition 7.1 shows that only (7.2)–(7.4) are needed. Rather, the set $\mathcal{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the condition $\mathcal{Q}>0$ replaces (7.2) by an open set constraint, the stability of $\tilde{\mathcal{A}}$ serves as a normality condition, and (A_c, B_c, C_c) minimal is a nondegeneracy condition.

For arbitrary $Q, P\in\mathbb{R}^{n\times n}$ define the following notation:

$$D\triangleq\sum_{i=1}^p D_i\hat{M}_iD_i^T,\quad E\triangleq\sum_{i=1}^p E_i^T\hat{N}_iE_i,$$

$$P_a\triangleq B^TP+R_{12}^T,\quad Q_a\triangleq QC^T+V_{12}+\sum_{i=1}^p D_i\hat{M}_iF_i^T,$$

$$A_P\triangleq A-BR_2^{-1}P_a,\quad A_Q\triangleq A-Q_aV_{2a}^{-1}C,\quad V_{2a}\triangleq V_2+\sum_{i=1}^p F_i\hat{M}_iF_i^T.$$

The following factorization lemma is needed. For details, see [8].

Lemma 8.1: If $\hat{Q}, \hat{P}\in\mathbb{N}^n$ and rank $\hat{Q}\hat{P}=n_c$, then there exist $n_c\times n$ G, Γ, and $n_c\times n_c$ invertible M such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \qquad (8.1)$$

$$\Gamma G^T=I_{n_c}. \qquad (8.2)$$

Furthermore, G, M, and Γ are unique except for a change of basis in $\mathbb{R}^{n_c}$.

As shown in [8], the matrix τ defined by

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}=G^T\Gamma \tag{8.3}$$

is an oblique projection where $(\ )^{\#}$ denotes group generalized inverse [8]. For convenience, define the complementary projection $\tau_\perp \triangleq I_n - \tau$.

Theorem 8.1: If $(\mathcal{Q}, A_c, B_c, C_c) \in \mathcal{S}$ solves the Auxiliary Minimization Problem with $\mathcal{U}$ given by (5.1), then there exist $Q, P, \hat{Q}, \hat{P} \in \mathfrak{N}^n$ such that

$$\mathcal{Q}=\begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}, \tag{8.4}$$

$$A_c=\Gamma(A-BR_{2}^{-1}P_a-Q_aV_{2a}^{-1}C+QE)G^T, \tag{8.5}$$

$$B_c=\Gamma Q_aV_{2a}^{-1}, \tag{8.6}$$

$$C_c=-R_2^{-1}P_aG^T, \tag{8.7}$$

and such that $Q, P, \hat{Q}, \hat{P}$ satisfy

$$0=AQ+QA^T+V_1+D+QEQ-Q_aV_{2a}^{-1}Q_a^T+\tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \tag{8.8}$$

$$0=[A+(Q+\hat{Q})E]^TP+P[A+(Q+\hat{Q})E] +R_1-P_a^TR_2^{-1}P_a+\tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{8.9}$$

$$0=(A_P+QE)\hat{Q}+\hat{Q}(A_P+QE)^T+\hat{Q}E\hat{Q}+Q_aV_{2a}^{-1}Q_a^T-\tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \tag{8.10}$$

$$0=(A_Q+QE)^T\hat{P}+\hat{P}(A_Q+QE)+P_a^TR_2^{-1}P_a-\tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{8.11}$$

$$\operatorname{rank}\hat{Q}=\operatorname{rank}\hat{P}=\operatorname{rank}\hat{Q}\hat{P}=n_c. \tag{8.12}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)=\operatorname{tr}[(Q+\hat{Q})R_1+\hat{Q}PBR_2^{-1}P_a-R_{12}R_2^{-1}P_a\hat{Q}]. \tag{8.13}$$

Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathfrak{N}^n$ satisfying (8.8)–(8.12), then $(\mathcal{Q}, A_c, B_c, C_c)$ given by (8.4)–(8.7) satisfy (7.2) and (7.3) with cost (8.13).

Proof: See [13], [14]. □

Remark 8.2: Theorem 8.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly characterize extremal quadruples $(\mathcal{Q}, A_c, B_c, C_c)$. These necessary conditions consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by both the optimal projection τ and uncertainty terms. Several special cases can immediately be discerned. For example, in the full-order case $n_c = n$, set $\tau = I_n$ so that $\tau_\perp = 0$. Now the last term in each of (8.8)–(8.11) can be deleted and G and Γ in (8.5)–(8.7) can be taken to be the identity. Furthermore, $\hat{P}$ plays no role so that (8.11) is superfluous. Note that in this case, (8.8) is independent of P and $\hat{Q}$. Setting further D_i, E_i, and F_i to zero, it can be seen that (8.10) and (8.11) drop out, while (8.8) and (8.9) reduce to the standard separated Riccati equations of LQG theory. If, alternatively, the reduced-order constraint is retained, but the uncertainty terms are deleted, then the results of [8] are recovered.

Remark 8.3: When solving (8.8)–(8.12) numerically, the uncertainty terms can be adjusted to examine tradeoffs between performance and robustness. Specifically, the bounds $\hat{M}_i$ and $\hat{N}_i$ and structure matrices D_i, E_i, and F_i appearing in Q_a, D, E, and V_{2a} can be varied systematically to determine the region of solvability of (8.8)–(8.12).

IX. Sufficient Conditions for Robust Stability and Performance

Theorem 9.1: Suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathfrak{N}^n$ satisfying (8.8)–(8.12), and assume that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathcal{U}$ with A_c, B_c, C_c given by (8.5)–(8.7) and $\mathcal{U}$ given by (5.1). Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathcal{U}$ and the closed-loop performance is bounded by (8.13).

Proof: Theorem 8.1 implies that $\mathcal{Q}$ given by (8.4) satisfies (7.2) and (7.3). With the stabilizability assumption, the result follows from Proposition 7.1. □

X. The Dual Case

In place of (5.1), assume now that $\Delta C = 0$, $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, and define

$$\mathcal{U}=\left\{(\Delta A, \Delta B) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m} : \Delta A=\sum_{i=1}^{p} D_iM_iN_iE_i,\right.$$

$$\left.\Delta B=\sum_{i=1}^{p} D_iM_iN_iG_i, \quad M_iM_i^T\le\hat{M}_i,\ N_i^TN_i\le\hat{N}_i,\ i=1,\cdots,p\right\}, \tag{10.1}$$

where, for $i = 1, \cdots, p$: $D_i \in \mathbb{R}^{n\times q_i}$, $E_i \in \mathbb{R}^{l_i\times n}$, and $G_i \in \mathbb{R}^{l_i\times m}$ are fixed matrices denoting the structure of the uncertainty; and $\hat{M}_i$, $\hat{N}_i$, M_i, and N_i are as before. For arbitrary $Q, P \in \mathbb{R}^{n\times n}$ define the following notation:

$$\hat{P}_a \triangleq B^TP+R_{12}^T+\sum_{i=1}^{p} G_i^T\hat{N}_iE_i, \quad \hat{Q}_a \triangleq QC^T+V_{12},$$

$$\hat{A}_P \triangleq A-BR_{2a}^{-1}\hat{P}_a, \quad \hat{A}_Q \triangleq A-\hat{Q}_aV_2^{-1}C, \quad R_{2a} \triangleq R_2+\sum_{i=1}^{p} G_i^T\hat{N}_iG_i.$$

The main result guaranteeing robust stability and performance for the dual problem can now be stated. For details, see [13], [14].

Theorem 10.1: Suppose there exist $P, Q, \hat{P}, \hat{Q} \in \mathfrak{N}^n$ satisfying (8.12) and

$$0=A^TP+PA+R_1+E+PDP-\hat{P}_a^TR_{2a}^{-1}\hat{P}_a+\tau_\perp^T\hat{P}_a^TR_{2a}^{-1}\hat{P}_a\tau_\perp, \tag{10.2}$$

$$0=[A+D(P+\hat{P})]Q+Q[A+D(P+\hat{P})]^T +V_1-\hat{Q}_aV_2^{-1}\hat{Q}_a^T+\tau_\perp\hat{Q}_aV_2^{-1}\hat{Q}_a^T\tau_\perp^T, \tag{10.3}$$

$$0=(\hat{A}_Q+DP)^T\hat{P}+\hat{P}(\hat{A}_Q+DP)+\hat{P}D\hat{P}+\hat{P}_a^TR_{2a}^{-1}\hat{P}_a-\tau_\perp^T\hat{P}_a^TR_{2a}^{-1}\hat{P}_a\tau_\perp, \tag{10.4}$$

$$0=(\hat{A}_P+DP)\hat{Q}+\hat{Q}(\hat{A}_P+DP)^T+\hat{Q}_aV_2^{-1}\hat{Q}_a^T-\tau_\perp\hat{Q}_aV_2^{-1}\hat{Q}_a^T\tau_\perp^T, \tag{10.5}$$

and assume that $(\tilde{R}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B) \in \mathcal{U}$ with A_c, B_c, C_c given by

$$A_c=\Gamma(A-\hat{Q}_aV_2^{-1}C-BR_{2a}^{-1}\hat{P}_a+DP)G^T, \tag{10.6}$$

$$B_c=\Gamma\hat{Q}_aV_2^{-1}, \tag{10.7}$$

$$C_c=-R_{2a}^{-1}\hat{P}_aG^T, \tag{10.8}$$

and $\mathcal{U}$ given by (10.1). Then, with (10.6)–(10.8), $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B) \in \mathcal{U}$ and the performance of the closed-loop system satisfies

$$J(A_c, B_c, C_c)\le\operatorname{tr}[(P+\hat{P})V_1+\hat{Q}_aV_2^{-1}CQ\hat{P}-\hat{P}\hat{Q}_aV_2^{-1}V_{12}^T]. \tag{10.9}$$

Remark 10.1: Even in the case $\Delta B = 0$, $\Delta C = 0$, the performance bounds (8.13) and (10.9) are generally different.

Remark 10.2: The case in which ΔB and ΔC are simultaneously nonzero also appears to be tractable and leads to additional terms in the design equations. The bound considered in [11] also permits this case.

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Robust Stability and Performance via Fixed-Order Dynamic Compensation with Guaranteed Cost Bounds*

Dennis S. Bernstein† and Wassim M. Haddad‡

Abstract. A feedback control-design problem involving structured plant parameter uncertainties is considered. Two robust control-design issues are addressed. The Robust Stability Problem involves deterministic bounded structured parameter variations, while the Robust Performance Problem includes, in addition, a quadratic performance criterion averaged over stochastic disturbances and maximized over the admissible parameter variations. The optimal projection approach to fixed-order dynamic compensation is merged with the guaranteed cost control approach to robust stability and performance to obtain a theory of full- and reduced-order robust control design. The principle result is a sufficient condition for characterizing dynamic controllers of fixed dimension which are guaranteed to provide both robust stability and performance. The sufficient conditions involve a system of modified Riccati and Lyapunov equations coupled by an oblique projection and the uncertainty bounds. The full-order result involves a system of two modified Riccati equations and two modified Lyapunov equations coupled by the uncertainty bounds. The coupling illustrates the breakdown of the separation principle for LQG control with structured plant parameter variations.

Key words. Robust stability, Robust performance, LQG control, Dynamic compensation, structured uncertainty.

1. Introduction

The direct method of Lyapunov has proven to be an effective approach to robust analysis and design of feedback control laws. References [B1], [B2], [BCL], [BG2], [CL], [CP], [ER], [GB], [H], [KB], [KBH], [L], [PH], [TB], and [VW] comprise a representative collection of the literature in this area. In performing robust synthesis there are two principal issues, namely, stability robustness and performance robustness. Stability robustness addresses the problem of guaranteeing stability of the closed-loop system for plant perturbations within a specified class of uncertainties. In addition to guaranteeing robust stability, it is often desirable to minimize the worst-case performance degradation within a given robust stability

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† Harris Corporation, Government Aerospace Systems Division, MS 22.4842, Melbourne, Florida 32902, U.S.A.

‡ Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, Florida 32901, U.S.A.

range. Although both robust stability and performance are of interest in practice, most of the literature involving quadratic Lyapunov functions deals only with the problem of robust stability. A notable exception is the early work of Chang and Peng [CP] which also provides bounds on worst-case quadratic performance within full-state feedback control design.

The contribution of this paper is a methodology for designing controllers which provide both robust stability and robust performance over a prescribed range of structured plant parameter variations. The feedback law is in the form of a fixed-order (i.e., full- or reduced-order) strictly proper dynamic compensator. The overall approach is based upon the merging of two distinct control-design techniques, namely, the guaranteed cost control approach to robust performance [CP] and the optimal projection approach to fixed-order dynamic compensation [BH3], [HB]. The principal motivation for our approach is to permit greater flexibility in the design of robust feedback laws by providing an alternative to full-state feedback and full-order dynamic compensation.

The guaranteed cost control approach [CP] adopted in this paper utilizes a performance bound to provide robust performance in addition to robust stability. Here, robust performance refers to a guaranteed bound on the worst-case value of the expectation of a quadratic cost criterion over a prescribed uncertainty set. This quadratic criterion is precisely the standard cost functional of linear-quadratic-Gaussian (LQG) control theory. By bounding the worst-case value of this criterion over a specified range of plant uncertainties, we effectively bound the variances of specified states and control signals.

To bound the worst-case closed-loop performance, we require a bound on the effect of plant uncertainties on the steady-state closed-loop covariance matrix. The form of the guaranteed cost control bound utilized herein was originally motivated by the effect of multiplicative white noise on the state covariance [B2], [BG2]. Since this bound is differentiable with respect to the covariance matrix and compensator gains, it permits optimal design via first-order necessary conditions. This approach is not possible using the nondifferentiable bound ordinally proposed in [CP]. An alternative differentiable bound proposed in [PH] for full-state feedback has been extended to fixed-order dynamic compensation in [BH1].

In this paper the guaranteed cost technique is used to bound the closed-loop performance and characterize robustly stabilizing controllers. This performance bound is then interpreted as an *auxiliary cost* which is to be minimized by the choice of compensator gains. The *actual* performance for a given realization of the parameter uncertainty is thus guaranteed to lie below this bound. Assuming stabilizability (disturbability), the robust performance bound automatically implies robust stability. The auxiliary cost and the Lyapunov equation constraint together form the Auxiliary Minimization Problem. Since the Auxiliary Minimization Problem is a nonconvex mathematical programming problem with differentiable data, it is amenable to first-order necessary conditions.

One feature of this approach is that since the necessary conditions are obtained for the Auxiliary Minimization Problem rather than the original problem, extremals are guaranteed to provide both robust stability and performance. Note that this is true for every extremal of the Auxiliary Minimization Problem whether it corre-

sponds to a local minimum, local maximum, or otherwise. Of course, the global minimum is most likely to provide the best worst-case performance over the robust stability range. In any case, necessary conditions for the Auxiliary Minimization Problem effectively serve as *sufficient* conditions for robust stability with a guaranteed performance bound.

This paper presents a rigorous development of sufficient conditions for robust stability and performance via fixed-order dynamic compensation. These sufficient conditions are in the form of a coupled system of algebraic matrix equations consisting of two modified Riccati equations and two modified Lyapunov equations. The coupling is due to the optimal projection, which characterizes reduced-order controllers, and the uncertainty bounds, which account for the effect of parameter uncertainties on the performance functional. When the compensator order is constrained to be equal to the dimension of the plant and the uncertainty bounds are absent, the equations specialize to the usual pair of separated Riccati equations of steady-state LQG theory.

We emphasize that our approach is constructive in nature rather than existential. Our sufficient conditions provide explicit formulae for robust, fixed-order feedback gains when the Auxiliary Minimization Problem has a solution, and in this case our constructive conditions are complementary to existential results on robust stabilizability. The existence of a solution to the Auxiliary Minimization Problem and associated design equations depends upon stabilizability via fixed-order controllers and on the sharpness of the quadratic Lyapunov bounds. The stabilizability problem has been studied using independent methods (see, e.g., [BHK]), while the conservatism of the bounds is considered in [BH2]. Here we state a local existence result for solvability of the design equations which assumes only nominal stabilizability.

The contents and scope of this paper are as follows. In Section 2 we state the robust stability and performance problems for fixed-order dynamic compensation with plant parameter uncertainty. In Section 3 a modified Lyapunov equation is introduced whose solution, when it exists, is guaranteed to bound the steady-state closed-loop covariance over the specified range of plant uncertainty. A performance bound is then given in terms of the covariance bound. In Section 4 we view the performance bound as an auxiliary cost and consider the problem of minimizing the auxiliary cost subject to the modified Lyapunov equation and a definiteness condition as side constraints. These side constraints have the property that all admissible elements provide robust stability and performance (Proposition 4.1). In Section 5 the uncertainty set and bound for constructing the modified Lyapunov equation are given concrete forms. Specifically, the uncertainty set has the form of an ellipsoidal region in parameter space while the modified Lyapunov equation includes additional linear terms to bound the uncertainty. A sufficient condition involving Kronecker sums and products implies the existence of a unique, nonnegative-definite solution to the modified Lyapunov equation. Section 6 presents the first-order necessary conditions (Theorem 6.1) for the Auxiliary Minimization Problem under minor additional technical conditions to ensure the applicability of the Lagrange multiplier technique. As discussed above, these necessary conditions are in the form of extended optimal projection equations. A partial

converse of the necessary conditions shows that solutions of these algebraic equations provide, by construction, a solution of the original modified Lyapunov equation. This result is combined in Section 7 (Theorem 7.1) with a stabilizability assumption to guarantee robust stability with a robust performance bound. In addition, we state an existence result for local solvability of the design equations by applying a result from [R1] and [R2] (Theorem 7.2). To draw connections with standard LQG theory, in Section 8 we specialize Theorem 7.1 to the full-order case. In contrast to the pair of separated Riccati equations of standard LQG theory, the full-order result in the presence of plant parameter variations is given by a coupled system of *four* modified Riccati/Lyapunov equations. In Section 9 the theory is illustrated by means of an example due to Doyle [D]. This problem was also considered in [BG1] before the robustness theory developed herein was available. Hence this paper can be viewed as the rigorous mathematical foundation which legitimizes the heretofore *ad hoc* robustness approach of [BG1].

Notation. Note: All matrices have real entries

$\mathbb{E}$	expected value
I_r, $(\ )^{\mathrm{T}}$, $0_{r\times s}$, 0_r	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
$Z_{(i,j)}$	(i, j)-element of matrix Z
$\oplus$, $\otimes$	Kronecker sum, Kronecker product [B3]
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$
A_α, $A_{c\alpha}$	$A + (\alpha/2)I_n$, $A_c + (\alpha/2)I_{n_c}$
R_1, R_2	state, control weighting matrices; $R_1 \in \mathbb{N}^n$, $R_2 \in \mathbb{P}^m$
R_{12}	$n \times m$ cross-weighting matrix; $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \ge 0$
$\tilde{R}$	$\begin{bmatrix} R_1 & R_{12}C_c \\ C_c^{\mathrm{T}}R_{12}^{\mathrm{T}} & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}$
$w_1(\cdot)$, $w_2(\cdot)$	n, l-dimensional white noise
V_1, V_2	intensity of $w_1(\cdot)$, $w_2(\cdot)$; $V_1 \in \mathbb{N}^n$, $V_2 \in \mathbb{P}^l$
V_{12}	$n \times l$ cross intensity of $w_1(\cdot)$, $w_2(\cdot)$
$\tilde{w}(\cdot)$, $\tilde{V}$	$\begin{bmatrix} w_1(\cdot) \\ B_cw_2(\cdot) \end{bmatrix}$, $\begin{bmatrix} V_1 & V_{12}B_c^{\mathrm{T}} \\ B_cV_{12}^{\mathrm{T}} & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}$

2. Robust Stability and Robust Performance Problems

In this section we state the Robust Stability Problem and Robust Performance Problem. Both problems involve a set $\mathscr{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times m}$ of uncertain perturbations $(\Delta A, \Delta B, \Delta C, \Delta D)$ of the nominal system matrices (A, B, C, D). The goal of the Robust Stability Problem is to determine a fixed-order, strictly proper dynamic compensator (A_c, B_c, C_c) which stabilizes the plant for all variations in $\mathscr{U}$ In this section and the following section no explicit assumptions are required for the set $\mathscr{U}$. In Section 5 the structure of variations in $\mathscr{U}$ will be specified.

Robust Stability Problem. For fixed $n_c \leq n$ determine (A_c, B_c, C_c) such that the closed-loop system consisting of the nth-order controlled plant,

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t), \qquad t \in [0, \infty), \tag{2.1}$$

measurements

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t), \tag{2.2}$$

and n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t), \tag{2.4}$$

is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$.

The Robust Performance Problem involves, in addition, white plant disturbances and measurement noise. The goal of this problem is to determine a fixed-order, strictly proper compensator (A_c, B_c, C_c) which minimizes the worst-case value over the uncertainty set $\mathcal{U}$ of a steady-state average quadratic performance criterion.

Robust Performance Problem. For fixed $n_c \leq n$, determine (A_c, B_c, C_c) such that, for the closed-loop system consisting of the nth-order controlled and disturbed plant

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + w_1(t), \qquad t \in [0, \infty), \tag{2.5}$$

noisy measurements

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t) + w_2(t), \tag{2.6}$$

and n_cth-order dynamic compensator (2.3), (2.4), the performance criterion

$$J(A_c, B_c, C_c) \triangleq$$

$$\sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}} \limsup_{t \to \infty} \mathbb{E}[x^{\mathrm{T}}(t) R_1 x(t) + 2x^{\mathrm{T}}(t) R_{12} u(t) + u^{\mathrm{T}}(t) R_2 u(t)] \tag{2.7}$$

is minimized.

Remark 2.1. The cost functional (2.7) is identical to the standard LQG criterion with the exception of the supremum for evaluating worst-case quadratic performance over $\mathcal{U}$. Note that (2.7) can also be written in terms of an averaged integral, i.e.,

$$J(A_c, B_c, C_c) =$$

$$\sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}} \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}\left\{\int_0^t [x^{\mathrm{T}}(s) R_1 x(s) + 2x^{\mathrm{T}}(s) R_{12} u(s) + u^{\mathrm{T}}(s) R_2 u(s)]\, ds\right\}.$$

For practical application, the cost (2.7) provides the means for minimizing the variances of selected state variables and control signals. This can be achieved by appropriate selection of the matrices R_1 and R_2 which serve as design weights. For robust performance the goal is to minimize the worst-case variances of selected variables over the plant uncertainty.

For each uncertain variation $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}$, the undisturbed closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t), \qquad t \in [0, \infty), \tag{2.8}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c + B_cDC_c \end{bmatrix}, \qquad \Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & B_c\Delta DC_c \end{bmatrix}.$$

Similarly, the disturbed closed-loop system (2.3)–(2.6) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{w}(t), \qquad t \in [0, \infty), \tag{2.9}$$

where the closed-loop disturbace $\tilde{w}(t)$ has intensity $\tilde{V} \in \mathbb{N}^{\tilde{n}}$.

3. Sufficient Conditions for Robust Stability and Performance

In practice, steady-state performance is only of interest when the undisturbed closed-loop system (2.8) is robustly stable over $\mathscr{U}$. The following result, which expresses the performance in terms of the steady-state closed-loop second-moment matrix, is immediate.

Lemma 3.1. *Let (A_c, B_c, C_c) be given and assume the system (2.8) is stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}$. Then*

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}} \tilde{R}, \tag{3.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \tilde{V}. \tag{3.2}$$

Remark 3.1. When $\mathscr{U}$ is compact, "sup" in (3.1) can be replaced by "max."

The key step in guaranteeing robust stability and performance is to replace the uncertain terms in the covariance Lyapunov equation (3.2) by a bounding function Ω. Note that since $\Delta\tilde{A}$ is independent of A_c, the bounding function need only depend upon B_c and C_c.

Theorem 3.1. *Let $\Omega\colon \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \to \mathbb{S}^{\tilde{n}}$ be such that*

$$\Delta\tilde{A}\mathscr{Q} + \mathscr{Q}\Delta\tilde{A}^{\mathrm{T}} \le \Omega(\mathscr{Q}, B_c, C_c),$$
$$(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}, \qquad (\mathscr{Q}, B_c, C_c) \in \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}, \tag{3.3}$$

and, for given (A_c, B_c, C_c), assume there exists $\mathscr{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \Omega(\mathscr{Q}, B_c, C_c) + \tilde{V}. \tag{3.4}$$

Then

$$(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathscr{Q}, B_c, C_c) - (\Delta\tilde{A}\mathscr{Q} + \mathscr{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2}) \quad \textit{is stabilizable},$$
$$(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathscr{U}, \tag{3.5}$$

if and only if

$$\tilde{A} + \Delta\tilde{A} \quad \textit{is asymptotically stable}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}. \tag{3.6}$$

In this case,

$$\tilde{Q}_{\Delta\tilde{A}} \leq \mathcal{Q}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}, \tag{3.7}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ *is given by* (3.2), *and*

$$J(A_c, B_c, C_c) \leq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{3.8}$$

Proof. First note for clarity that in (3.3) $\mathcal{Q}$ denotes an arbitrary element of $\mathbb{N}^{\tilde{n}}$ since (3.3) holds for all $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$, while in (3.4) $\mathcal{Q}$ denotes a specific solution to (3.4). Now for $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, (3.4) is equivalent to

$$0 = (\tilde{A} + \Delta\tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A} + \Delta\tilde{A}^{\mathrm{T}}) + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}}) + \tilde{V}. \tag{3.9}$$

Hence, by assumption, (3.9) has a solution $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$ and, by (3.3), $\Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})$ is nonnegative definite. Now if the stabilizability condition (3.5) holds for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, it follows from Lemma 12.2 of [W] that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. Conversely, if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, then (3.5) holds. Next, subtracting (3.2) from (3.9) yields

$$0 = (\tilde{A} + \Delta\tilde{A})(\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}}) + (\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}})(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}}),$$

or, equivalently, since $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$,

$$\mathcal{Q} - \tilde{Q}_{\Delta\tilde{A}} = \int_0^\infty e^{(\tilde{A}+\Delta\tilde{A})t}[\Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]e^{(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}t}\, dt \geq 0,$$

which implies (3.7). The performance bound (3.8) is now an immediate consequence of (3.7). ■

Remark 3.2. In applying Theorem 3.1 it may be convenient to replace condition (3.5) with a stronger condition which is easier to verify in practice. Clearly, (3.5) is satisfied if $\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})$ is positive definite for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. This will be the case, for example, if either $\tilde{V}$ is positive definite or strict inequality holds in (3.3). Also, it follows from Theorem 3.6 of [W] that (3.5) is implied by the stronger condition

$$(\tilde{A} + \Delta\tilde{A}, \tilde{V}^{1/2}) \quad \text{is stabilizable}, \qquad (\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}. \tag{3.10}$$

Remark 3.3. The covariance bound (3.7) can also be used to analyze the effect of disturbances on specified state variables. For example, if $E_1 \in \mathbb{R}^{q \times n}$, then (3.7) implies

$$[E_1 \quad 0_{q \times n_c}]\tilde{Q}_{\Delta\tilde{A}}\begin{bmatrix} E_1^{\mathrm{T}} \\ 0_{n_c \times q} \end{bmatrix} \leq [E_1 \quad 0_{q \times n_c}]\mathcal{Q}\begin{bmatrix} E_1^{\mathrm{T}} \\ 0_{n_c \times q} \end{bmatrix} \tag{3.11}$$

so that the right-hand side of (3.11) serves as a bound on selected state variances. For control-design purposes we effectively set $R_1 = E_1^{\mathrm{T}} E_1$. Similar remarks apply to obtaining bounds on the variances of control signals.

4. The Auxiliary Minimization Problem

The key step in our development involves consideration of the performance bound (3.8) in place of the actual worst-case performance $J(A_c, B_c, C_c)$. This leads to the following problem.

Auxiliary Minimization Problem. Determine $(\mathcal{Q}, A_c, B_c, C_c)$ which minimizes

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R} \tag{4.1}$$

subject to (3.4) and

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}}. \tag{4.2}$$

The relationship between the Auxiliary Minimization Problem and the Robust Stability and Performance Problems is straightforward as shown by the following observation.

Proposition 4.1. *If $(\mathcal{Q}, A_c, B_c, C_c)$ satisfies (3.4) and (4.2) and the stabilizability condition (3.5) holds, then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$, and*

$$J(A_c, B_c, C_c) \le \mathcal{J}(\mathcal{Q}, A_c, B_c, C_c). \tag{4.3}$$

Proof. Since (3.4) has a solution $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ and the stabilizability condition (3.5) holds, the hypotheses of Theorem 3.1 are satisfied so that robust stability with robust performance bound (3.8) is guaranteed; (4.3) is merely a restatement of (3.8). ■

Several comments are in order. Since the *auxiliary cost* (4.1) is an upper bound for the actual cost (2.7), it is clearly desirable to minimize (4.1) over $\mathcal{Q}$ and the controller gains. Note, however, that the Auxiliary Minimization Problem is a nonconvex mathematical programming problem on a noncompact set. Hence existence of solutions and sufficient conditions for global optimality cannot be obtained without imposing additional restrictive assumptions. To develop nonrestrictive results, we proceed in Section 6 by deriving necessary conditions for optimality which require no further assumptions except that Ω be differentiable and that the minimization be performed over an open set. In the next section we construct a bound Ω which possesses the required smoothness.

5. Uncertainty Structure and the Guaranteed Cost Bound

Having established the theoretical basis for our approach, we now assign explicit structure to the set $\mathcal{U}$ and bounding function Ω. Specifically, the uncertainty set $\mathcal{U}$ is assumed to be of the form

$$\mathcal{U} = \left\{(\Delta A, \Delta B, \Delta C, \Delta D): \Delta A = \sum_{i=1}^{p} \sigma_i A_i, \Delta B = \sum_{i=1}^{p} \sigma_i B_i, \Delta C = \sum_{i=1}^{p} \sigma_i C_i, \right.$$
$$\left. \Delta D = \sum_{i=1}^{p} \sigma_i D_i, \sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \le 1 \right\}, \tag{5.1}$$

where for $i = 1, \ldots, p$: $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{l \times n}$ and $D_i \in \mathbb{R}^{l \times m}$ are fixed matrices denoting the structure of the parametric uncertainty; α_i is a given positive number; and σ_i is an uncertain real parameter. Note that the uncertain parameters σ_i are assumed to lie in a specified ellipsoidal region in $\mathbb{R}^p$. The closed-loop system (2.8) thus has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{p} \sigma_i \tilde{A}_i, \tag{5.2}$$

where

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & B_c D_i C_c \end{bmatrix}, \qquad i = 1, \ldots, p.$$

The uncertainty set $\mathscr{U}$ is general in the sense that no explicit assumptions such as the matching conditions used in [BCL] will be made with regard to the structure of A_i, B_i, C_i, and D_i. We do, however, require (as is evident from (5.1)) that the uncertain parameters σ_i appear linearly in the off-nominal perturbations which is more confining than matching assumptions. Note that the symmetry of the uncertainty set entails no loss of generality by requiring only a redefinition of the nominal plant matrices.

In order to obtain *explicit* gain expressions for (A_c, B_c, C_c) in Section 6, we shall require one additional technical assumption. Specifically, we assume that, for each $i \in \{1, \ldots, p\}$, at most one of the matrices B_i, C_i, and D_i is nonzero. This condition thus implies that a given uncertain parameter σ_i may appear explicitly in both ΔA and ΔB, or both ΔA and ΔC, or both ΔA and ΔD, or only ΔA, but not (say) in both ΔB and ΔD. Thus we can account partially (but not totally) for *correlated* parameter uncertainties in different plant matrices. If a given uncertain parameter does arise in both (say) ΔB and ΔD, then it must be represented by two distinct uncertain parameters. If this assumption is not imposed, then optimality conditions can still be derived, but at the expense of closed-form gain expressions.

For the structure of $\mathscr{U}$ as specified by (5.1), the bound Ω satisfying (3.3) can now be given a concrete form.

Proposition 5.1. *Let α be an arbitrary positive scalar. Then the function*

$$\Omega(\mathscr{Q}, B_c, C_c) = \alpha\mathscr{Q} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} \tag{5.3}$$

satisfies (3.3) *with $\mathscr{U}$ given by* (5.1).

Proof. Note that

$$0 \le \sum_{i=1}^{p} [(\alpha^{1/2}\sigma_i/\alpha_i) I_{\tilde{n}} - (\alpha_i/\alpha^{1/2})\tilde{A}_i] \mathscr{Q} [(\alpha^{1/2}\sigma_i/\alpha_i) I_{\tilde{n}} - (\alpha_i/\alpha^{1/2})\tilde{A}_i]^{\mathrm{T}}$$

$$= \alpha \sum_{i=1}^{p} (\sigma_i^2/\alpha_i^2)\mathscr{Q} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} - \sum_{i=1}^{p} \sigma_i (\tilde{A}_i \mathscr{Q} + \mathscr{Q}\tilde{A}_i^{\mathrm{T}}),$$

which, since $\sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \le 1$, implies (3.3). ∎

Remark 5.1. Note that the bound Ω given by (5.3) consists of two distinct terms. The first term $\alpha\mathscr{Q}$ can be thought of as arising from an exponential time weighting

of the cost, or, equivalently, from a uniform right shift of the open-loop dynamics [AM]. The second term $\alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i\mathcal{Q}\tilde{A}_i^{\mathrm{T}}$ arises naturally from a multiplicative white-noise model [BG1], [BG2], [B]. Such interpretations have no bearing on the results obtained here since only the bound Ω defined by (5.3) is required. Note that the bound (5.3) is valid for all positive α. A similar bound was also considered in [KB].

With Ω defined by (5.3), the modified Lyapunov equation (3.4) becomes

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \alpha\mathcal{Q} + \alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i\mathcal{Q}\tilde{A}_i^{\mathrm{T}} + \tilde{V} \tag{5.4}$$

or, equivalently,

$$0 = \tilde{A}_\alpha\mathcal{Q} + \mathcal{Q}\tilde{A}_\alpha^{\mathrm{T}} + \sum_{i=1}^{p}\gamma_i\tilde{A}_i\mathcal{Q}\tilde{A}_i^{\mathrm{T}} + \tilde{V}, \tag{5.5}$$

where

$$\tilde{A}_\alpha \triangleq \tilde{A} + \frac{\alpha}{2}I_{\tilde{n}} = \begin{bmatrix} A_\alpha & BC_c \\ B_cC & A_{c\alpha} + B_cDC_c \end{bmatrix} \tag{5.6}$$

and $\gamma_i \triangleq \alpha_i^2/\alpha$. Note that (5.5) is equivalent to

$$0 = \mathscr{A}\operatorname{vec}\mathcal{Q} + \operatorname{vec}\tilde{V}, \tag{5.7}$$

where "vec" is the column-stacking operation defined in [B3] and $\mathscr{A}$ is defined by

$$\mathscr{A} \triangleq \tilde{A}_\alpha \oplus \tilde{A}_\alpha + \sum_{i=1}^{p}\gamma_i\tilde{A}_i \otimes \tilde{A}_i.$$

Next, using the bound Ω given by (5.3) and $\mathscr{U}$ given by (5.1) we present a result which guarantees the existence of a nonnegative-definite solution to (3.4) or, equivalently, (5.5) for a given controller (A_c, B_c, C_c). For the converse we view $\tilde{V}$ as an arbitrary element of $\mathbb{N}^{\tilde{n}}$.

Proposition 5.2. *Let (A_c, B_c, C_c) be given and let $\alpha > 0$. If $\mathscr{A}$ is asymptotically stable, then there exists a unique $\tilde{n}\times\tilde{n}$ $\mathcal{Q}$ satisfying (5.5) and, furthermore, $\mathcal{Q} \geq 0$. Conversely, if for all $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ there exists $\mathcal{Q} \geq 0$ satisfying (5.5), then $\mathscr{A}$ is asymptotically stable.*

Proof. Since (5.5) is equivalent to

$$\mathcal{Q} = -\operatorname{vec}^{-1}[\mathscr{A}^{-1}\operatorname{vec}\tilde{V}], \tag{5.8}$$

existence and uniqueness hold. To prove that $\mathcal{Q}$ is nonnegative definite, we rewrite (5.8) as

$$\mathcal{Q} = \int_0^\infty \operatorname{vec}^{-1}[e^{\mathscr{A}t}\operatorname{vec}\tilde{V}]\,dt \tag{5.9}$$

and show that the integrand is nonnegative definite for all $t \in [0, \infty)$. (Note that the following argument does not require that $\mathscr{A}$ be stable). Using the Lie exponential product formula, the exponential in (5.9) can be written as

$$e^{\mathscr{A}t} = \lim_{k\to\infty}\left\{\exp\left[\frac{1}{k}(\tilde{A}_\alpha \oplus \tilde{A}_\alpha)t\right]\prod_{i=1}^{p}\exp\left[\frac{1}{k}\gamma_i(\tilde{A}_i \otimes \tilde{A}_i)t\right]\right\}^k. \tag{5.10}$$

For convenience, let S and N be $r \times r$ matrices with $N \geq 0$. Since (see [B3])

$$\mathrm{vec}^{-1}[(S \otimes S)\,\mathrm{vec}\,N] = SNS^{\mathrm{T}} \geq 0 \tag{5.11}$$

and

$$(S^k \otimes S^k)(S \otimes S) = S^{k+1} \otimes S^{k+1}, \tag{5.12}$$

it follows that

$$\mathrm{vec}^{-1}[e^{S \oplus S}\,\mathrm{vec}\,N] = \sum_{k=0}^{\infty} (k!)^{-1} S^k N S^{k\mathrm{T}} \geq 0. \tag{5.13}$$

Furthermore,

$$\mathrm{vec}^{-1}[e^{S \oplus S}\,\mathrm{vec}\,N] = \mathrm{vec}^{-1}[(e^S \otimes e^S)\,\mathrm{vec}\,N] = e^S N e^{S^{\mathrm{T}}} \geq 0. \tag{5.14}$$

Applying (5.13) and (5.14) alternately with (5.10) and using induction on k it follows that the integrand of (5.9) is nonnegative definite. To prove the converse, note that it follows from (5.5) that $\mathcal{Q}$ satisfies

$$\mathcal{Q} = \mathrm{vec}^{-1}[e^{\mathscr{A}t}\,\mathrm{vec}\,\mathcal{Q}] + \int_0^t \mathrm{vec}^{-1}[e^{\mathscr{A}s}\,\mathrm{vec}\,\tilde{V}]\,ds, \qquad t \in [0, \infty). \tag{5.15}$$

Since the integral term on the right-hand side of (5.15) is nonnegative definite, is bounded from above by $\mathcal{Q}$, and $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ is arbitrary, it follows that $\mathscr{A}$ is asymptotically stable. ■

Proposition 5.2 shows that a solution of (5.5) exists as long as $\alpha_1, \ldots, \alpha_p$ are sufficiently small so that $\mathscr{A}$ remains stable for some $\alpha > 0$. The following result characterizes values of $\alpha_1, \ldots, \alpha_p$ for which $\mathscr{A}$ is asymptotically stable. Let $\|\cdot\|$ denote an arbitrary vector norm and its induced matrix norm.

Proposition 5.3. *Let (A_c, B_c, C_c) be given, assume $\tilde{A}$ is asymptotically stable, and let $\alpha, \alpha_1, \ldots, \alpha_p > 0$. If*

$$\left\| (\tilde{A} \oplus \tilde{A})^{-1} \left(\alpha I_{\tilde{n}^2} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \otimes \tilde{A}_i \right) \right\| < 1, \tag{5.16}$$

then there exists $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying (5.5) and $\mathscr{A}$ is asymptotically stable.

Proof. Define $\{\mathcal{Q}_k\}_{k=0}^{\infty}$ where $\mathcal{Q}_0$ satisfies

$$0 = \tilde{A}\mathcal{Q}_0 + \mathcal{Q}_0\tilde{A}^{\mathrm{T}} + \tilde{V},$$

and $\mathcal{Q}_{k+1}$ satisfies

$$0 = \tilde{A}\mathcal{Q}_{k+1} + \mathcal{Q}_{k+1}\tilde{A}^{\mathrm{T}} + \Omega(\mathcal{Q}_k, B_c, C_c) + \tilde{V}.$$

Note that $\mathcal{Q}_k \geq 0$, $k = 1, 2, \ldots$. Hence it follows that

$$\mathrm{vec}\,\mathcal{Q}_{k+1} - \mathrm{vec}\,\mathcal{Q}_k = -(\tilde{A} \oplus \tilde{A})^{-1}[\mathrm{vec}\,\Omega(\mathcal{Q}_k, B_c, C_c) - \mathrm{vec}\,\Omega(\mathcal{Q}_{k-1}, B_c, C_c)]$$

and thus

$$\|\mathrm{vec}\,\mathcal{Q}_{k+1} - \mathrm{vec}\,\mathcal{Q}_k\| \leq \left\| (\tilde{A} \oplus \tilde{A})^{-1} \left(\alpha I_{\tilde{n}^2} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \otimes \tilde{A}_i \right) \right\| \|\mathrm{vec}\,\mathcal{Q}_k - \mathrm{vec}\,\mathcal{Q}_{k-1}\|.$$

Using (5.16) it follows that $\mathscr{Q} \triangleq \lim_{k\to\infty} \mathscr{Q}_k$ exists. Thus $\mathscr{Q} \geq 0$ satisfies (5.5). Furthermore, since $\tilde{V} \in \mathbb{N}^{\tilde{n}}$ can be considered arbitrary, Proposition 5.2 implies that $\mathscr{A}$ is asymptotically stable. ■

6. Necessary Conditions for the Auxiliary Minimization Problem

The derivation of the necessary conditions for the Auxiliary Minimization Problem is based upon the Fritz John form of the Lagrange multiplier theorem. Application of this theorem requires that we further restrict $(\mathscr{Q}, A_c, B_c, C_c)$ to the open set

$$\mathscr{S} \triangleq \{(\mathscr{Q}, A_c, B_c, C_c)\colon \mathscr{Q} \in \mathbb{P}^{\tilde{n}}, \mathscr{A} \text{ is asymptotically stable,}$$
$$\text{and } (A_c, B_c, C_c) \text{ is controllable and observable}\}.$$

As will be seen, the constraint $(\mathscr{Q}, A_c, B_c, C_c) \in \mathscr{S}$ is not required for either robust stability or robust performance since Proposition 4.1 shows that only (3.4), (3.5), and (4.2) are needed. Rather, the set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the condition $\mathscr{Q} > 0$ replaces (4.2) by an open set constraint, the asymptotic stability of $\mathscr{A}$ serves as a normality condition which further implies that the dual $\mathscr{P}$ of $\mathscr{Q}$ is nonnegative definite, and (A_c, B_c, C_c) minimal is a nondegeneracy condition which implies that the lower right $n_c \times n_c$ subblocks of $\mathscr{Q}$ and $\mathscr{P}$ are positive definite thus yielding explicit expressions for B_c and C_c. Note that by Proposition 5.2 the condition that $\mathscr{A}$ be asymptotically stable also implies that (5.5) has a unique, nonnegative solution. Finally, we point out that the stabilizability condition (3.5) and stability condition (3.6) play no role in determining solutions of the Auxiliary Minimization Problem.

In order to state the main results we require some additional notation and a lemma concerning pairs of nonnegative-definite matrices. For a real $n \times n$ matrix Z define the set of real diagonalizing matrices

$$\mathscr{D}(Z) \triangleq \{\Psi \in \mathbb{R}^{n\times n}\colon \Psi^{-1} Z \Psi \text{ is diagonal}\},$$

and, for a pair of $n \times n$ symmetric matrices, X, Y define the set of real *contragrediently diagonalizing* matrices

$$\mathscr{C}(X, Y) \triangleq \{\Psi \in \mathscr{D}(XY)\colon \Psi^{-1} X \Psi^{-\mathrm{T}} \text{ and } \Psi^{\mathrm{T}} Y \Psi \text{ are diagonal}\}$$

and the subset of real *balancing* transformations

$$\mathscr{B}(X, Y) \triangleq \{\Psi \in \mathscr{C}(X, Y)\colon \Psi^{-1} X \Psi^{-\mathrm{T}} = \Psi^{\mathrm{T}} Y \Psi\}.$$

Of course, a necessary condition for $\mathscr{B}(X, Y)$ to be nonempty is that X, Y, and XY all have the same rank. Note that in general

$$\mathscr{B}(X, Y) \subset \mathscr{C}(X, Y) \subset \mathscr{D}(XY). \tag{6.1}$$

Obviously, a diagonalizable matrix is either invertible (has no zero eigenvalues) or has semisimple zero eigenvalues. Hence if $\mathscr{D}(Z) \neq \varnothing$, then the group generalized inverse $Z^{\#}$ exists as a special case of the Drazin generalized inverse [CM]. Note that we limit our consideration to diagonalizable matrices with real eigenvalues.

Also, note that there is no assumption here that Z is symmetric. Of course, when Z is symmetric the group, Drazin, and Moore–Penrose generalized inverses coincide.

Lemma 6.1. *Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and let $r = \operatorname{rank} \hat{Q}\hat{P}$. Then the following statements hold:*

(i) *$\hat{Q}\hat{P}$ has nonnegative eigenvalues.*
(ii) *$\mathscr{C}(\hat{Q}, \hat{P}) \neq \varnothing$.*
(iii) *$\hat{Q}\hat{P}$ is diagonalizable.*
(iv) *The $n \times n$ matrix*

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{*} = (\hat{Q}\hat{P})^{*}\hat{Q}\hat{P} \tag{6.2}$$

is idempotent, i.e., τ is an oblique projection, and

$$\operatorname{rank} \tau = r. \tag{6.3}$$

(v) *There exist $G, \Gamma \in \mathbb{R}^{r \times n}$ and invertible $M \in \mathbb{R}^{r \times r}$ such that*

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \tag{6.4}$$

$$\Gamma G^{\mathrm{T}} = I_r. \tag{6.5}$$

(vi) *If $G, \Gamma \in \mathbb{R}^{r \times n}$ and $M \in \mathbb{R}^{r \times r}$ satisfy (6.4) and (6.5), then*

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} M = r, \tag{6.6}$$

$$(\hat{Q}\hat{P})^{*} = G^{\mathrm{T}} M^{-1} \Gamma, \tag{6.7}$$

$$\tau = G^{\mathrm{T}} \Gamma, \tag{6.8}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \qquad \Gamma \tau = \Gamma. \tag{6.9}$$

(vii) *The matrices $G, \Gamma \in \mathbb{R}^{r \times n}$ and $M \in \mathbb{R}^{r \times r}$ satisfying (6.4) and (6.5) are unique except for a change of basis in $\mathbb{R}^r$. Furthermore, all such M are diagonalizable with positive eigenvalues.*
(viii) *If $\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = r$, then $\mathscr{B}(\hat{Q}, \hat{P}) \neq \varnothing$ and*

$$\hat{Q} = \tau\hat{Q} = \hat{Q}\tau^{\mathrm{T}} = \tau\hat{Q}\tau^{\mathrm{T}}, \tag{6.10}$$

$$\hat{P} = \tau^{\mathrm{T}}\hat{P} = \hat{P}\tau = \tau^{\mathrm{T}}\hat{P}\tau. \tag{6.11}$$

Proof. See Appendix A. ■

A triple (G, M, Γ) satisfying (6.4) and (6.5) with $G, \Gamma \in \mathbb{R}^{r \times n}$, $M \in \mathbb{R}^{r \times r}$, and $r = \operatorname{rank} \hat{Q}\hat{P}$ is called a *projective factorization* of $\hat{Q}\hat{P}$. In particular, we set $r = n_c$. Furthermore, define the complementary projection

$$\tau_{\perp} \triangleq I_n - \tau, \tag{6.12}$$

and, for arbitrary $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$, $G, \Gamma \in \mathbb{R}^{n_c \times n}$, $B_c \in \mathbb{R}^{n_c \times l}$, $C_c \in \mathbb{R}^{m \times n_c}$, and $\alpha > 0$, define the following notation:

$$V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i [C_i(Q + \hat{Q})C_i^{\mathrm{T}} + D_i C_c \Gamma \hat{Q} \Gamma^{\mathrm{T}} C_c^{\mathrm{T}} D_i^{\mathrm{T}}].$$

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i [B_i^{\mathrm{T}}(P + \hat{P})B_i + D_i^{\mathrm{T}} B_c^{\mathrm{T}} G \hat{P} G^{\mathrm{T}} B_c D_i].$$

$$Q_s \triangleq QC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} \gamma_i[A_i(Q + \hat{Q})C_i^{\mathrm{T}} + A_i\hat{Q}\Gamma^{\mathrm{T}}C_c^{\mathrm{T}}D_i^{\mathrm{T}}],$$

$$P_s \triangleq B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i[B_i^{\mathrm{T}}(P + \hat{P})A_i - D_i^{\mathrm{T}}B_c^{\mathrm{T}}G\hat{P}A_i],$$

$$A_Q \triangleq A_x - Q_sV_{2s}^{-1}C, \qquad A_P \triangleq A_x - BR_{2s}^{-1}P_s.$$

The above definitions are for convenience in stating the necessary conditions for the Auxiliary Minimization Problem. This result provides explicit formulae for extremals $(\mathcal{Q}, A_c, B_c, C_c)$ of the Auxiliary Minimization Problem. A partial converse shows that this form of the necessary conditions represents no loss of generality with regard to the constraint equation (5.5).

Theorem 6.1.

(I) *Suppose $(\mathcal{Q}, A_c, B_c, C_c) \in \mathcal{S}$ solves the Auxiliary Minimization Problem with $\mathcal{U}$ given by (5.1) and Ω given by (5.3). Then there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that, for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, $(\mathcal{Q}, A_c, B_c, C_c)$ are given by*

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}, \tag{6.13}$$

$$A_c = \Gamma(A - BR_{2s}^{-1}P_s - Q_sV_{2s}^{-1}C + Q_sV_{2s}^{-1}DR_{2s}^{-1}P_s)G^{\mathrm{T}}, \tag{6.14}$$

$$B_c = \Gamma Q_sV_{2s}^{-1}, \tag{6.15}$$

$$C_c = -R_{2s}^{-1}P_sG^{\mathrm{T}}, \tag{6.16}$$

and such that $Q, P, \hat{Q}$, and $\hat{P}$ satisfy

$$0 = A_xQ + QA_x^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i[A_iQA_i^{\mathrm{T}} + (A_i - B_iR_{2s}^{-1}P_s)\hat{Q}(A_i - B_iR_{2s}^{-1}P_s)^{\mathrm{T}}]$$
$$- Q_sV_{2s}^{-1}Q_s^{\mathrm{T}} + \tau_\perp Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{6.17}$$

$$0 = A_x^{\mathrm{T}}P + PA_x + R_1 + \sum_{i=1}^{p} \gamma_i[A_i^{\mathrm{T}}PA_i + (A_i - Q_sV_{2s}^{-1}C_i)^{\mathrm{T}}\hat{P}(A_i - Q_sV_{2s}^{-1}C_i)]$$
$$- P_s^{\mathrm{T}}R_{2s}^{-1}P_s + \tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp, \tag{6.18}$$

$$0 = A_P\hat{Q} + \hat{Q}A_P^{\mathrm{T}} + Q_sV_{2s}^{-1}Q_s^{\mathrm{T}} - \tau_\perp Q_sV_{2s}^{-1}Q_s^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{6.19}$$

$$0 = A_Q^{\mathrm{T}}\hat{P} + \hat{P}A_Q + P_s^{\mathrm{T}}R_{2s}^{-1}P_s - \tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp, \tag{6.20}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c. \tag{6.21}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)$$
$$= \operatorname{tr}[(Q + \hat{Q})R_1 - 2R_{12}R_{2s}^{-1}P_s\hat{Q} + P_s^{\mathrm{T}}R_{2s}^{-1}R_2R_{2s}^{-1}P_s\hat{Q}]. \tag{6.22}$$

(II) *Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (6.17)–(6.21) with B_c and C_c given by (6.15) and (6.16), then $(\mathcal{Q}, A_c, B_c, C_c)$ given by (6.13)–(6.16) satisfy (4.2) and (5.5) with $\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c)$ given by (6.22).*

Proof. See Appendix B. ■

Remark 6.1. Theorem 6.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly characterize extremal quadruples $(\mathcal{Q}, A_c, B_c, C_c)$. These necessary conditions consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by both the optimal projection τ and uncertainty bounds. If the uncertainty bounds are deleted, then the resuls of [HB] are recovered.

Remark 6.2. When solving (6.17)–(6.21) numerically, the uncertainty terms can be adjusted to examine tradeoffs between performance and robustness. Specifically, the bounds α_i and the structure matrices A_i, B_i, C_i, and D_i appearing in V_{2s}, R_{2s}, Q_s, and P_s can be varied systematically to determine the region of solvability of (6.17)–(6.21).

Remark 6.3. Although (6.17)–(6.21) appear formidable, they are, in fact, quite numerically tractable. For related problems involving coupled Riccati equations, homotopic continuation methods have been shown to be effective [KLJ], [MB]. Similar algorithms for solving (6.17)–(6.21) have been developed in [GH], [R1], and [R2], while iterative algorithms are discussed in [G2], [GV], and [CY].

Remark 6.4. Because of the presence of B_c and C_c in the definitions of V_{2s}, R_{2s}, Q_s, and P_s, the optimality conditions (6.17)–(6.20) are coupled with the gain expressions (6.15) and (6.16) for B_c and C_c. When the problem is specialized to the case $D_i = 0$, $i = 1, \ldots, p$, this coupling disappears and (6.17)–(6.20) can be solved without reference to the gain expressions (6.15) and (6.16).

7. Sufficient Conditions for Robust Stability and Performance

In this section we combine Theorem 3.1 with Theorem 6.1(II) to obtain our main result guaranteeing robust stability and performance.

Theorem 7.1. *Assume there exist Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^{\hat{n}}$ satisfying* (6.17)–(6.21) *with B_c and C_c given by* (6.15) *and* (6.16). *Then, with $(\mathcal{Q}, A_c, B_c, C_c)$ given by* (6.13)–(6.16), *$(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. In this case, the performance* (2.7) *of the closed-loop system* (2.9) *satisfies the bound*

$$J(A_c, B_c, C_c) \leq \operatorname{tr}[(Q + \hat{Q})R_1 - 2R_{12}R_{2s}^{-1}P_s\hat{Q} + P_s^{\mathrm{T}}R_{2s}^{-1}R_2R_{2s}^{-1}P_s\hat{Q}]. \quad (7.1)$$

Proof. The converse portion to Theorem 6.1 implies that $\mathcal{Q}$ given by (6.13) is nonnegative definite and satisfies (5.5) or, equivalently, (3.4). It now follows from Theorem 3.1 that the stabilizability condition (3.5) is equivalent to the asymptotic stability of $\tilde{A} + \Delta\tilde{A}$ for all $(\Delta A, \Delta B, \Delta C, \Delta D) \in \mathcal{U}$. In this case Proposition 4.1 yields robust stability and performance. The robust performance bound (7.1) is a restatement of (4.3) utilizing (6.22). ■

Note that Theorem 7.1 is constructive in nature rather than existential. Specifically, Theorem 7.1 involves a coupled system of modified Riccati/Lyapunov equations (6.17)–(6.21) whose solutions, when they exist, are used explicitly to construct the dynamic feedback gains (6.14)–(6.16) which are guaranteed to provide both robust stability and performance. The following existence result concerns the solvability of (6.17)–(6.21). Let n_u denote the dimension of the unstable subspace of the plant dynamics matrix A.

Theorem 7.2. *Assume $n_c \geq n_u$, $R_1 > 0$, $V_1 > 0$, suppose the nominal plant, i.e.,* (2.1), (2.2) *with $\alpha_i = 0$, $i = 1, \ldots, p$, is stabilizable and detectable and, in addition, is stabilizable by means of an n_cth-order strictly proper dynamic compensator* (2.3), (2.4). *Then there exist $\bar{\alpha}_1, \ldots, \bar{\alpha}_p > 0$ such that if $\alpha_i \in [0, \bar{\alpha}_i)$, $i = 1, \ldots, p$, then* (6.17)–(6.21) *have a solution $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ for which (A_c, B_c, C_c) given by* (6.14)–(6.16) *solves the Robust Stability Problem with robust performance bound* (6.22).

Proof. From Theorem 3.1 of [R1] and [R2] it follows that there exists a solution to (6.17)–(6.21) which stabilizes the nominal plant. By continuity there exists a neighborhood over which robust stability with performance bound (6.22) holds. ■

Theorem 7.2 is an existence result which guarantees solvability of the sufficiency conditions over a range of parameter uncertainties. The actual range of uncertainty which can be bounded and the conservatism of the performance bound are, of course, problem dependent.

8. Specialization to Full-Order Dynamic Compensation

To draw connections with standard full-order LQG theory, we specialize the results of Sections 6 and 7 to the full-order case, i.e., $n_c = n$. As discussed in [HB], in the full-order case $G = \Gamma^{-1}$ and thus $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. To develop further connections with standard LQG theory assume

$$R_{12} = 0, \qquad V_{12} = 0, \qquad D = \Delta D = 0. \tag{8.1}$$

Since $\Delta D = 0$ we write $(\Delta A, \Delta B, \Delta C)$ in place of $(\Delta A, \Delta B, \Delta C, \Delta D)$. Also, for arbitrary $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$ and $\alpha > 0$ define the following notation:

$$\hat{V}_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i C_i (Q + \hat{Q}) C_i^{\mathrm{T}}, \qquad \hat{R}_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} (P + \hat{P}) B_i,$$

$$\hat{Q}_s \triangleq Q C^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i A_i (Q + \hat{Q}) C_i^{\mathrm{T}}, \qquad \hat{P}_s \triangleq B^{\mathrm{T}} P + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} (P + \hat{P}) A_i,$$

$$\hat{A}_Q \triangleq A_\alpha - \hat{Q}_s \hat{V}_{2s}^{-1} C, \qquad \hat{A}_P \triangleq A_\alpha - B \hat{R}_{2s}^{-1} \hat{P}_s.$$

Theorem 8.1. *Let $n_c = n$, assume* (8.1) *is satisfied, and assume there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying*

$$0 = A_\alpha Q + Q A_\alpha^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i [A_i Q A_i^{\mathrm{T}} + (A_i - B_i \hat{R}_{2s}^{-1} \hat{P}_s) \hat{Q} (A_i - B_i \hat{R}_{2s}^{-1} \hat{P}_s)^{\mathrm{T}}]$$
$$- \hat{Q}_s \hat{V}_{2s}^{-1} \hat{Q}_s^{\mathrm{T}}, \tag{8.2}$$

$$0 = A_x^{\mathrm{T}}P + PA_x + R_1 + \sum_{i=1}^{p} \gamma_i[A_i^{\mathrm{T}}PA_i + (A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i)^{\mathrm{T}}\hat{P}(A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i)]$$
$$-\hat{P}_s^{\mathrm{T}}\hat{R}_{2s}^{-1}\hat{P}_s, \tag{8.3}$$
$$0 = \hat{A}_P\hat{Q} + \hat{Q}\hat{A}_P^{\mathrm{T}} + \hat{Q}_s\hat{V}_{2s}^{-1}\hat{Q}_s^{\mathrm{T}}, \tag{8.4}$$
$$0 = \hat{A}_Q^{\mathrm{T}}\hat{P} + \hat{P}\hat{A}_Q + \hat{P}_s^{\mathrm{T}}\hat{R}_{2s}^{-1}\hat{P}_s, \tag{8.5}$$

and let $(\mathcal{Q}, A_c, B_c, C_c)$ *be given by*

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix}, \tag{8.6}$$
$$A_c = A - B\hat{R}_{2s}^{-1}\hat{P}_s - \hat{Q}_s\hat{V}_{2s}^{-1}C, \tag{8.7}$$
$$B_c = \hat{Q}_s\hat{V}_{2s}^{-1}, \tag{8.8}$$
$$C_c = -\hat{R}_{2s}^{-1}\hat{P}_s. \tag{8.9}$$

Then, $(\tilde{A} + \Delta\tilde{A}, [\tilde{V} + \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2}])$ *is stabilizable for all* $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$ *if and only if* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. *In this case the performance of the closed-loop system* (2.9) *satisfies the bound*

$$J(A_c, B_c, C_c) \leq \mathrm{tr}[(Q + \hat{Q})R_1 + \hat{P}_s^{\mathrm{T}}\hat{R}_{2s}^{-1}R_2\hat{R}_{2s}^{-1}\hat{P}_s\hat{Q}]. \tag{8.10}$$

Proof. The proof follows from the reduced-order case given in Appendix B. ■

Remark 8.1. Theorem 8.1 presents sufficient conditions for robust stability and performance for full-order dynamic compensation. These sufficient conditions comprise a system of two modified Riccati equations and two Lyapunov equations coupled by the uncertainty bounds. This coupling illustrates the breakdown of regulator/estimator separation and shows that the certainty equivalence principle is no longer valid for the LQG result with real-valued structured plant parameter variations. If, however, the uncertainty terms A_i, B_i, C_i are set to zero, it can be seen that (8.4) and (8.5) drop out, while (8.2) and (8.3) reduce to the standard separated Riccati equations of LQG theory.

9. Illustrative Numerical Example

To demonstrate the above results we present an illustrative numerical example. The example chosen was originally used by Doyle [D] to illustrate the lack of a guaranteed gain margin for LQG controllers. This example was also considered in [BG1] for a preliminary robustness study. Define

$$n = 2, \qquad m = l = p = 1,$$
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = [1 \quad 0], \qquad D = 0,$$
$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = [0 \quad 0], \qquad D_1 = 0,$$

$$R_1 = V_1 = \begin{bmatrix} 60 & 60 \\ 60 & 60 \end{bmatrix}, \qquad R_{12} = V_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad R_2 = V_2 = 1.$$

Note that the system is open-loop unstable and becomes unstabilizable at $\sigma_1 = -1$. As can easily be seen using root locus, a strictly proper stabilizing controller must be of at least second order. Hence we consider (6.17)–(6.21) with $n_c = n$ and thus $\tau_\perp = 0$. Using algorithms described in [GH] and [R1], controllers were obtained for $(\alpha, \alpha_1) = (0.1, 0.1), (0.4, 0.2)$, and $(1.6, 0.4)$. Figure 1 compares the guaranteed robust stability region to the actual robust stability region. Note that the design approach yields greater stability than is guaranteed *a priori*. This phenomenon is not surprising since even the LQG result may provide arbitrarily high levels of robustness for particular problems while failing to guarantee even minimal robustness for all problems. These results thus demonstrate the ability of the theory to robustify the LQG result. Interestingly, the form of the actual stability region mimics the classical 6 dB downward/infinite dB upward gain margin of full-state-feedback LQR controllers. Finally, Figure 2 compares guaranteed closed-loop performance to actual closed-loop performance over the guaranteed closed-loop robust stability region. Controller gains are given in Table 1.

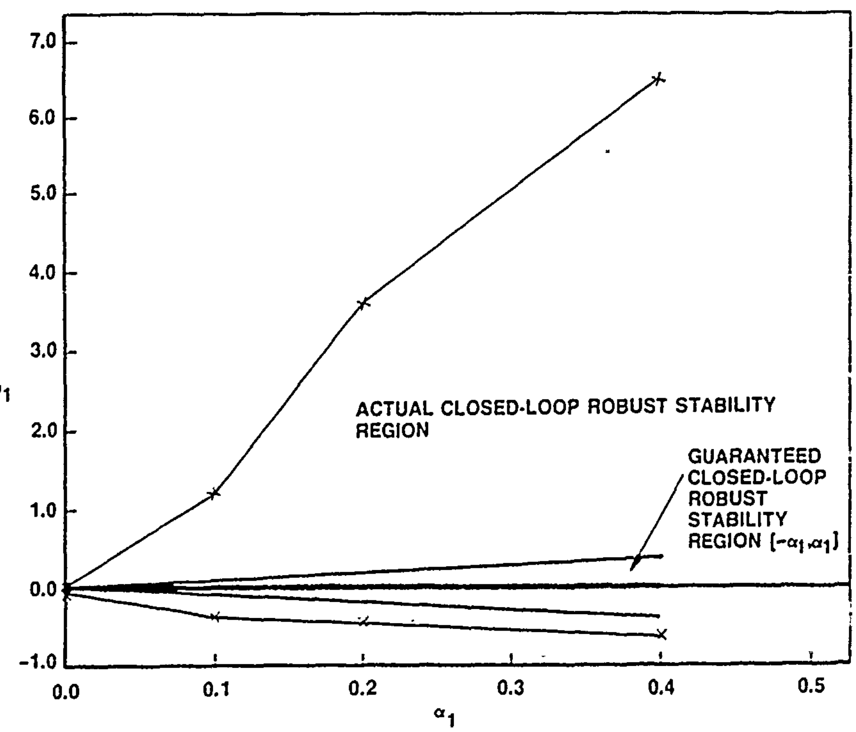

Fig 1

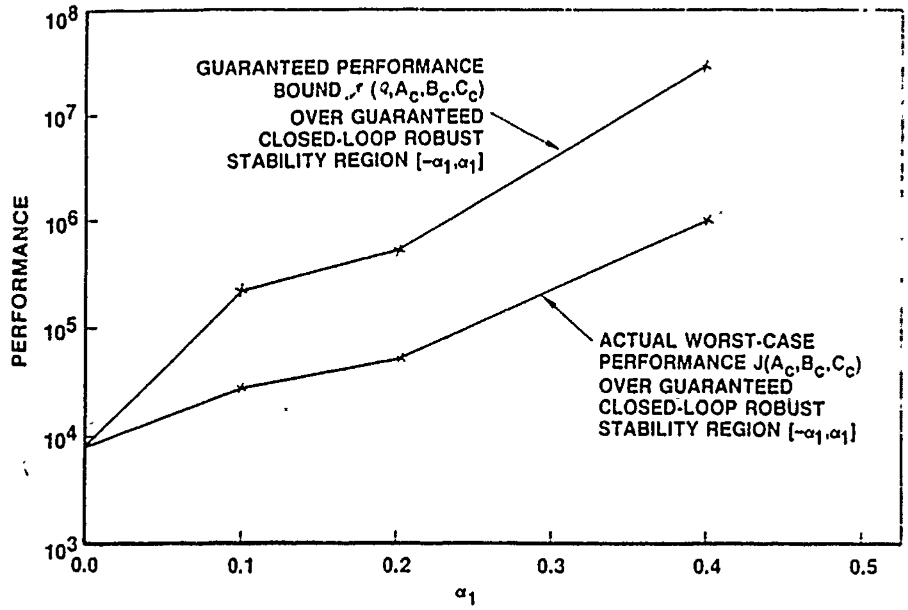

Fig. 2

Table 1

(α, α_1)	A_c	B_c	C_c
(0.1, 0.1)	$\begin{bmatrix} -14.917 & 1.0 \\ -85.177 & 3.9657 \end{bmatrix}$	$\begin{bmatrix} 15.917 \\ 79.959 \end{bmatrix}$	$[-15.2182 \quad -4.9657]$
(0.4, 0.2)	$\begin{bmatrix} -17.963 & 1.0 \\ -133.65 & -4.4614 \end{bmatrix}$	$\begin{bmatrix} 18.963 \\ 127.05 \end{bmatrix}$	$[-6.6011 \quad -5.4614]$
(1.6, 0.4)	$\begin{bmatrix} -47.813 & 1.0 \\ -1087.3 & -6.5463 \end{bmatrix}$	$\begin{bmatrix} 48.813 \\ 1073.5 \end{bmatrix}$	$[-13.766 \quad -7.5463]$

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Appendix A. Proof of Lemma 6.1

(i) Clearly $\hat{Q}\hat{P}$ and $\hat{P}^{1/2}\hat{Q}\hat{P}^{1/2}$ have the same nonzero eigenvalues. Since $\hat{P}^{1/2}\hat{Q}\hat{P}^{1/2}$ is nonnegative definite, $\hat{Q}\hat{P}$ has nonnegative eigenvalues.

(ii) The result follows from Theorem 6.2.5 of [RM], p. 123. See also Theorem 4.3 of [G1].

(iii) This result follows from (ii) and (6.1).

(iv) This result follows from the definition of the group generalized inverse (see [CM]). Alternatively, let $\hat{Q}\hat{P} = \Psi D \Psi^{-1}$, where $\Psi \in \mathscr{D}(\hat{Q}\hat{P})$, $D = \mathrm{diag}(d_1, \ldots, d_n)$. Then $(\hat{Q}\hat{P})^{\#} = \Psi D^{\#} \Psi^{-1}$, where $D^{\#}_{(i,i)} = 1/d_i$ if $d_i \neq 0$, and $D^{\#}_{(i,i)} = 0$, if $d_i = 0$, $i = 1, \ldots, n$. Hence $\hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = \Psi E \Psi^{-1}$ is idempotent, where E is a diagonal matrix with r ones and $n - r$ zeros on the diagonal. Clearly, (6.3) is valid.

(v) Without loss of generality choose Ψ in the preceding argument so that $D = \text{block-diag}(\hat{D}, 0_{n-r})$, where $\hat{D} = \mathrm{diag}(\hat{d}_1, \ldots, \hat{d}_r)$, $\hat{d}_i > 0$, $i = 1, \ldots, r$. Hence

$$\hat{Q}\hat{P} = \Psi \begin{bmatrix} \hat{D} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{-1},$$

and thus (6.5) holds with

$$G = [I_r \quad 0_{r \times (n-r)}]\Psi^{\mathrm{T}}, \qquad M = \hat{D}, \qquad \Gamma = [I_r \quad 0_{r \times (n-r)}]\Psi^{-1}.$$

(vi) Sylvester's inequality and (6.4) imply that

$$r = \operatorname{rank} \hat{Q}\hat{P} \leq \{\operatorname{rank} G, \operatorname{rank} M, \operatorname{rank} \Gamma\} \leq r,$$

which yields (6.6). The expression (6.7) for $(\hat{Q}\hat{P})^{\#}$ follows directly from the definition of the group generalized inverse. Furthermore, (6.2), (6.5), and (6.7) imply (6.8), while (6.5) and (6.8) imply (6.9).

(vii) Let both (G, M, Γ) and an identically dimensioned triple $(\hat{G}, \hat{M}, \hat{\Gamma})$ satisfy (6.4). Then it is easy to verify that $\hat{G} = S^{-1}G$, $\hat{M} = SMS^{-1}$, and $\hat{\Gamma} = S\Gamma$, where $S = \hat{\Gamma}G^{\mathrm{T}}$ and $S^{-1} = \Gamma\hat{G}^{\mathrm{T}}$.

(viii) It follows from (ii) that there exists $\Psi \in \mathscr{C}(\hat{Q}, \hat{P})$ such that

$$\hat{Q} = \Psi \begin{bmatrix} D_{\hat{Q}} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{\mathrm{T}}, \qquad \hat{P} = \Psi^{-\mathrm{T}} \begin{bmatrix} D_{\hat{P}} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix} \Psi^{-1},$$

where $D_{\hat{Q}}$ and $D_{\hat{P}}$ are positive diagonal. Define

$$\hat{\Psi} = \Psi \begin{bmatrix} (D_{\hat{Q}} D_{\hat{P}}^{-1})^{1/4} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & I_{n-r} \end{bmatrix}$$

so that

$$\hat{\Psi}^{-1}\hat{Q}\hat{\Psi}^{-\mathrm{T}} = \hat{\Psi}^{\mathrm{T}}\hat{P}\hat{\Psi} = \begin{bmatrix} (D_{\hat{Q}} D_{\hat{P}})^{1/2} & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{n-r} \end{bmatrix}$$

and thus $\hat{\Psi} \in \mathscr{B}(\hat{Q}, \hat{P})$. Finally, (6.10) and (6.11) are immediate. ∎

Appendix B. Proof of Theorem 6.1

To optimize (4.10) over the open set $\mathscr{S}$ subject to the constraint (5.5), form the Lagrangian

$$\mathscr{L}(\mathscr{Q}, A_c, B_c, C_c, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\left\{\lambda\mathscr{Q}\tilde{R} + [\tilde{A}_s\mathscr{Q} + \mathscr{Q}\tilde{A}_s^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \mathscr{Q} \tilde{A}_i^{\mathrm{T}} + \tilde{V}]\mathscr{P}\right\}, \quad \text{(B.1)}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = \tilde{A}_s^{\mathrm{T}}\mathscr{P} + \mathscr{P}\tilde{A}_s + \sum_{i=1}^{p} \gamma_i \tilde{A}_i^{\mathrm{T}}\mathscr{P}\tilde{A}_i + \lambda\tilde{R}. \tag{B.2}$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = \tilde{A}_s^{\mathrm{T}}\mathscr{P} + \mathscr{P}\tilde{A}_s + \sum_{i=1}^{p} \gamma_i \tilde{A}_i^{\mathrm{T}}\mathscr{P}\tilde{A}_i + \lambda\tilde{R} \tag{B.3}$$

or, equivalently,

$$\mathscr{A}^{\mathrm{T}} \operatorname{vec} \mathscr{P} = -\lambda \operatorname{vec} \tilde{R}.$$

Since $\mathscr{A}$ is assumed to be stable, $\mathscr{A}^{\mathrm{T}}$ is invertible, and thus $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, it follows from Proposition 5.2 with $\mathscr{A}$, $\tilde{V}$ replaced by $\mathscr{A}^{\mathrm{T}}$, $\tilde{R}$ that $\mathscr{P}$ is nonnegative definite.

Now partition $\tilde{n} \times \tilde{n}$ $\mathscr{Q}$, $\mathscr{P}$ into $n \times n$, $n \times n_c$, and $n_c \times n_c$ subblocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \qquad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix},$$

and define the positive-definite matrices

$$V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i [C_i Q_1 C_i^{\mathrm{T}} + D_i C_c Q_2 C_c^{\mathrm{T}} D_i^{\mathrm{T}}],$$

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i [B_i^{\mathrm{T}} P_1 B_i + D_i^{\mathrm{T}} B_c^{\mathrm{T}} P_2 B_c D_i].$$

Thus, the stationarity conditions for A_c, B_c, C_c are given by

$$\frac{\partial \mathscr{L}}{\partial A_c} = P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2 = 0, \tag{B.4}$$

$$\frac{\partial \mathscr{L}}{\partial B_c} = P_2 B_c V_{2s} + (P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}) C^{\mathrm{T}} + P_{12}^{\mathrm{T}}\left[V_{12} + \sum_{i=1}^{p} \gamma_i (A_i Q_1 C_i^{\mathrm{T}} + A_i Q_{12} C_c^{\mathrm{T}} D_i^{\mathrm{T}})\right] = 0. \tag{B.5}$$

$$\frac{\partial \mathscr{L}}{\partial C_c} = R_{2s} C_c Q_2 + B^{\mathrm{T}}(P_1 Q_{12} + P_{12} Q_2) + \left[R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i (B_i^{\mathrm{T}} P_1 A_i + D_i^{\mathrm{T}} B_c^{\mathrm{T}} P_{12}^{\mathrm{T}} A_i)\right] Q_{12} = 0. \tag{B.6}$$

Expanding (5.5) and (B.3) yields

$$0 = A_s Q_1 + Q_1 A_s^{\mathrm{T}} + B C_c Q_{12}^{\mathrm{T}} + Q_{12} C_c^{\mathrm{T}} B^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i [A_i Q_1 A_i^{\mathrm{T}} + B_i C_c Q_{12}^{\mathrm{T}} A_i^{\mathrm{T}} + A_i Q_{12} C_c^{\mathrm{T}} B_i^{\mathrm{T}} + B_i C_c Q_2 C_c^{\mathrm{T}} B_i^{\mathrm{T}}] + V_1, \tag{B.7}$$

$$0 = A_{\mathrm{s}}Q_{12} + Q_{12}A_{\mathrm{cs}}^{\mathrm{T}} + Q_{12}C_{\mathrm{c}}^{\mathrm{T}}D^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}} + Q_1C^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}} + BC_{\mathrm{c}}Q_2 + \sum_{i=1}^{p}\gamma_i[A_iQ_1C_i^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}} + A_iQ_{12}C_{\mathrm{c}}^{\mathrm{T}}D_i^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}] + V_{12}B_{\mathrm{c}}^{\mathrm{T}}, \tag{B.8}$$

$$0 = A_{\mathrm{cs}}Q_2 + Q_2A_{\mathrm{cs}}^{\mathrm{T}} + B_{\mathrm{c}}CQ_{12} + Q_{12}^{\mathrm{T}}C^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}} + B_{\mathrm{c}}DC_{\mathrm{c}}Q_2 + Q_2C_{\mathrm{c}}^{\mathrm{T}}D^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}} + B_{\mathrm{c}}V_{2s}B_{\mathrm{c}}^{\mathrm{T}}, \tag{B.9}$$

$$0 = A_{\mathrm{s}}^{\mathrm{T}}P_1 + P_1A_{\mathrm{s}} + C^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}B_{\mathrm{c}}C + \sum_{i=1}^{p}\gamma_i[A_i^{\mathrm{T}}P_1A_i + C_i^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_{12}^{\mathrm{T}}A_i + A_i^{\mathrm{T}}P_{12}B_{\mathrm{c}}C_i + C_i^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_2B_{\mathrm{c}}C_i] + R_1, \tag{B.10}$$

$$0 = A_{\mathrm{s}}^{\mathrm{T}}P_{12} + P_{12}A_{\mathrm{cs}} + P_{12}B_{\mathrm{c}}DC_{\mathrm{c}} + P_1BC_{\mathrm{c}} + C^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_2 + \sum_{i=1}^{p}\gamma_i[A_i^{\mathrm{T}}P_1B_iC_{\mathrm{c}} + A_i^{\mathrm{T}}P_{12}B_{\mathrm{c}}D_iC_{\mathrm{c}}] + R_{12}C_{\mathrm{c}}, \tag{B.11}$$

$$0 = A_{\mathrm{cs}}^{\mathrm{T}}P_2 + P_2A_{\mathrm{cs}} + C_{\mathrm{c}}^{\mathrm{T}}B^{\mathrm{T}}P_{12} + P_{12}^{\mathrm{T}}BC_{\mathrm{c}} + C_{\mathrm{c}}^{\mathrm{T}}D^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_2 + P_2B_{\mathrm{c}}DC_{\mathrm{c}} + C_{\mathrm{c}}^{\mathrm{T}}R_{2s}C_{\mathrm{c}}. \tag{B.12}$$

Lemma B.1. *Q_2 and P_2 are positive definite.*

Proof. By a minor extension of results from [A], (B.9) can be rewritten as

$$0 = (A_{\mathrm{cs}} + B_{\mathrm{c}}DC_{\mathrm{c}} + B_{\mathrm{c}}CQ_{12}Q_2^{+})Q_2 + Q_2(A_{\mathrm{cs}} + B_{\mathrm{c}}DC_{\mathrm{c}} + B_{\mathrm{c}}CQ_{12}Q_2^{+})^{\mathrm{T}} + B_{\mathrm{c}}V_{2s}B_{\mathrm{c}}^{\mathrm{T}},$$

where Q_2^{+} is the Moore–Penrose or Drazin generalized inverse of Q_2. Next note that since $(A_{\mathrm{cs}}, B_{\mathrm{c}})$ is controllable then, by Theorem 3.6 of [W], $(A_{\mathrm{cs}} + B_{\mathrm{c}}DC_{\mathrm{c}} + B_{\mathrm{c}}CQ_{12}Q_2^{+}, B_{\mathrm{c}}V_{2s}^{1/2})$ is also controllable. Now, since Q_2 and $B_{\mathrm{c}}V_{2s}B_{\mathrm{c}}^{\mathrm{T}}$ are nonnegative definite, it follows from Lemma 12.2 of [W], that Q_2 is positive definite. Using (B.12), similar arguments show that P_2 is positive definite. ■

Since R_{2s}, V_{2s}, Q_2, and P_2 are invertible, (B.4)–(B.6) can be written as

$$-P_2^{-1}P_{12}^{\mathrm{T}}Q_{12}Q_2^{-1} = I_{n_c}, \tag{B.13}$$

$$B_{\mathrm{c}} = -P_2^{-1}\left\{(P_{12}^{\mathrm{T}}Q_1 + P_2Q_{12}^{\mathrm{T}})C^{\mathrm{T}} + P_{12}^{\mathrm{T}}\left[V_{12} + \sum_{i=1}^{p}\gamma_i(A_iQ_1C_i^{\mathrm{T}} + A_iQ_{12}C_{\mathrm{c}}^{\mathrm{T}}D_i^{\mathrm{T}})\right]\right\}V_{2s}^{-1}, \tag{B.14}$$

$$C_{\mathrm{c}} = -R_{2s}^{-1}\left\{B^{\mathrm{T}}(P_1Q_{12} + P_{12}Q_2) + \left[R_{12}^{\mathrm{T}} + \sum_{i=1}^{p}\gamma_i(B_i^{\mathrm{T}}P_1A_i + D_i^{\mathrm{T}}B_{\mathrm{c}}^{\mathrm{T}}P_{12}^{\mathrm{T}}A_i)\right]Q_{12}\right\}Q_2^{-1}. \tag{B.15}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}},$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^{\mathrm{T}},$$

and the $n_c \times n$, $n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^{\mathrm{T}}, \qquad M \triangleq Q_2 P_2, \qquad \Gamma \triangleq -P_2^{-1} P_{12}^{\mathrm{T}}.$$

Note that $\tau = G^{\mathrm{T}}\Gamma$.

Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and $\hat{Q}$ and $\hat{P}$ are nonnegative definite. To show that Q and P are also nonnegative definite, note that Q is the upper left-hand block of the nonnegative-definite matrix $\tilde{\mathcal{Q}}\mathcal{Q}\tilde{\mathcal{Q}}^{\mathrm{T}}$, where

$$\tilde{\mathcal{Q}} = \begin{bmatrix} I_n & -Q_{12}Q_2^{-1} \\ 0_{n_c \times n} & I_{n_c} \end{bmatrix}.$$

Similarly, P is nonnegative definite.

Next note that with the above definitions, (B.13) is equivalent to (6.5) and that (6.4) holds. Hence $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Furthermore, it is helpful to note the identities

$$\hat{Q} = Q_{12}G = G^{\mathrm{T}}Q_{12}^{\mathrm{T}} = G^{\mathrm{T}}Q_2 G, \qquad \hat{P} = -P_{12}\Gamma = -\Gamma^{\mathrm{T}}P_{12}^{\mathrm{T}} = \Gamma^{\mathrm{T}}P_2\Gamma, \tag{B.16}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \qquad \Gamma\tau = \Gamma, \tag{B.17}$$

$$\hat{Q} = \tau\hat{Q}, \qquad \hat{P} = \hat{P}\tau, \tag{B.18}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^{\mathrm{T}}. \tag{B.19}$$

Using (B.13) and Sylvester's inequality, it follows that

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} Q_{12} = \operatorname{rank} P_{12} = n_c.$$

Now using (B.16) and Sylvester's inequality yields

$$n_c = \operatorname{rank} Q_{12} + \operatorname{rank} G - n_c \le \operatorname{rank} \hat{Q} \le \operatorname{rank} Q_{12} = n_c,$$

which implies that rank $\hat{Q} = n_c$. Similarly, rank $\hat{P} = n_c$, and rank $\hat{Q}\hat{P} = n_c$ follows from (B.19). The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \tag{B.20}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \qquad P_{12} = -\hat{P}G^{\mathrm{T}}, \tag{B.21}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \qquad P_2 = G\hat{P}G^{\mathrm{T}}. \tag{B.22}$$

The expressions (6.13), (6.15), and (6.16) follow from the definition of $\mathcal{Q}$, (B.14) and (B.15). Substituting (B.20)–(B.22) into (B.7)–(B.12) yields

$$0 = A_x Q + QA_x^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i [A_i Q A_i^{\mathrm{T}} + (A_i - B_i R_{2s}^{-1} P_s)\hat{Q}(A_i - B_i R_{2s}^{-1} P_s)^{\mathrm{T}}] \\ + A_P\hat{Q} + \hat{Q}A_P^{\mathrm{T}}, \tag{B.23}$$

$$0 = [A_P\hat{Q} + \hat{Q}(G^{\mathrm{T}}A_{cx}\Gamma + Q_s V_{2s}^{-1}C)^{\mathrm{T}} + Q_s V_{2s}^{-1} Q_s^{\mathrm{T}}]\Gamma^{\mathrm{T}}, \tag{B.24}$$

$$0 = \Gamma[(G^{\mathrm{T}}A_{cx}\Gamma + Q_s V_{2s}^{-1}C)\hat{Q} + \hat{Q}(G^{\mathrm{T}}A_{cx}\Gamma + Q_s V_{2s}^{-1}C)^{\mathrm{T}} + Q_s V_{2s}^{-1}Q_s^{\mathrm{T}}]\Gamma^{\mathrm{T}}, \tag{B.25}$$

$$0 = A_x^{\mathrm{T}}P + PA_x + R_1 + \sum_{i=1}^{p} \gamma_i [A_i^{\mathrm{T}} P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^{\mathrm{T}}\hat{P}(A_i - Q_s V_{2s}^{-1} C_i)] \\ + A_Q^{\mathrm{T}}\hat{P} + \hat{P}A_Q, \tag{B.26}$$

$$0 = [A_Q^{\mathrm{T}}\hat{P} + \hat{P}(G^{\mathrm{T}}A_{cs}\Gamma + BR_{2s}^{-1}P_s) + P_s^{\mathrm{T}}R_{2s}^{-1}P_s]G^{\mathrm{T}}, \tag{B.27}$$

$$0 = G[(G^{\mathrm{T}}A_{cs}\Gamma + BR_{2s}^{-1}P_s)^{\mathrm{T}}\hat{P} + \hat{P}(G^{\mathrm{T}}A_{cs}\Gamma + BR_{2s}^{-1}P_s) + P_s^{\mathrm{T}}R_{2s}^{-1}P_s]G^{\mathrm{T}}. \tag{B.28}$$

Next, computing either Γ(B.24)–(B.25) or G(B.27)–(B.28) yields (6.14). Substituting this expression for A_c into (B.23), (B.24), (B.27), and (B.28) it follows that (B.25) $= \Gamma$(B.24) and (B.28) $= G$(B.27). Thus, (B.25) and (B.28) are superfluous and can be omitted. Next, using (B.23) + $G^{\mathrm{T}}\Gamma$(B.24)G – (B.24)G – [(B.24)G]$^{\mathrm{T}}$ and $G^{\mathrm{T}}\Gamma$(B.24)G – (B.24)G – [(B.24)G]$^{\mathrm{T}}$ yields (6.17) and (6.19). Using (B.26) + $\Gamma^{\mathrm{T}}G$(B.27)Γ – (B.27)Γ – [(B.27)Γ]$^{\mathrm{T}}$ and $\Gamma^{\mathrm{T}}G$(B.27)Γ – (B.27)Γ – [(B.27)Γ]$^{\mathrm{T}}$ yields (6.18) and (6.20).

Finally, to prove the converse we use (6.13)–(6.21) to obtain (5.5) and (B.3)–(B.6). Let $A_c, B_c, C_c, G, \Gamma, \tau, Q, P, \hat{Q}, \hat{P}, \mathscr{Q}$ be as in the statement of Theorem 6.1 and define $Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2$ by (B.20)–(B.22). Using (6.5), (6.15), and (6.16), it is easy to verify (B.5), (B.6). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, and τ into (6.17)–(6.20), reverse the steps taken earlier in the proof, and use (6.13)–(6.16) along with (6.5) and (6.8)–(6.11) to obtain (5.5) and (B.3). Finally, note that

$$\mathscr{Q} = \begin{bmatrix} Q & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q} [I_n \quad \Gamma^{\mathrm{T}}],$$

which show that $\mathscr{Q} \geq 0$ thus verifying (4.2). ■

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Robust H_∞ control design for systems with structured parameter uncertainty

Appasaheb N. MADIWALE *
Massachusetts Institute of Technology, Lincoln Laboratory, P.O. Box 73, Lexington, MA 02173, U.S.A.

Wassim M. HADDAD **
Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, FL 32901, U.S.A.

Dennis S. BERNSTEIN **
Government Aerospace Systems Division, Harris Corporation, MS 22/4848, Melbourne, FL 32901, U.S.A.

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Abstract: In a recent paper a unification of the H_2 (LQG) and H_∞ control-design problems was obtained in terms of modified algebraic Riccati equations. In the present paper these results are extended to guarantee robust H_2 and H_∞ performance in the presence of structured real-valued parameter variations (ΔA, ΔB, ΔC) in the state space model. For design flexibility the paper considers two distinct types of uncertainty bounds for both full- and reduced-order dynamic compensation. An important special case of these results generates *H_2/H_∞ controller designs with guaranteed gain margins.*

Keywords: Robust control; parameter uncertainty; H-infinity design.

1. Introduction

It has recently been shown that the solution to the optimal H_∞ disturbance attenuation problem can be expressed in terms of a pair of modified Riccati equations [3,4]. Furthermore, it was shown in [3] that H_2/H_∞ design tradeoffs can be achieved by solving a coupled system consisting of *three* modified Riccati equations. As is well known, the disturbance attenuation problem can be used to guarantee robustness with respect to unstructured plant uncertainties. However, if plant uncertainty is present in the form of structured parametric variations of the state space model, then alternative bounding techniques are required. The goal of the present paper is thus to extend the results of [3] to include bounds on the effects of real-valued structured parameter variations.

In the *absence* of an H_∞ design constraint, robust stability and H_2 performance for dynamic compensator design were guaranteed in [1,2] by incorporating quadratic Lyapunov bounds within LQG design theory. Two distinct bounds were considered. In [1] a quadratic bound was used while in [2] a linear bound was employed. In each case full- and reduced-order dynamic compensators were characterized by means of coupled systems of modified Riccati and Lyapunov equations.

To design H_∞ controllers which are robust with respect to structured real-valued parameter variations we proceed by combining the results of [3] with those of [1,2]. That is, we derive coupled systems of

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modified Riccati and Lyapunov equations whose solutions yield controllers *which are guaranteed to satisfy a prespecified H_∞ attenuation constraint for all variations* (ΔA, ΔB, ΔC), *belonging to a given uncertainty set*. If the uncertainty is absent (i.e., $\Delta A = 0$, etc.), then the results of [3] are recovered, while if the H_∞ constraint is relaxed, then the results of [1,2] are obtained. Thus the results of [3] can be viewed as a specialization of a broader design theory which accounts for structured real-valued parameter uncertainty. Finally, we state all results for the case of a fixed-order (i.e., reduced-order) controller for maximal design flexibility. Extensions to even more general design problems are mentioned in Section 9 but omitted here for lack of space.

Notation

Note: all matrices have real entries.
$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$: real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.
I_r, $()^T$, $0_{r\times s}$, 0_r: $r\times r$ identity matrix, transpose, $r\times s$ zero matrix, $0_{r\times r}$.
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$: $r\times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2$, $Z_1 < Z_2$: $Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$.
n, m, l, n_c: positive integers.
p, d, d_∞, q, q_∞; $\tilde{n}$: positive integers; $n + n_c$ $(n_c \le n)$.
x, u, y, x_c, $\tilde{x}$: n, m, l, n_c, $\tilde{n}$-dimensional vectors.
A, ΔA; B, ΔB; C, ΔC: $n\times n$; $n\times m$; $l\times n$ matrices.
A_c, B_c, C_c: $n_c\times n_c$, $n_c\times l$, $m\times n_c$ matrices.

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \qquad \Delta\tilde{A} = \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & 0_{n_c} \end{bmatrix}.$$

$w(\cdot)$: d-dimensional standard white noise.
D_1, D_2: $n\times d$, $l\times d$ matrices; $D_1D_2^T = 0$.
V_1, V_2: $D_1D_1^T$, $D_2D_2^T$; $V_2 \in \mathbb{P}^l$.

$$\tilde{D} = \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}, \qquad \tilde{V} = \begin{bmatrix} V_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & B_cV_2B_c^T \end{bmatrix} = \tilde{D}\tilde{D}^T.$$

E_1, E_2: $q\times n$, $q\times m$ matrices; $E_1^TE_2 = 0$.
$\tilde{E}$, R_1, R_2: $[E_1 \;\; E_2C_c]$, $E_1^TE_1$, $E_2^TE_2$; $R_2 \in \mathbb{P}^m$.

$$\tilde{R} = \begin{bmatrix} R_1 & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_2C_c \end{bmatrix} = \tilde{E}^T\tilde{E}.$$

$E_{1\infty}$, $E_{2\infty}$: $q_\infty\times n$, $q_\infty\times m$ matrices; $E_{1\infty}^TE_{2\infty} = 0$.
$\tilde{E}_\infty$, $R_{1\infty}$, $R_{2\infty}$: $[E_{1\infty} \;\; E_{2\infty}C_c]$, $E_{1\infty}^TE_{1\infty}$, $E_{2\infty}^TE_{2\infty}$.

$$\tilde{R}_\infty = \begin{bmatrix} R_{1\infty} & 0_{n\times n_c} \\ 0_{n_c\times n} & C_c^TR_{2\infty}C_c \end{bmatrix} = \tilde{E}_\infty^T\tilde{E}_\infty.$$

$D_{1\infty}$, $D_{2\infty}$: $n\times d_\infty$, $l\times d_\infty$ matrices; $D_{1\infty}D_{2\infty}^T = 0$.
$V_{1\infty}$, $V_{2\infty}$: $D_{1\infty}D_{1\infty}^T$, $D_{2\infty}D_{2\infty}^T$.

$$\tilde{D}_\infty = \begin{bmatrix} D_{1\infty} \\ B_cD_{2\infty} \end{bmatrix}, \qquad \tilde{V}_\infty = \begin{bmatrix} V_{1\infty} & 0_{n\times n_c} \\ 0_{n_c\times n} & B_cV_{2\infty}B_c^T \end{bmatrix}.$$

β; γ, α: nonnegative constant; positive constants.
$A_\alpha = A + \tfrac{1}{2}\alpha I_n$, $A_{c\alpha} = A_c + \tfrac{1}{2}\alpha I_{n_c}$.

2. Robust stability and H_2 performance with a robust H_∞ constraint

In this section we state the robust stability and H_2 performance problem with an H_∞ disturbance attenuation constraint. Specifically, we consider a fixed-order dynamic output-feedback control-design problem with structured real-valued plant parameter uncertainties [illegible] constrained H_∞ disturbance attenuation. This problem involves a set $\mathrm{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}$ [illegible] of uncertain perturbations $(\Delta A, \Delta B, \Delta C)$ of the nominal system matrices (A, B, C). The goal of the problem is to determine a fixed-order, strictly proper dynamic compensator (A_c, B_c, C_c) which (i) stabilizes the plant for all variations in U, (ii) satisfies an H_∞ constraint on disturbance rejection for all variations in U, and (iii) minimizes the worst-case value over the uncertainty set U of a steady-state H_2 performance criterion. In this and the following section no explicit structure is assumed for the elements of U. In Sections 4 and 7, two specific structures of variations in U will be introduced.

H_∞-constrained robust dynamic compensation problem. Given the n-th-order stabilizable and detectable plant with structured real-valued plant parameter variations

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + D_1 w(t), \tag{2.1}$$

$$y(t) = (C + \Delta C)x(t) + D_2 w(t), \tag{2.2}$$

determine an n_c-th-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t), \tag{2.4}$$

which satisfies the following design criteria:

(i) the closed-loop system (2.1)–(2.4) is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$, i.e., $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$;

(ii) the $q_\infty \times d$ closed-loop transfer function

$$H_{\Delta\tilde{A}}(s) \triangleq \tilde{E}_\infty\left[sI_{\tilde{n}} - (\tilde{A} + \Delta\tilde{A})\right]^{-1}\tilde{D} \tag{2.5}$$

from $w(t)$ to $E_{1\infty}x(t) + E_{2\infty}u(t)$, satisfies the constraint

$$\| H_{\Delta\tilde{A}}(s) \|_\infty \le \gamma, \qquad (\Delta A, \Delta B, \Delta C) \in \mathrm{U}, \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

(iii) the performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \ \limsup_{t\to\infty} \mathbb{E}\left[x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)\right] \tag{2.7}$$

is minimized.

Note that for each uncertain variation $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$, the closed-loop system can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty), \tag{2.8}$$

and that (2.7) becomes

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \ \limsup_{t\to\infty} \mathbb{E}\left[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)\right]. \tag{2.9}$$

Furthermore, by defining the transfer function

$$\tilde{H}_{\Delta\tilde{A}}(s) \triangleq \tilde{E}\left[sI_{\tilde{n}} - (\tilde{A} + \Delta\tilde{A})\right]^{-1}\tilde{D},$$

it can be shown that when (i) is satisfied, (2.7) is given by

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \| \tilde{H}_{\Delta\tilde{A}}(s) \|_2^2.$$

Note that the problem statement involves both H_2 and H_∞ performance weights. In particular, the matrices R_1 and R_2 are the H_2 weights for the state and control variables. By introducing the variables

$$z(t) = E_1 x(t), \qquad v(t) = E_2 u(t),$$

the cost (2.7) can be written as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \limsup_{t \to \infty} \mathbb{E}\left[z^{\mathrm{T}}(t) z(t) + v^{\mathrm{T}}(t) v(t) \right].$$

For convenience we thus define $R_1 \triangleq E_1^{\mathrm{T}} E_1$ and $R_2 \triangleq E_2^{\mathrm{T}} E_2$ which appear in subsequent expressions. Although an H_2 cross-weighting term of the form $2x^{\mathrm{T}}(t) R_{12} u(t)$ can also be included, we shall not do so here to facilitate the presentation.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$ and $E_{2\infty}$ for the state and control variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^{\mathrm{T}} E_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}} E_{2\infty}$ are thus the H_∞ counterparts of the H_2 weights R_1 and R_2. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2, we shall require for technical reasons that $R_{2\infty} = \beta^2 R_2$, where the nonnegative scalar β is a design variable. We further note that the assumption $E_{1\infty}^{\mathrm{T}} E_{2\infty} = 0$ precludes an H_∞ cross-weighting term which again facilitates the presentation. Finally, similar remarks apply to the disturbance and sensor noise intensities $V_1 \triangleq D_1 D_1^{\mathrm{T}}$, $V_2 \triangleq D_2 D_2^{\mathrm{T}}$, $V_{1\infty} \triangleq D_{1\infty} D_{1\infty}^{\mathrm{T}}$ and $V_{2\infty} \triangleq D_{2\infty} D_{2\infty}^{\mathrm{T}}$ for the H_2 and H_∞ designs respectively. As in [3], $w(t)$ is interpreted as white noise for the H_2 design and as an L_2 signal for the H_∞ design aspect.

Before continuing it is useful to note that if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$ for a given compensator (A_c, B_c, C_c), then the performance (2.7) is given by

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}} \tilde{R}, \tag{2.10}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t \to \infty} \mathbb{E}\left[\tilde{x}(t) \tilde{x}^{\mathrm{T}}(t) \right] \tag{2.11}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = (\tilde{A} + \Delta\tilde{A}) \tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}} (\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \tilde{V}. \tag{2.12}$$

The key step in guaranteeing robust stability and performance is to replace the uncertain terms in the covariance Lyapunov equation (2.12) by a bounding function Ω. Note that since $\Delta\tilde{A}$ is independent of A_c, the bounding function Ω need only depend on B_c and C_c. Furthermore, the H_∞ disturbance attenuation constraint (2.6) is enforced for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$ by replacing the modified algebraic Lyapunov equation (2.12) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1. *Let* $\Omega : \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \times \mathbb{N}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}}$ *be such that*

$$\Delta\tilde{A} \mathrm{Q} + \mathrm{Q} \Delta\tilde{A}^{\mathrm{T}} \le \Omega(B_c, C_c, \mathrm{Q}), \quad (\Delta A, \Delta B, \Delta C) \in \mathrm{U}, \ (B_c, C_c, \mathrm{Q}) \in \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c} \times \mathbb{N}^{\tilde{n}}, \tag{2.13}$$

and, for a given (A_c, B_c, C_c), *suppose there exists* $\mathrm{Q} \in \mathbb{N}^{\tilde{n}}$ *satisfying*

$$0 = \tilde{A} \mathrm{Q} + \mathrm{Q} \tilde{A}^{\mathrm{T}} + \gamma^{-2} \mathrm{Q} \tilde{R}_\infty \mathrm{Q} + \Omega(B_c, C_c, \mathrm{Q}) + \tilde{V}. \tag{2.14}$$

Then

$$(\tilde{A}+\Delta\tilde{A},\ \tilde{D}) \text{ is stabilizable}, \quad (\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}, \tag{2.15}$$

if and only if

$$\tilde{A}+\Delta\tilde{A} \text{ is asymptotically stable}, \quad (\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}. \tag{2.16}$$

In this case,

$$\| H_{\Delta\tilde{A}}(s)\|_\infty \le \gamma, \quad (\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}, \tag{2.17}$$

and

$$\tilde{Q}_{\Delta\tilde{A}} \le \mathrm{Q}, \quad (\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}, \tag{2.18}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ *is given by* (2.12). *Consequently,*

$$J(A_c,\ B_c,\ C_c) \le \mathrm{J}(A_c,\ B_c,\ C_c,\ \mathrm{Q}), \tag{2.19}$$

where

$$\mathrm{J}(A_c,\ B_c,\ C_c,\ \mathrm{Q}) \triangleq \operatorname{tr} \mathrm{Q}\tilde{R}. \tag{2.20}$$

Proof. First note for clarity that in (2.13) Q denotes an arbitrary element of $\mathbb{N}^{\tilde{n}}$ since (2.13) holds for all $\mathrm{Q}\in\mathbb{N}^{\tilde{n}}$, while in (2.14) Q denotes a specific solution to (2.14). Now for $(\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}$, (2.14) is equivalent to

$$0=(\tilde{A}+\Delta\tilde{A})\mathrm{Q}+\mathrm{Q}(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}+\gamma^{-2}\mathrm{Q}\tilde{R}_\infty\mathrm{Q}+\Omega(B_c,\ C_c,\ \mathrm{Q})-(\Delta\tilde{A}\mathrm{Q}+\mathrm{Q}\Delta\tilde{A}^{\mathrm{T}})+\tilde{V}. \tag{2.21}$$

Hence, by assumption, (2.21) has a solution $\mathrm{Q}\in\mathbb{N}^{\tilde{n}}$ for all $(\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}$ and, by (2.13), $\Omega(B_c,\ C_c,\ \mathrm{Q})-(\Delta\tilde{A}\mathrm{Q}+\mathrm{Q}\Delta\tilde{A}^{\mathrm{T}})$ is nonnegative definite. Now it follows from Theorem 3.6 of [7] and (2.15) that $(\tilde{A}+\Delta\tilde{A},\ [\tilde{V}+\gamma^{-2}\mathrm{Q}\tilde{R}_\infty\mathrm{Q}+\Omega(B_c,\ C_c,\ \mathrm{Q})-(\Delta\tilde{A}\mathrm{Q}+\mathrm{Q}\Delta\tilde{A}^{\mathrm{T}})]^{1/2})$ is stabilizable for all $(\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}$. Thus it follows from (2.21) and Lemma 12.2 of [7] that $\tilde{\mathrm{A}}+\Delta\tilde{\mathrm{A}}$ is asymptotically stable for all $(\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}$. Conversely, if $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A,\ \Delta B,\ \Delta C)\in \mathbf{U}$, then (2.16) holds. The proof of (2.17) follows from a standard manipulation of (2.14). Next, subtracting (2.12) from (2.20) yields

$$\begin{aligned} 0=&(\tilde{A}+\Delta\tilde{A})(\mathrm{Q}-\tilde{Q}_{\Delta\tilde{A}})+(\mathrm{Q}-\tilde{Q}_{\Delta\tilde{A}})(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}\\ &+\gamma^{-2}\mathrm{Q}\tilde{R}_\infty\mathrm{Q}+\Omega(B_c,\ C_c,\ \mathrm{Q})-(\Delta\tilde{A}\mathrm{Q}+\mathrm{Q}\Delta\tilde{A}^{\mathrm{T}}), \end{aligned}$$

or, equivalently, since $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A,\ \Delta B,\ \Delta C)\in \Omega$,

$$\mathrm{Q}-\tilde{Q}_{\Delta\tilde{A}}=\int_0^\infty \mathrm{e}^{(\tilde{A}+\Delta\tilde{A})t}\left[\gamma^{-2}\mathrm{Q}\tilde{R}_\infty\mathrm{Q}+\Omega(B_c,\ C_c,\ \mathrm{Q})-(\Delta\tilde{A}\mathrm{Q}+\mathrm{Q}\Delta\tilde{A}^{\mathrm{T}})\right]\mathrm{e}^{(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}t}\,\mathrm{d}t\ge 0$$

which implies (2.18). The performance bound (2.19) is now an immediate consequence of (2.18). □

Remark 2.1. Note that (2.15) is actually a closed-loop 'disturbability' condition which is not concerned with control as such. This condition guarantees that the closed-loop system does not possess unstable undisturbed modes.

3. The auxiliary minimization problem

As shown in the previous section, the replacement of (2.12) by (2.14) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the worst case H_2 performance criterion. That is,

given a compensator (A_c, B_c, C_c) for which there exists a nonnegative-definite solution to (2.14), the *actual* worst case H_2 performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathrm{J}(A_c, B_c, C_c, \mathrm{Q})$. Hence, $\mathrm{J}(A_c, B_c, C_c, \mathrm{Q})$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, \mathrm{Q})$ which minimizes $\mathrm{J}(A_c, B_c, C_c, \mathrm{Q})$ subject to (2.14) with $\mathrm{Q} \in \mathbb{N}^{\tilde{n}}$.

It follows from Lemma 2.1 that the satisfaction of (2.14) for $\mathrm{Q} \in \mathbb{N}^{\tilde{n}}$ along with the generic condition (2.15) leads to (i) closed-loop stability for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$; (ii) prespecified H_∞ performance attenuation for all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$; and (iii) an upper bound for the worst case H_2 performance criterion. Hence, it remains to determine (A_c, B_c, C_c) which *minimizes* $\mathrm{J}(A_c, B_c, C_c, \mathrm{Q})$ and thus provides an optimized bound for the actual worst case H_2 performance $J(A_c, B_c, C_c)$ over all $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$.

4. Uncertainty structure: Linear bound

Having established the theoretical basis for our approach, we now assign explicit structure to the set of U and bounding function Ω. Specifically, the uncertainty set U is assumed to be of the form

$$\mathrm{U} = \left\{(\Delta A, \Delta B, \Delta C) \colon \Delta A = \sum_{i=1}^{p} \sigma_i A_i,\ \Delta B = \sum_{i=1}^{p} \sigma_i B_i,\ \Delta C = \sum_{i=1}^{p} \sigma_i C_i,\ \sum_{i=1}^{p} \sigma_i^2/\alpha_i^2 \le 1\right\}, \tag{4.1}$$

where, for $i = 1, \ldots, p$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, and $C_i \in \mathbb{R}^{l \times n}$ are fixed matrices denoting the structure of the parametric uncertainty; α_i is a given positive number; and σ_i is an uncertain real parameter. Note that the uncertain parameters σ_i are assumed to lie in a specified ellipsoidal region in $\mathbb{R}^p$. The closed-loop system (2.8) thus has structured uncertainty of the form

$$\Delta \tilde{A} = \sum_{i=1}^{p} \sigma_i \tilde{A}_i, \quad \text{where} \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & 0_{n_c} \end{bmatrix}, \quad i = 1, \ldots, p. \tag{4.2}$$

Note that the symmetry of the uncertainty set entails no loss of generality by requiring only a redefinition of the nominal plant matrices.

In order to obtain explicit gain expressions for (A_c, B_c, C_c) in Sections 5 and 6, we shall require that at most one of the perturbations ΔB and ΔC is nonzero. We thus consider the cases $(\Delta A, \Delta C) \in \mathrm{U}$ or $(\Delta A, \Delta B) \in \mathrm{U}$. If this assumption is not imposed, then optimality conditions can still be derived, but at the expense of closed-form gain expressions. In this section and Section 5 we will assume that $\Delta B = 0$ (i.e., $B_i = 0$, $i = 1, \ldots, p$) so that $\Omega(B_c, C_c, \mathrm{Q})$ becomes $\Omega(B_c, \mathrm{Q})$. The dual case $\Delta B \neq 0$, $\Delta C = 0$ (i.e., $C_i = 0$, $i = 1, \ldots, p$) will be considered in Section 6.

For the structure of U specified by (4.1), the bound Ω satisfying (2.13) can now be given a concrete form.

Proposition 4.1. *Let α be an arbitrary positive scalar. Then the function*

$$\Omega(B_c, \mathrm{Q}) \triangleq \alpha \mathrm{Q} + \alpha^{-1} \sum_{i=1}^{p} \alpha_i^2 \tilde{A}_i \mathrm{Q} \tilde{A}_i^{\mathrm{T}} \tag{4.3}$$

satisfies (2.13) *with* U *given by* (4.1).

Proof. See [2]. □

Remark 4.1. Note that the bound Ω given by (4.3) consists of two distinct terms. The first term αQ can be thought of as arising from an exponential time weighting of the cost, or, equivalently, from a uniform right shift of the open-loop dynamics. The second term $\alpha^{-1}\Sigma_{i=1}^{p}\alpha_i^2\tilde{A}_i Q\tilde{A}_i^{\mathrm{T}}$ arises naturally from a multiplicative white noise model. Such interpretations have no bearing on the results obtained here since only the bound Ω defined by (4.3) is required. We call (4.3) the *linear bound* since it is linear in Q. For a more detailed discussion on (4.3) see [2].

With Ω defined by (4.3), the modified Riccati equation (2.14) becomes

$$0=\tilde{A}Q+Q\tilde{A}^{\mathrm{T}}+\gamma^{-2}Q\tilde{R}_\infty Q+\alpha Q+\alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i Q\tilde{A}_i^{\mathrm{T}}+\tilde{V} \tag{4.4}$$

or, equivalently,

$$0=\tilde{A}_\alpha Q+Q\tilde{A}_\alpha^{\mathrm{T}}+\gamma^{-2}Q\tilde{R}_\infty Q+\sum_{i=1}^{p}\delta_i\tilde{A}_i Q\tilde{A}_i^{\mathrm{T}}+\tilde{V}, \tag{4.5}$$

where $\delta_i\triangleq\alpha_i^2/\alpha$ and

$$\tilde{A}_\alpha\triangleq\tilde{A}+\tfrac{1}{2}\alpha I_{\tilde{n}}=\begin{bmatrix}A_\alpha & BC_c\\ B_cC & A_{c\alpha}\end{bmatrix}.$$

5. Sufficient conditions for robust stability and performance with robust H_∞ disturbance attenuation: Linear bound

In this section we state sufficient conditions for characterizing fixed-order (i.e., full- and reduced-order) controllers guaranteeing closed-loop stability for all $(\Delta A,\ \Delta C)\in\mathrm{U}$, constrained H_∞ disturbance attenuation for all $(\Delta A,\ \Delta C)\in\mathrm{U}$, and an optimized worst case H_2 performance bound. In order to state the main results we require some additional notation and a factorization lemma.

Lemma 5.1. *Let $\hat{Q},\ \hat{P}\in\mathbb{N}^n$ and suppose* rank $\hat{Q}\hat{P}=n_c$. *Then there exist $n_c\times n$ G, Γ and $n_c\times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that*

$$\hat{Q}\hat{P}=G^{\mathrm{T}}M\Gamma,\qquad \Gamma G^{\mathrm{T}}=I_{n_c}. \tag{5.1), (5.2}$$

Furthermore, the $n\times n$ matrices

$$\tau\triangleq G^{\mathrm{T}}\Gamma,\qquad \tau_\perp\triangleq I_n-\tau \tag{5.3), (5.4}$$

are idempotent and have rank n_c and $n-n_c$, respectively. Finally, if $P\in\mathbb{N}^n$ and $\beta\geq 0$ then the inverse

$$S\triangleq\left(I_n+\beta^2\gamma^{-2}\hat{Q}P\right)^{-1} \tag{5.5}$$

exists.

For arbitrary $Q,\ \hat{Q}\in\mathbb{R}^{n\times n}$ and $\alpha>0$ define the following notation:

$$V_{2s}\triangleq V_2+\sum_{i=1}^{p}\delta_iC_i(Q+\hat{Q})C_i^{\mathrm{T}},\qquad Q_s\triangleq QC^{\mathrm{T}}+\sum_{i=1}^{p}\delta_iA_i(Q+\hat{Q})C_i^{\mathrm{T}},\qquad \Sigma\triangleq BR_2^{-1}B^{\mathrm{T}}.$$

Theorem 5.1. *Suppose there exists Q, P, $\hat{Q}$, $\hat{P} \in \mathbb{N}^n$ satisfying*

$$0 = A_\alpha Q + QA_\alpha^{\mathrm{T}} + \gamma^{-2} QR_{1\infty}Q + V_1 + \sum_{i=1}^{p} \delta_i A_i (Q + \hat{Q}) A_i^{\mathrm{T}} - Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} + \tau_\perp Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \tag{5.6}$$

$$0 = \left(A_\alpha + \gamma^{-2}[Q + \hat{Q}] R_{1\infty}\right)^{\mathrm{T}} P + P\left(A_\alpha + \gamma^{-2}[Q+\hat{Q}] R_{1\infty}\right) + R_1$$
$$+ \sum_{i=1}^{p} \delta_i \left[A_i^{\mathrm{T}} P A_i + \left(A_i - Q_s V_{2s}^{-1} C_i\right)^{\mathrm{T}} \hat{P} \left(A_i - Q_s V_{2s}^{-1} C_i\right)\right] - S^{\mathrm{T}} P \Sigma P S + \tau_\perp^{\mathrm{T}} S^{\mathrm{T}} P \Sigma P S \tau_\perp, \tag{5.7}$$

$$0 = \left(A_\alpha - \Sigma PS + \gamma^{-2} QR_{1\infty}\right)\hat{Q} + \hat{Q}\left(A_\alpha - \Sigma PS + \gamma^{-2} QR_{1\infty}\right)^{\mathrm{T}}$$
$$+ \gamma^{-2} \hat{Q}\left(R_{1\infty} + \beta^2 S^{\mathrm{T}} P \Sigma P S\right)\hat{Q} + Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} - \tau_\perp Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} \tau_\perp^{\mathrm{T}}, \tag{5.8}$$

$$0 = \left(A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2} QR_{1\infty}\right)^{\mathrm{T}} \hat{P} + \hat{P}\left(A_\alpha - Q_s V_{2s}^{-1} C + \gamma^{-2} QR_{1\infty}\right) + S^{\mathrm{T}} P \Sigma P S - \tau_\perp^{\mathrm{T}} S^{\mathrm{T}} P \Sigma P S \tau_\perp, \tag{5.9}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c, \tag{5.10}$$

and let $(A_c, B_c, C_c, \mathbf{Q})$ be given by

$$A_c = \Gamma\left(A - \Sigma PS - Q_s V_{2s}^{-1} C + \gamma^{-2} QR_{1\infty}\right) G^{\mathrm{T}}, \tag{5.11}$$

$$B_c = \Gamma Q_s V_{2s}^{-1}, \tag{5.12}$$

$$C_c = -R_2^{-1} B^{\mathrm{T}} PSG^{\mathrm{T}}, \tag{5.13}$$

$$\mathbf{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}. \tag{5.14}$$

Then $(\tilde{A} + \Delta\tilde{A}, \tilde{D})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathbf{U}$ if and only if $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathbf{U}$. In this case, the closed-loop transfer function $H_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\| H_{\Delta\tilde{A}}(s) \|_\infty \le \gamma, \quad (\Delta A, \Delta C) \in \mathbf{U}, \tag{5.15}$$

and the worst-case H_2 performance criterion (2.10) *satisfies the bound*

$$J(A_c, B_c, C_c) \le \operatorname{tr}\left[(Q + \hat{Q}) R_1 + \hat{Q} S^{\mathrm{T}} P \Sigma P S\right]. \tag{5.16}$$

Proof. The proof follows from Lemma 2.1 by combining the proofs of Theorem 6.1 of [3] and Theorem 6.1 of [2]. □

Remark 5.1. Theorem 5.1 presents sufficient conditions for designing controllers yielding robust stability and performance with a constraint on the H_∞ norm of the closed-loop transfer function for a state-space system with structured real-valued plant parameter variations. These sufficient conditions comprise a system of three modified Riccati equations and one modified Lyapunov equation coupled by the optimal projection τ, the linear uncertainty bound, and the H_∞ constraint. If the uncertainty bound is deleted, then the results of [3] are recovered. If, alternatively, the uncertainty terms are retained but the H_∞ constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then the results of [2] are recovered for the case $B_i = 0$, $i = 1, \ldots, p$.

Remark 5.2. To specialize Theorem 5.1 to the full-order case $n_c = n$, it is only necessary to set $G^{\mathrm{T}} = \Gamma^{-1}$ so that $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$ without loss of generality. Now the last term in each of (5.6)–(5.9) can be deleted and G and Γ in (5.11)–(5.14) can be taken to be the identity. It is interesting to note that in the full-order case the H_∞ design problem with structured parameter variations is comprised of four coupled

Riccati/Lyapunov equations. This coupling illustrates the breakdown of regulator/estimator separation and shows that the certainty equivalence principle is no longer valid. This is not surprising since separation also breaks down for the full-order H_2 result with parameter uncertainties [2].

Remark 5.3. When solving (5.6)–(5.10) numerically, the uncertainty terms and the H_∞ constraint can be adjusted to examine tradeoffs among performance, robustness, and disturbance rejection. Specifically, the uncertainty range α_i and the structure matrices A_i, C_i appearing in Q_s and V_{2s} along with γ can be varied systematically to determine the region of solvability of the design equations (5.6)–(5.9).

Remark 5.4. We point out that if $\beta = 0$ or, equivalently, $E_{2\infty} = 0$, which corresponds to the 'cheap' H_∞ control case (i.e., H_∞ attenuation between disturbances and controls is not constrained), it is possible to obtain closed-form gains (A_c, B_c, C_c) given by a modified set of design equations when all three of ΔA, ΔB, and ΔC are nonzero. Because of space limitations this result will be given in a future paper.

Remark 5.5. An important special case of the results of Section 5 is obtained by setting $\Delta A = 0$, $\Delta B = 0$, $\Delta C = \sigma_1 C_1$, and $C_1 = C$. The resulting H_2/H_∞ design is guaranteed to possess a gain margin of $\pm 100\alpha_1$ percent at the sensor output.

6. The dual case: Linear bound

Unlike the standard LQG result involving a pair of uncoupled Riccati equations, the new result guaranteeing robust stability, robust performance, and H_∞ disturbance rejection involves a coupled system of *four* modified Riccati/Lyapunov equations. The asymmetry of these equations suggests the possibility of a *dual* result in which the modifications to the standard Riccati equations are reversed. One motivation for developing such dual results is that for certain problems the dual bounds may be sharper than the primal bound introduced in Section 4. Furthermore, the dual theory permits distinct H_∞ disturbance weights ($V_{1\infty}$ and $V_{2\infty}$), although we now require $R_{1\infty} = R_1$. Finally, the dual theory allows for uncertainty in the control matrix B (i.e., $\Delta B \neq 0$), although we now require $\Delta C = 0$, (i.e., $C_i = 0$, $i = 1, \ldots, p$) to obtain closed-form gain expressions for (A_c, B_c, C_c). We begin with the following lemma.

Lemma 6.1. *Suppose the system* (2.8) *is asymptotically stable for all* $(\Delta A, \Delta B, \Delta C) \in \mathrm{U}$ *for a given* (A_c, B_c, C_c). *Then*

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathrm{U}} \operatorname{tr} \tilde{P}_{\Delta \tilde{A}} \tilde{V}, \tag{6.1}$$

where $\tilde{P}_{\Delta \tilde{A}} \in \mathbb{N}^{\tilde{n}}$ *is the unique solution to*

$$0 = (\tilde{A} + \Delta \tilde{A})^{\mathrm{T}} \tilde{P}_{\Delta \tilde{A}} + \tilde{P}_{\Delta \tilde{A}} (\tilde{A} + \Delta \tilde{A}) + \tilde{R}. \tag{6.2}$$

Proof. See [1]. □

Utilizing (6.1) in place of (2.10), the H_∞ disturbance attenuation constraint from plant and sensor disturbances to the state and control variables given by

$$\| \hat{H}_{\Delta \tilde{A}}(s) \|_\infty = \| \tilde{E}[sI_{\tilde{n}} - (\tilde{A} + \Delta \tilde{A})]^{-1} \tilde{D}_\infty \|_\infty \le \gamma \tag{6.3}$$

can now be enforced by replacing (2.14) by the modified Riccati equation

$$0 = \tilde{A}^{\mathrm{T}} \mathbf{P} + \mathbf{P} \tilde{A} + \gamma^{-2} \mathbf{P} \tilde{V}_\infty \mathbf{P} + \hat{\Omega}(C_c, \mathbf{P}) + \tilde{R}, \tag{6.4}$$

where

$$\Delta\tilde{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\Delta\tilde{A}\le\hat{\Omega}(C_c,\mathbf{P}),\quad(\Delta A,\Delta B)\in\mathrm{U}. \tag{6.5}$$

Note that (6.4) is merely the dual of (2.14). We also require the condition dual to (2.15) given by

$$(\tilde{E},\ \tilde{A}+\Delta\tilde{A})\text{ is detectable for all }(\Delta A,\Delta B)\in\mathrm{U}. \tag{6.6}$$

For the structure of U as specified by (4.1) with $\Delta C=0$, the bound $\hat{\Omega}$ satisfying (6.5) can now be given a concrete form.

Proposition 6.1. *Let α be an arbitrary positive scalar. Then the function*

$$\hat{\Omega}(C_c,\mathbf{P})\triangleq\alpha\mathbf{P}+\alpha^{-1}\sum_{i=1}^{p}\alpha_i^2\tilde{A}_i^{\mathrm{T}}\mathbf{P}\tilde{A}_i \tag{6.7}$$

satisfies (6.5) *with* U *given by* (4.1) *and* $\Delta C=0$. *With $\hat{\Omega}$ defined by* (6.7), *the modified dual Riccati equation* (6.4) *becomes*

$$0=\tilde{A}_\alpha^{\mathrm{T}}\mathbf{P}+\mathbf{P}\tilde{A}_\alpha+\gamma^{-2}\mathbf{P}\tilde{V}_\infty\mathbf{P}+\sum_{i=1}^{p}\delta_i\tilde{A}_i^{\mathrm{T}}\mathbf{P}\tilde{A}_i+\tilde{R}. \tag{6.8}$$

We can now state sufficient conditions for robust stability, robust H_2 performance, and robust disturbance attenuation for the dual problem. For arbitrary, Q, P, $\hat{P}\in\mathbb{R}^{n\times n}$ and $\alpha>0$ define the following notation:

$$R_{2s}\triangleq R_2+\sum_{i=1}^{p}\delta_iB_i^{\mathrm{T}}(P+\hat{P})B_i,\qquad P_s\triangleq B^{\mathrm{T}}P+\sum_{i=1}^{p}\delta_iB_i^{\mathrm{T}}(P+\hat{P})A_i,$$

$$\hat{S}\triangleq\left(I_n+\gamma^{-2}\beta^2Q\hat{P}\right)^{-1},\qquad\bar{\Sigma}\triangleq C^{\mathrm{T}}V_2^{-1}C.$$

Theorem 6.1. *Suppose there exist P, Q, $\hat{P}$, $\hat{Q}\in\mathbb{N}^n$ satisfying* (5.10) *and*

$$0=A_\alpha^{\mathrm{T}}P+PA_\alpha+\gamma^{-2}PV_{1\infty}P+R_1+\sum_{i=1}^{p}\delta_iA_i^{\mathrm{T}}(P+\hat{P})A_i-P_s^{\mathrm{T}}R_{2s}^{-1}P_s+\tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp, \tag{6.9}$$

$$\begin{aligned}0={}&\left(A_\alpha+\gamma^{-2}V_{1\infty}[P+\hat{P}]\right)Q+Q\left(A_\alpha+\gamma^{-2}V_{1\infty}[P+\hat{P}]\right)^{\mathrm{T}}+V_1\\&+\sum_{i=1}^{p}\delta_i\left[A_iQA_i^{\mathrm{T}}+(A_i-B_iR_{2s}^{-1}P_s)\hat{Q}(A_i-B_iR_{2s}^{-1}P_s)^{\mathrm{T}}\right]-\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}+\tau_\perp\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}\tau_\perp^{\mathrm{T}},\end{aligned} \tag{6.10}$$

$$\begin{aligned}0={}&\left(A_\alpha-\hat{S}Q\bar{\Sigma}+\gamma^{-2}V_{1\infty}P\right)^{\mathrm{T}}\hat{P}+\hat{P}\left(A_\alpha-\hat{S}Q\bar{\Sigma}+\gamma^{-2}V_{1\infty}P\right)+\gamma^{-2}\hat{P}\left(V_{1\infty}+\beta^2\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}\right)\hat{P}\\&+P_s^{\mathrm{T}}R_{2s}^{-1}P_s-\tau_\perp^{\mathrm{T}}P_s^{\mathrm{T}}R_{2s}^{-1}P_s\tau_\perp,\end{aligned} \tag{6.11}$$

$$0=\left(A_\alpha-BR_{2s}^{-1}P_s+\gamma^{-2}V_{1\infty}P\right)\hat{Q}+\hat{Q}\left(A_\alpha-BR_{2s}^{-1}P_s+\gamma^{-2}V_{1\infty}P\right)^{\mathrm{T}}+\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}-\tau_\perp\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{6.12}$$

and let $(A_c, B_c, C_c, \mathbf{P})$ be given by

$$A_c=\Gamma\left(A-\hat{S}Q\bar{\Sigma}-BR_{2s}^{-1}P_s+\gamma^{-2}V_{1\infty}P\right)G^{\mathrm{T}}, \tag{6.13}$$

$$B_c=\Gamma\hat{S}QC^{\mathrm{T}}V_2^{-1}, \tag{6.14}$$

$$C_c=-R_{2s}^{-1}P_sG^{\mathrm{T}}, \tag{6.15}$$

$$\mathbf{P}=\begin{bmatrix}P+\hat{P} & -\hat{P}G^{\mathrm{T}}\\ -G\hat{P} & G\hat{P}G^{\mathrm{T}}\end{bmatrix}. \tag{6.16}$$

Then $(\tilde{E}, \tilde{A}+\Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B) \in \mathrm{U}$ if and only if $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B) \in \mathrm{U}$. In this case, the closed-loop transfer function $\hat{H}_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint

$$\|\hat{H}_{\Delta\tilde{A}}(s)\|_\infty \leq \gamma, \quad (\Delta A, \Delta B) \in \mathrm{U}, \tag{6.17}$$

and the worst-case H_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c) \leq \operatorname{tr}\left[(P+\hat{P})V_1 + \hat{P}\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}\right]. \tag{6.18}$$

Remark 6.1. The dual case of Remark 5.5 is obtained by setting $\Delta A = 0$, $\Delta B = \sigma_1 B_1$, $\Delta C = 0$, and $B_1 = B$. The resulting H_2/H_∞ design is guaranteed to possess a guaranteed gain margin of $\pm 100\alpha_1$ percent at the input.

7. Uncertainty structure and sufficient conditions for robust stability and performance with H_∞ disturbance attenuation: Quadratic bound

We now assign a different structure to the uncertainty set U and the bounding function Ω. Specifically, the uncertainty set U is assumed to be of the form

$$\mathrm{U} = \Big\{(\Delta A, \Delta B, \Delta C): \Delta A = \sum_{i=1}^{p} F_i M_i N_i G_i,\ \Delta B = \sum_{i=1}^{p} F_i M_i N_i H_i,$$
$$\Delta C = \sum_{i=1}^{p} K_i M_i N_i G_i,\ M_i M_i^{\mathrm{T}} \leq \overline{M}_i,\ N_i^{\mathrm{T}} N_i \leq \overline{N}_i,\ i = 1, \ldots, p\Big\}, \tag{7.1}$$

where, for $i = 1, \ldots, p$, $F_i \in \mathbb{R}^{n \times r_i}$, $G_i \in \mathbb{R}^{t_i \times n}$, $H_i \in \mathbb{R}^{t_i \times m}$, and $K_i \in \mathbb{R}^{l \times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\overline{M}_i \in \mathbb{N}^{r_i}$ and $\overline{N}_i \in \mathbb{N}^{t_i}$ are given uncertainty bounds; and $M_i \in \mathbb{R}^{r_i \times s_i}$ and $N_i \in \mathbb{R}^{s_i \times t_i}$ are uncertain matrices.

In order to obtain explicit gain expressions for (A_c, B_c, C_c) we again consider two cases: (1) $(\Delta A, \Delta C) \in \mathrm{U}$ with $\Delta B = 0$, and (2) $(\Delta A, \Delta B) \in \mathrm{U}$ with $\Delta C = 0$. When $\Delta B = 0$ the closed-loop system has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{p} \tilde{F}_i M_i N_i \tilde{G}_i, \tag{7.2}$$

where

$$\tilde{F}_i \triangleq \begin{bmatrix} F_i \\ B_c K_i \end{bmatrix}, \qquad \tilde{G}_i \triangleq \begin{bmatrix} G_i & 0_{t_i \times n_c} \end{bmatrix}.$$

In this case the quadratic bound Ω satisfying (2.13) can now be given a concrete form.

Proposition 7.1. *The function*

$$\Omega(B_c, \mathrm{Q}) \triangleq \sum_{i=1}^{p} \tilde{F}_i \overline{M}_i \tilde{F}_i^{\mathrm{T}} + \mathrm{Q}\tilde{G}_i^{\mathrm{T}} \overline{N}_i \tilde{G}_i \mathrm{Q} \tag{7.3}$$

satisfies (2.13) *with* U *given by* (7.1) *and* $\Delta B = 0$.

Proof. See [1]. □

Thus, with Ω defined by (7.3), the modified Riccati equation (2.14) becomes

$$0=\tilde{A}\mathrm{Q}+\mathrm{Q}\tilde{A}^{\mathrm{T}}+\gamma^{-2}\mathrm{Q}\tilde{R}_{\infty}\mathrm{Q}+\sum_{i=1}^{p}\left[\tilde{F}_i\overline{M}_i\tilde{F}_i^{\mathrm{T}}+\mathrm{Q}\tilde{G}_i^{\mathrm{T}}\overline{N}_i\tilde{G}_i\mathrm{Q}\right]+\tilde{V}. \tag{7.4}$$

For arbitrary $Q\in\mathbf{R}^{n\times n}$ define:

$$Q_{\mathrm{a}}\triangleq QC^{\mathrm{T}}+\sum_{i=1}^{p}F_i\overline{M}_iK_i,\qquad D\triangleq\sum_{i=1}^{p}F_i\overline{M}_iF_i^{\mathrm{T}},$$

$$V_{2\mathrm{a}}\triangleq V_2+\sum_{i=1}^{p}K_i\overline{M}_iK_i^{\mathrm{T}},\qquad E\triangleq\sum_{i=1}^{p}G_i^{\mathrm{T}}\overline{N}_iG_i.$$

Theorem 7.1. *Suppose these exist Q, P, $\hat{Q}$, $\hat{P}\in\mathbf{N}^n$ satisfying* (5.10) *and*

$$0=AQ+QA^{\mathrm{T}}+\gamma^{-2}QR_{1\infty}Q+V_1+QEQ+D-Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}Q_{\mathrm{a}}^{\mathrm{T}}+\tau_{\perp}Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}Q_{\mathrm{a}}^{\mathrm{T}}\tau_{\perp}^{\mathrm{T}}, \tag{7.5}$$

$$\begin{aligned}0=&\left(A+[Q+\hat{Q}][\gamma^{-2}R_{1\infty}+E]\right)^{\mathrm{T}}P+P\left(A+[Q+\hat{Q}][\gamma^{-2}R_{1\infty}+E]\right)\\&+R_1-S^{\mathrm{T}}P\Sigma PS+\tau_{\perp}^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_{\perp},\end{aligned} \tag{7.6}$$

$$\begin{aligned}0=&\left(A-\Sigma PS+Q[\gamma^{-2}R_{1\infty}+E]\right)\hat{Q}+\hat{Q}\left(A-\Sigma PS+Q[\gamma^{-2}R_{1\infty}+E]\right)^{\mathrm{T}}\\&+\hat{Q}\left(\gamma^{-2}[R_{1\infty}+\beta^2S^{\mathrm{T}}P\Sigma PS]+E\right)\hat{Q}+Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}Q_{\mathrm{a}}^{\mathrm{T}}-\tau_{\perp}Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}Q_{\mathrm{a}}^{\mathrm{T}}\tau_{\perp}^{\mathrm{T}},\end{aligned} \tag{7.7}$$

$$\begin{aligned}0=&\left(A-Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}C+Q[\gamma^{-2}R_{1\infty}+E]\right)^{\mathrm{T}}\hat{P}+\hat{P}\left(A-Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}C+Q[\gamma^{-2}R_{1\infty}+E]\right)+S^{\mathrm{T}}P\Sigma PS\\&-\tau_{\perp}^{\mathrm{T}}S^{\mathrm{T}}P\Sigma PS\tau_{\perp},\end{aligned} \tag{7.8}$$

and let Q *be given by* (5.14) *and* $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ *by*

$$A_{\mathrm{c}}=\Gamma\left(A-\Sigma PS-Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}C+Q[\gamma^{-2}R_{1\infty}+E]\right)G^{\mathrm{T}}, \tag{7.9}$$

$$B_{\mathrm{c}}=\Gamma Q_{\mathrm{a}}V_{2\mathrm{a}}^{-1}, \tag{7.10}$$

$$C_{\mathrm{c}}=-R_2^{-1}B^{\mathrm{T}}PSG^{\mathrm{T}}. \tag{7.11}$$

Then $(\tilde{A}+\Delta\tilde{A},\ \tilde{D})$ is stabilizable for all $(\Delta A,\ \Delta C)\in\mathrm{U}$ if and only if $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A,\ \Delta C)\in\mathrm{U}$. In this case, the closed-loop transfer function $H_{\Delta\tilde{A}}(s)$ satisfies the H_∞ disturbance attenuation constraint ·

$$\|H_{\Delta\tilde{A}}(s)\|_{\infty}\le\gamma,\quad(\Delta A,\ \Delta C)\in\mathrm{U}, \tag{7.12}$$

and the worst-case H_2 performance criterion (2.10) *satisfies the bound*

$$J(A_{\mathrm{c}},B_{\mathrm{c}},C_{\mathrm{c}})\le\mathrm{tr}\left[(Q+\hat{Q})R_1+\hat{Q}S^{\mathrm{T}}P\Sigma PS\right]. \tag{7.13}$$

Proof. The proof follows by combining the proofs of Theorems 6.1 of [3] and Theorem 8.1 of [1]. □

Remark 7.1. It is interesting to note that in the full-order case $n_{\mathrm{c}}=n$ with $G=\Gamma=\tau=I_n$ and $\tau_{\perp}=0$ (see Remark 5.1), $\hat{P}$ plays no role so that (7.8) is superfluous. Thus, unlike the full-order result for the linear bound involving four equations, the full-order quadratic bound involves *three* modified Riccati equations coupled by the quadratic bound and the H_∞ constraint. If, alternatively, the reduced-order constraint is retained, but the uncertainty terms are deleted, then the results of [3] are recovered. If, furthermore, the uncertainty terms are retained, but the H_∞ constraint is sufficiently relaxed, i.e., $\gamma\to\infty$, then the results of [1] are recovered.

8. The dual case: Quadratic bound

For the structure of U as specified by (7.1) with $\Delta C = 0$, the closed-loop system has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{p} \tilde{\mathbf{F}}_i M_i N_i \tilde{\mathbf{G}}_i, \tag{8.1}$$

where

$$\tilde{\mathbf{F}}_i \triangleq \begin{bmatrix} F_i \\ 0_{n_c \times r_i} \end{bmatrix}, \qquad \tilde{\mathbf{G}}_i \triangleq [G_i \quad H_i C_c].$$

Proposition 8.1. *The function*

$$\hat{\Omega}(C_c, \mathbf{P}) \triangleq \sum_{i=1}^{p} \tilde{\mathbf{G}}_i^{\mathrm{T}} \overline{N}_i \tilde{\mathbf{G}}_i + \mathbf{P}\tilde{\mathbf{F}}_i \overline{M}_i \tilde{\mathbf{F}}_i^{\mathrm{T}} \mathbf{P} \tag{8.2}$$

satisfies (6.5) *with* U *given by* (7.1) *and* $\Delta C = 0$.

With $\hat{\Omega}$ defined by (8.2), the modified dual equation (6.4) becomes

$$0 = \tilde{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{A} + \gamma^{-2}\mathbf{P}\tilde{V}_\infty\mathbf{P} + \sum_{i=1}^{p}\left[\tilde{\mathbf{G}}_i^{\mathrm{T}}\overline{N}_i\tilde{\mathbf{G}}_i + \mathbf{P}\tilde{\mathbf{F}}_i\overline{M}_i\tilde{\mathbf{F}}_i^{\mathrm{T}}\mathbf{P}\right] + \tilde{R}. \tag{8.3}$$

For arbitrary $P \in \mathbb{R}^{n \times n}$ define:

$$P_a \triangleq B^{\mathrm{T}}P + \sum_{i=1}^{p} H_i^{\mathrm{T}}\overline{N}_i G_i, \qquad R_{2a} \triangleq R_2 + \sum_{i=1}^{p} H_i^{\mathrm{T}}\overline{N}_i H_i.$$

Theorem 8.1. *Suppose there exist* $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ *satisfying* (5.10) *and*

$$0 = A^{\mathrm{T}}P + PA + \gamma^{-2}PV_{1\infty}P + R_1 + E + PDP - P_a^{\mathrm{T}}R_{2a}^{-1}P_a + \tau_\perp^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}P_a\tau_\perp, \tag{8.4}$$

$$0 = \left(A + [\gamma^{-2}V_{1\infty} + D][P + \hat{P}]\right)Q + Q\left(A + [\gamma^{-2}V_{1\infty} + D][P + \hat{P}]\right)^{\mathrm{T}} + V_1 - \hat{S}Q\overline{\Sigma}Q\hat{S}^{\mathrm{T}} + \tau_\perp\hat{S}Q\overline{\Sigma}Q\hat{S}^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{8.5}$$

$$0 = \left(A - \hat{S}Q\overline{\Sigma} + [\gamma^{-2}V_{1\infty} + D]P\right)^{\mathrm{T}}\hat{P} + \hat{P}\left(A - \hat{S}Q\overline{\Sigma} + [\gamma^{-2}V_{1\infty} + D]P\right) + \hat{P}\left(\gamma^{-2}[V_{1\infty} + \beta^2\hat{S}Q\overline{\Sigma}Q\hat{S}^{\mathrm{T}}] + D\right)\hat{P} + P_a^{\mathrm{T}}R_{2a}^{-1}P_a - \tau_\perp^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}P_a\tau_\perp, \tag{8.6}$$

$$0 = \left(A - BR_{2a}^{-1}P_a + [\gamma^{-2}V_{1\infty} + D]P\right)\hat{Q} + \hat{Q}\left(A - BR_{2a}^{-1}P_a + [\gamma^{-2}V_{1\infty} + D]P\right)^{\mathrm{T}} + \hat{S}Q\overline{\Sigma}Q\hat{S}^{\mathrm{T}} - \tau_\perp\hat{S}Q\overline{\Sigma}Q\hat{S}^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{8.7}$$

and let $\mathbf{P}$ *be given by* (6.16) *and* (A_c, B_c, C_c) *by*

$$A_c = \Gamma\left(A - \hat{S}Q\overline{\Sigma} - BR_{2a}^{-1}P_a + [\gamma^{-2}V_{1\infty} + D]P\right)G^{\mathrm{T}}, \tag{8.8}$$

$$B_c = \Gamma\hat{S}QC^{\mathrm{T}}V_2^{-1}, \tag{8.9}$$

$$C_c = -R_{2a}^{-1}P_aG^{\mathrm{T}}. \tag{8.10}$$

Then $(\tilde{E}, \tilde{A} + \Delta\tilde{A})$ *is detectable for all* $(\Delta A, \Delta B) \in$ U *if and only if* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $(\Delta A, \Delta B) \in$ U. *In this case, the closed-loop transfer function* $\hat{H}_{\Delta\tilde{A}}(s)$ *satisfies the* H_∞ *disturbance attenuation constraint*

$$\|\hat{H}_{\Delta\tilde{A}}(s)\|_\infty \le \gamma, \quad (\Delta A, \Delta B) \in \mathrm{U}, \tag{8.11}$$

and the worst case L_2 performance criterion (2.7) *satisfies the bound*

$$J(A_c, B_c, C_c) \le \mathrm{tr}\left[(P+\hat{P})V_1 + \hat{P}\hat{S}Q\bar{\Sigma}Q\hat{S}^{\mathrm{T}}\right]. \tag{8.12}$$

9. Numerical solution of the design equation

One of the principal motivations for the Riccati equation approach is the opportunity it provides for developing efficient computational algorithms for control design. In particular, the goal is to develop numerical methods which exploit the structure of the Riccati equations. It turns out however, that methods for solving standard Riccati equations cannot account for the additional terms which appear in the modified equations such as (5.6)–(5.9). Therefore, a new class of numerical algorithms has been developed based upon homotopic continuation methods. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path which connects the starting problem to the original problem. These ideas have been illustrated for the reduced order problem in [5] and the H_∞ constrained problem in [3] where the coupling terms preclude standard Riccati techniques. A complete description of the homotopy algorithm will appear in [6].

10. Further extensions

The results of this paper can readily be extended in several directions:

(1) Mixed bounds, i.e., letting $\Delta A = \Delta A_1 + \Delta A_2$ and bounding ΔA_1 with the linear bound and ΔA_2 with the quadratic bound (this would unify the linear and quadratic bound results).

(2) H_2 and H_∞ cross weighting terms (e.g., $x^{\mathrm{T}}R_{12}u$) as well as correlated plant disturbance and sensor noise.

(3) Nonstrictly proper plant model, i.e., (2.2) replaced by

$$y(t) = (C+\Delta C)x(t) + (D+\Delta D)u(t) + D_2w(t). \tag{10.1}$$

(4) Nonstrictly proper controller, i.e., (2.4) replaced by

$$u(t) = C_c x_c(t) + D_c y(t) \tag{10.2}$$

and the related problems of singular control weighting ($R_2 \ge 0$) and singular measurement noise ($V_2 \ge 0$).

(5) Discrete-time and sampled-data design.

11. Conclusions

The Riccati equation approach to fixed-order H_∞ constrained LQG design has been extended to account for the presence of parameter uncertainties in the state space plant model. Specifically, by embedding quadratic Lyapunov bounds within the design equations, the resulting controllers are guaranteed to provide robust stability and robust H_2/H_∞ performance over a specified range of parameter uncertainty. Two distinct bounds were considered, namely, a linear bound and a quadratic bound. Among the open problems which remain to be examined are the necessity of the design equations, the conservatism of the bounds, and the existence of solutions.

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Appendix E

OPTIMAL CONTROL APPLICATIONS & METHODS, VOL. 11, 1–20 (1990)

FINITE-DIMENSIONAL APPROXIMATION FOR OPTIMAL FIXED-ORDER COMPENSATION OF DISTRIBUTED PARAMETER SYSTEMS

DENNIS S. BERNSTEIN

Government Aerospace Systems Division, Harris Corporation, MS 22/4848, Melbourne, FL 32902, U.S.A.

I. GARY ROSEN

Department of Mathematics, University of Southern California, Los Angeles, CA 90089, U.S.A.

SUMMARY

In controlling distributed parameter systems it is often desirable to obtain low-order, finite-dimensional controllers in order to minimize real-time computational requirements. Standard approaches to this problem employ model/controller reduction techniques in conjunction with LQG theory. In this paper we consider the finite-dimensional approximation of the infinite-dimensional Bernstein/Hyland optimal projection theory. Our approach yields fixed-finite-order controllers which are optimal with respect to high-order, approximating, finite-dimensional plant models. We illustrate the technique by computing a sequence of first-order controllers for one-dimensional, single-input/single-output parabolic (heat/diffusion) and hereditary systems using a spline-based, Ritz–Galerkin, finite element approximation. Our numerical studies indicate convergence of the feedback gains with less than 2% performance degradation over full-order LQG controllers for the parabolic system and 10% degradation for the hereditary system.

KEY WORDS Finite-dimensional compensation Distributed parameter systems Optimal control

1. INTRODUCTION

Approximation methods for the optimal control of distributed parameter systems have been widely studied. In particular, the approach taken in References 1–12 involves approximating the original distributed parameter system by a sequence of finite-dimensional systems and then using finite-dimensional control design techniques to obtain a sequence of approximating, suboptimal control laws, observers or compensators. Furthermore, in these treatments it was demonstrated that if the open-loop system is approximated appropriately, then it is possible to guarantee convergence of the sequence of suboptimal controllers, observers or compensators respectively to the optimal controller, observer or compensator for the original infinite-dimensional system. In addition, it can be shown that when the approximating suboptimal control laws or estimators are applied to the original system, near-optimal performance can frequently be obtained. These ideas have been pursued in the context of both open- and closed-loop control, in both continuous and discrete time, and for both full-state-feedback control and LQG (i.e. Kalman-filter-based) state estimation and compensation.

In practical situations, however, it is often of interest to obtain the simplest (i.e. the lowest-order) controller which provides a given desired feedback performance. This is usually achieved in one of two ways: either the plant approximation order is reduced prior to controller design or reduction techniques are applied to a given high-order control law. Unfortunately, th-

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former approach may result in undesirable spillover effects while the latter may yield low-order controllers of low authority which perform unacceptably. In fact, with the second approach this may occur even when a suitable controller is known to exist. For example, as is shown in Reference 13, controller reduction techniques may even destabilize the closed-loop system.

A third, more direct approach involves fixing the controller order *a priori* and then optimizing a performance criterion over the class of fixed-order controllers. In a finite-dimensional setting a set of necessary conditions in the form of four coupled matrix equations (as a direct extension of the pair of separated Riccati equations of LQG theory) which characterize the optimal fixed-order compensator was derived in Reference 14. These four equations are coupled via an oblique projection (idempotent) matrix. In the full-order case this projection becomes the identity, thus effectively eliminating the additional two equations, and the necessary conditions reduce to the standard LQG Riccati equations.

The notion that this direct (i.e. fixed-finite-order) approach can be applied to distributed parameter systems was first suggested by Johnson[15] and further developed in References 16 and 17. To realize such an approach, however, would require a suitable generalization of the optimality conditions for the finite-dimensional fixed-order theory. This result was subsequently obtained in Reference 18, where the matrix optimal projection equations obtained in Reference 14 for finite-dimensional systems were extended to a set of four coupled *operator* Riccati and Lyapunov equations characterizing optimal fixed-finite-order controllers for infinite-dimensional systems.

In developing numerical schemes to actually compute fixed-finite-order compensators for infinite-dimensional systems, one might consider an approach wherein LQG reduction procedures are applied to a sequence of controllers obtained by using finite-dimensional full-order design techniques in conjunction with high-order finite-dimensional plant approximations. However, such an approach is unappealing for two reasons. First, since such methods are not predicated on the minimization of a performance index, prospects for convergence are slim. Secondly, controller reduction methods have not proven to be reliable in producing stabilizing compensators (see e.g. Reference 13).

Hence, as an alternative, we develop an abstract approximation framework (and ultimately computational schemes) which combines the infinite-dimensional optimal projection theory of Reference 18 with the approximation ideas developed in References 9–12 for infinite-dimensional LQG problems. More precisely, our approach involves constructing a sequence of approximating finite-dimensional subspaces of the original, underlying, infinite-dimensional Hilbert state space along with corresponding sequences of bounded linear operators which approximate the given input, output and system operators. Then, by choosing bases for these approximating subspaces and applying the finite-dimensional optimal projection theory, a sequence of matrix equations characterizing a sequence of approximating optimal fixed-finite-order compensators for the distributed system is obtained. Finally, numerical techniques for solving the matrix optimal projection equations (e.g. the homotopic continuation algorithm described in References 19 and 20) can be used to compute the sequence of approximating gains.

Our primary aim in this paper is to describe the general approach we are proposing, to discuss its implementation and to demonstrate its feasibility and practicality. We offer no convergence arguments here but rather hope to treat them in a more theoretical paper to follow. Instead we consider the application of our technique to two examples. One involves the control of a one-dimensional, single-input/single-output parabolic (heat/diffusion) system while the other involves a single-input/single-output one-dimensional hereditary control system. These relatively simple examples have been used throughout the distributed parameter control

literature to illustrate the application of new theories and techniques. A detailed discussion of the application of our ideas to more complex control systems, e.g. the vibration control of flexible structures, will also appear elsewhere. We use spline-based Ritz–Galerkin finite element schemes to approximate the open-loop systems (one for which convergence can be demonstrated in the LQG case) and present and discuss some of the numerical results which we have obtained using our general approximation framework.

We now outline the remainder of the paper. In Section 2 we briefly review the infinite-dimensional optimal projection theory from Reference 18, describe the approximation framework and derive the corresponding equivalent matrix equations and feedback gains which characterize the approximating fixed-finite-order compensator. In Section 3 we consider the examples, construct the approximation schemes and discuss our numerical findings. Section 4 contains a summary and some concluding remarks.

2. OPTIMAL PROJECTION THEORY AND FINITE-DIMENSIONAL APPROXIMATION

We consider the following fixed-finite-order dynamic compensation problem. Given the infinite-dimensional control system

$$\dot{x}(t) = Ax(t) + Bu(t) + H_1 w(t) \tag{1}$$

with measurements

$$y(t) = Cx(t) + H_2 w(t) \tag{2}$$

where $t \in [0, \infty)$, design a finite-dimensional, n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3}$$

$$u(t) = C_c x_c(t) \tag{4}$$

which minimizes the steady-state performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t \to \infty} \mathbb{E}[\langle R_1 x(t), x(t)\rangle + u(t)^{\mathrm{T}} R_2 u(t)] \tag{5}$$

For convenience we denote the infinite-dimensional plant by Π; that is,

$$\Pi \triangleq \{A, B, C, R_1, R_2, V_1, V_2\}$$

Here $x(t)$ lies in a real, separable Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle$, A: $\mathrm{Dom}(A) \subset \mathscr{H} \to \mathscr{H}$ is a closed, densely defined operator which generates a C_0 semigroup $\{T(t): t \geqslant 0\}$ of bounded linear operators on $\mathscr{H}$, $B \in \mathscr{L}(\mathbb{R}^m, \mathscr{H})$ and $C \in \mathscr{L}(\mathscr{H}, \mathbb{R}^l)$. We assume that the state and measurement are corrupted by a white noise signal $w(t)$ in a real, separable Hilbert space $\hat{\mathscr{H}}$ (see Reference 21 or 22), that $H_1 \in \mathscr{L}(\hat{\mathscr{H}}, \mathscr{H})$, $H_2 \in \mathscr{L}(\hat{\mathscr{H}}, \mathbb{R}^l)$, $R_1 \in \mathscr{L}(\mathscr{H})$ is (self-adjoint) non-negative definite and that R_2 is an $m \times m$ (symmetric) positive definite matrix. We define $V_1 = H_1 H_1^*$ and $V_2 = H_2 H_2^*$, where $(\ )^*$ denotes adjoint, and assume for convenience that $H_1 H_2^* = 0$ and that V_2 is positive definite. In addition we make the assumption that either the open-loop semigroup $\{T(t): t \geqslant 0\}$ is Hilbert–Schmidt or the operator V_1 is trace class. Recall that a linear semigroup $\{S(t): t \geqslant 0\}$ is said to be Hilbert–Schmidt if the operators $S(t)$ are Hilbert–Schmidt for $t > 0$. Note also that H_1 Hilbert–Schmidt is sufficient for V_1 to be trace class. The compensator is assumed to be of fixed finite order n_c (i.e. $x_c(t) \in \mathbb{R}^{n_c}$) and A_c, B_c and C_c are matrices of appropriate dimension. For further details and discussion on the problem statement and the above assumptions, see Reference 18.

We summarize here the primary result from Reference 18 characterizing optimal fixed-finite-order controllers. For convenience define $\Sigma \triangleq BR_2^{-1}B^*$ and $\bar{\Sigma} \triangleq C^*V_2^{-1}C$. Also let I_{n_c} and $I_{\mathcal{H}}$ denote respectively the $n_c \times n_c$ identity matrix and the identity operator on $\mathcal{H}$.

Theorem 1

Let n_c be given and suppose there exists a controllable and observable n_cth-order dynamic compensator (A_c, B_c, C_c) which minimizes J given by (5) and for which the closed-loop semigroup generated by

$$\mathcal{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \tag{6}$$

is uniformly exponentially stable. Then there exist non-negative definite operators $Q, P, \hat{Q}, \hat{P}$ on $\mathcal{H}$ such that A_c, B_c and C_c are given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^* \tag{7}$$

$$B_c = \Gamma Q C^* V_2^{-1} \tag{8}$$

$$C_c = -R_2^{-1}B^*PG^* \tag{9}$$

where $\{\mathcal{T}(t)\colon t \geq 0\}$ is the closed-loop semigroup on $\mathcal{H} \times \mathbb{R}^{n_c}$ generated by the operator $\mathcal{A}$ given by (6),

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{10}$$

$$\hat{Q}\hat{P} = G^*M\Gamma, \qquad \Gamma G^* = I_{n_c} \tag{11}$$

for some $M \in \mathbb{R}^{n_c \times n_c}$,

$$0 = AQ + QA^* + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^* \tag{12}$$

$$0 = A^*P + PA + R_1 - P\Sigma P + \tau_\perp^* P\Sigma P\tau_\perp \tag{13}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^* + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^* \tag{14}$$

$$0 = (A - Q\bar{\Sigma})^*\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^* P\Sigma P\tau_\perp \tag{15}$$

where

$$\tau \triangleq G^*\Gamma, \qquad \tau_\perp \triangleq I_{\mathcal{H}} - \tau$$

Furthermore, the resulting optimal closed-loop cost is given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \int_0^\infty \mathcal{T}(t)\,\mathcal{V}\,\mathcal{T}^*(t)\mathcal{R}\,\mathrm{d}t \tag{16}$$

where $\{\mathcal{T}(t)\colon t \geq 0\}$ is the closed-loop semigroup on $\mathcal{H} \times \mathbb{R}^n$ generated by the operator $\mathcal{A}$ given by (6)

$$\mathcal{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}$$

$$\mathcal{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}$$

It is shown in Reference 18 that t[illegible] actorization (11) for the non-negative definite operators $\hat{Q}$ and $\hat{P}$ satisfying rank $\hat{Q}\hat{P} -$ [illegible] lways exists and is unique except for a change of basis in $\mathbb{R}^n$. Also shown is that $G^*\colon \mathbb{R}^{n_c} \to \operatorname{Dom}(A)$ so that the expression (7) is well-defined.

Equations (12)–(15) are, in general, infinite-dimensional operator equations. To actually use them to compute the optimal fixed-order compensator, a finite-dimensional plant approximation is required. For each $N = 1, 2, \ldots$ let $\mathscr{X}^N$ denote a finite-dimensional subspace of $\mathscr{X}$ and let $\mathscr{P}^N : \mathscr{X} \to \mathscr{X}^N$ denote the corresponding orthogonal projection of $\mathscr{X}$ onto $\mathscr{X}^N$. Let $A^N \in \mathscr{L}(\mathscr{X}^N)$, $B^N \in \mathscr{L}(\mathbb{R}^m, \mathscr{X}^N)$, $C^N \in \mathscr{L}(\mathscr{X}^N, \mathbb{R}^l)$, $R_1^N \in \mathscr{L}(\mathscr{X}^N)$ and $V_1^N \in \mathscr{L}(\mathscr{X}^N)$. We consider the system (7)–(15) with the plant Π replaced by the plant Π^N given by

$$\Pi^N \triangleq \{A^N, B^N, C^N, R_1^N, R_2, V_1^N, V_2\}$$

Typically, the operators B^N, C^N, R_1^N and V_1^N are chosen as $B^N = \mathscr{P}^N B$, $C^N = C\mathscr{P}^N$, $R_1^N = \mathscr{P}^N R_1$ and $V_1^N = \mathscr{P}^N V_1$ with the requirement that $\mathscr{P}^N$ converge strongly to the identity $I_{\mathscr{X}}$ as $N \to \infty$. The operator A^N is chosen so that it and its adjoint satisfy the hypotheses of the Trotter–Kato semigroup approximation theorem (i.e. stability and consistency; see e.g. Reference 23). That is, A^N is chosen so that $\lim_{N\to\infty} T^N(t)\mathscr{P}^N\phi = T(t)\phi$ and $\lim_{N\to\infty} T^N(t)^*\mathscr{P}^N\phi = T(t)^*\phi$ uniformly in t for t in bounded intervals for each $\phi \in \mathscr{X}$, where $T^N(t) = \exp(tA^N)$, $t \geqslant 0$. We shall say more about these choices for A^N, B^N, C^N, R_1^N and V_1^N when we make some remarks concerning convergence questions below.

Although with the plant Π^N equations (12)–(15) are finite-dimensional, they are still operator equations. It is their *matrix* equivalents which are used in computations. Unless orthonormal bases are chosen for the subspaces $\mathscr{X}^N$ (which is typically not the case in practice), some care must be taken to obtain the appropriate matrix system.

For each $N = 1, 2, \ldots$ let $\{\phi_j^N\}_{j=1}^{k^N}$ be a basis for $\mathscr{X}^N$ and choose the standard bases for all Euclidean spaces. For a linear operator L with domain and range $\mathscr{X}^N$ or any Euclidean space, let $[L]$ denote its matrix representation with respect to the bases chosen above. Also, let Φ^N denote the k^N-square Gram matrix corresponding to the basis $\{\phi_j^N\}_{j=1}^{k^N}$; that is, $\Phi_{ij}^n = \langle \phi_i^N, \phi_j^N \rangle$, $i, j = 1, 2, \ldots, k^N$. Noting that

$$[(A^N)^*] = (\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N, \qquad [(B^N)^*] = [B^N]^{\mathrm{T}}\Phi^N, \qquad [(C^N)] = (\Phi^N)^{-1}[C^N]^{\mathrm{T}}$$
$$[(\tau_\perp^N)^*] = (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N, \qquad [\Sigma^N] = [B^N]R_2^{-1}[B^N]^{\mathrm{T}}\Phi^N,$$
$$[\bar{\Sigma}^N] = (\Phi^N)^{-1}[C^N]^{\mathrm{T}}V_2^{-1}[C^N]$$

the matrix equivalents of the operator equations (11)–(14) become

$$\begin{aligned} 0 = {} & [A^N][Q^N] + [Q^N](\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N + [V_1^N] - [Q^N][\bar{\Sigma}^N][Q^N] \\ & + [\tau_\perp^N][Q^N][\bar{\Sigma}^N][Q^N] - (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N \end{aligned} \tag{17}$$

$$\begin{aligned} 0 = {} & (\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N[P^N] + [P^N] - [A^N] + [R_1^N] - [P^N][\Sigma^N][P^N] \\ & + (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N[P^N] - [\Sigma^N][P^N][\tau_\perp^N] \end{aligned} \tag{18}$$

$$\begin{aligned} 0 = {} & ([A^N] - [\Sigma^N][P^N])[\hat{Q}^N] + [\hat{Q}^N](\Phi^N)^{-1}([A^N] - [\Sigma^N][P^N])^{\mathrm{T}}\Phi^N \\ & + [Q^N][\bar{\Sigma}^N][Q^N] - [\tau_\perp^N][Q^N][\bar{\Sigma}^N][Q^N](\Phi^N)^{-1} - [\tau_\perp^N]^{\mathrm{T}}\Phi^N \end{aligned} \tag{19}$$

$$\begin{aligned} 0 = {} & (\Phi^N)^{-1}([A^N] - [Q^N][\bar{\Sigma}^N])^{\mathrm{T}}\Phi^N[\hat{P}^N] + [\hat{P}^N]([A^N] - [Q^N][\bar{\Sigma}^N]) \\ & + [P^N][\Sigma^N][P^N] - (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N[P^N][\Sigma^N][P^N][\tau_\perp^N] \end{aligned} \tag{20}$$

Therefore, if we define the $k^N \times k^n$ non-negative definite matrices

$$\begin{aligned} Q_0^N &\triangleq [Q^N](\Phi^N)^{-1}, & P_0^N &\triangleq \Phi^N[P^N] \\ \hat{Q}_0^N &\triangleq [\hat{Q}^N](\Phi^N)^{-1}, & \hat{P}_0^N &\triangleq \Phi^N[\hat{P}^N] \\ V_0^N &\triangleq [V_1^N](\Phi^N)^{-1}, & R_0^N &\triangleq \Phi^N[R_1^N] \\ \Sigma_0^N &\triangleq [B^N]R_2^{-1}[B^N]^{\mathrm{T}}, & \bar{\Sigma}_0^N &\triangleq [C^N]^{\mathrm{T}}V_2^{-1}[C^N] \end{aligned}$$

we can solve the *matrix* optimal projection equations given in Reference 14 corresponding to the matrix plant model

$$\Pi_0^N \triangleq \{[A^N], [B^N], [C^N], R_0^N, R_2, V_0^N, V_2\}$$

to obtain the matrices $Q_0^N, P_0^N, \hat{Q}_0^N$ and $\hat{P}_0^N$. The approximating optimal n_cth-order dynamic compensator $\{A_c^N, B_c^N, C_c^N\}$ is then given by

$$A_c^N = \Gamma_0^N([A^N] - Q_0^N \bar{\Sigma}_0^N - \Sigma_0^N P_0^N)(G_0^N)^{\mathrm{T}}$$
$$B_c^N = \Gamma_0^N Q_0^N [C^N]^{\mathrm{T}} V_2^{-1}$$
$$C_c^N = -R_2^{-1}[B^N]^{\mathrm{T}} P_0^N (G_0^N)^{\mathrm{T}}$$

where $\Gamma_0^N, G_0^N \in \mathbb{R}^{n_c \times k^N}$ and $M^N \in \mathbb{R}^{n_c \times n_c}$ satisfy

$$\hat{Q}_0^N \hat{P}_0^N = G_0^N M^N \Gamma_0^N \qquad \Gamma_0^N (G_0^N)^{\mathrm{T}} = I_{n_c}$$
$$[\tau^N] = (G_0^N)^{\mathrm{T}} \Gamma_0^N, \qquad [\tau_\perp^N] = I_{k^N} - [\tau^N]$$

When an infinite-dimensional controller will suffice, $C_c = -R_2^{-1} B^* P \in \mathscr{L}(\mathscr{X}, \mathbb{R}^m)$ and $B_c = QC^* V_2^{-1} \in \mathscr{L}(\mathbb{R}^l, \mathscr{X})$ are the usual infinite-dimensional LQG controller and observer gains.[9] The operators $P, Q \in \mathscr{L}(\mathscr{X})$ are the non-negative definite solutions to the two decoupled operator algebraic Riccati equations (12) and (13) with τ and $\tau_\perp$ formally set to $I_{\mathscr{X}}$ and 0 respectively. Since C_c has range in $\mathbb{R}^m$ and B_c has domain $\mathbb{R}^l$, there exist vectors $c_c = (c_c^1, \ldots, c_c^m)^{\mathrm{T}} \in \times_{j=1}^m \mathscr{X}$ and $b_c = (b_c^1, \ldots, b_c^l)^{\mathrm{T}} \in \times_{j=1}^l \mathscr{X}$ such that

$$[C_c x]_i = \langle c_c^i, x \rangle, \quad i = 1, 2, \ldots, m \quad x \in \mathscr{X}$$
$$B_c y = b_c^{\mathrm{T}} y = \sum_{i=1}^{l} y_i b_c^i, \quad y = (y_1, \ldots, y_l) \in \mathbb{R}^l$$

The vectors c_c and b_c are referred to as the optimal LQG functional controller and observer gains respectively.

With regard to approximation for the full order LQG problem, for each $N = 1, 2, \ldots$ we take $n_c = k^N$. Then it is not difficult to show that

$$C_c^N [P^N x] = \langle c_c^N, x \rangle, \quad x \in \mathscr{X}$$
$$B_c^n y = (b_c^N)^{\mathrm{T}} y, \quad y \in \mathbb{R}^l$$

where $c_c^N \in \times_{j=1}^m \mathscr{X}^N$ and $b_c^N \in \times_{j=1}^l \mathscr{X}^N$ are given by $c_c^N = C_c^N (\Phi^N)^{-1} \phi^N$ and $b_c^N = (B_c^N)^{\mathrm{T}} \phi^N$ respectively with $\phi^n = (\phi_1^N, \ldots, \phi_{k^N}^N) \in \times_{j=1}^{k^N} \mathscr{X}^N$. The vectors c_c^N and b_c^N are referred to as the approximating optimal LQG functional controller and observer gains respectively. To compute them we need only solve two standard decoupled matrix algebraic Riccati equations for the $k^N \times k^N$ non-negative definite matrices Q_0^N and P_0^N.

A rather complete convergence theory for LQG approximation can be found in Reference 9. Essentially, it is shown there that if the approximating subspaces $\mathscr{X}^N$ are chosen so that the projections $\mathscr{P}^N$ converge strongly to the identity as $N \to \infty$, the operators A^N, B^N, C^N, R_1^N and V_1^N are chosen as was described above and the operators Q^N and P^N are uniformly bounded in N, then Q^N and P^N converge weakly to Q and P respectively as $N \to \infty$. This in turn implies that $C_c^N \to C_c$ strongly, $B_c^N \to B_c$ weakly, $c_c^N \to c_c$ and $b_c^N \to b_c$ weakly and the closed-loop semigroup for the approximating optimal LQG compensator converges weakly to the closed-loop semigroup for the optimal infinite-dimensional LQG compensator as $N \to \infty$. If, in addition, the operators $S^N(t) = T^N(t) + B^N C_c^N$ and $\hat{S}^N(t) = T^N(t) - B_c^N C^N$ are uniformly exponentially stable, uniformly in N, then $Q^N \to Q$ and $P^N \to P$ strongly, $C_c^N \to C_c$ and $B_c^N \to B_c$ in norm, $c_c^N \to c_c$ and $b_c^n \to b_c$ strongly and the closed-loop semigroups converge

strongly as $N \to \infty$. If R_1^N and V_1^N are coercive, uniformly in N, then $S^N(t)$ and $\hat{S}^N(t)$ will be uniformly exponentially stable. If it is also true that R_1 and V_1 are trace class and $R_1^N \mathscr{P}^N \to R_1$ and $V_1^N \mathscr{P}^N \to V_1$ in trace norm, then Q and P are trace class and $Q^N \mathscr{P}^N \to Q$ and $P^N \mathscr{P}^N \to P$ in trace norm as $N \to \infty$. The development of a complete convergence theory for the approximating fixed-order designs appears to be a much more difficult question. One would expect, however, that any such theory would require at least minimally that the sufficient conditions for convergence of the approximating LQG designs be satisfied.

Returning to the fixed-finite-order case we note that in general the approximating optimal projection equations may not possess a unique solution. However, Richter[19] shows for the finite-dimensional case that it is possible to obtain an upper bound for the number of stabilizing solutions. He uses topological degree theory to obtain the following result.

Theorem 2

Consider equations (12)–(15) with the infinite-dimensional plant Π replaced by the finite-dimensional plant Π^N. Let n_u denote the dimension of the unstable subspace of A^N and assume that $n_c \geqslant n_u$. Then in the class of non-negative definite operators $Q^N, P^N, \hat{Q}^N, \hat{P}^N$ on $\mathscr{X}^N$ satisfying rank $\hat{Q}^N =$ rank $\hat{P}^N =$ rank $\hat{Q}^N\hat{P}^N = n_c$ there exist at most

$$\begin{cases} \begin{pmatrix} \min(k^N, m, l) - n_u \\ n_c - n_u \end{pmatrix}, & n_c \leqslant \min(k^N, m, l) \\ 1, & \text{otherwise} \end{cases}$$

solutions of (12)–(15), each of which is stabilizing. If, in addition, the plant (A^N, B^N, C^N) is stabilizable by an n_cth-order controller, then there exists at least one stabilizing solution of (10)–(15).

Theorem 2 shows that while there may exist multiple solutions to the finite-dimensional optimal projection equations, in practice this number can be quite small. For example, if $n_c \geqslant n_u$ and the system is either single-input ($m = 1$) or single-output ($l = 1$), then there exists at most one solution to (10)–(15) for the plant Π^N. Moreover, the number of solutions of the approximating optimal projection equations remains bounded in N, since for N sufficiently large, $\min(k^N, m, l) = \min(m, l)$. The existence of at least one stabilizing solution of course depends upon whether or not the plant is stabilizable by an n_cth-order controller (for relevant results, see Reference 24). Finally, while it may be possible to stabilize a plant with $n_c < n_u$, this case lies outside the scope of the analysis given in Reference 19.

3. EXAMPLES AND NUMERICAL RESULTS

We first consider the one-dimensional, single-input/single-output parabolic (heat/diffusion) control system with Dirichlet boundary conditions given by

$$\frac{\partial v}{\partial t}(t,\eta) = a\frac{\partial^2 v}{\partial \eta^2}(t,\eta) + b(\eta)u(t) + h_1(\eta)w_1(t,\eta), \quad 0 < \eta < 1, \quad t > 0 \tag{21}$$

$$v(t,0) = 0, \qquad v(t,1) = 0, \quad t > 0 \tag{32}$$

$$y(t) = \int_0^1 c(\eta)v(t,\eta)\,\mathrm{d}\eta + h_2 w_2(t), \quad t > 0 \tag{23}$$

where $a > 0$, and $b(\cdot)$ and $c(\cdot)$ are given by

$$b(\eta) = \begin{cases} 1/(\beta_2 - \beta_1), & \beta_1 \leqslant \eta \leqslant \beta_2 \\ 0, & \text{elsewhere} \end{cases}$$

$$c(\eta) = \begin{cases} 1/(\gamma_2 - \gamma_1), & \gamma_1 \leqslant \eta \leqslant \gamma_2 \\ 0, & \text{elsewhere} \end{cases}$$

with $0 \leqslant \beta_1 < \beta_2 \leqslant 1$ and $0 \leqslant \gamma_1 < \gamma_2 \leqslant 1$. In (21) and (23), $h(\cdot) \in L_\infty(0,1)$, $w_1(t,\cdot) \in L_2(0,1)$, almost all $t \in [0,\infty)$ (see Reference 23, p. 314), h_2 is a non-zero constant and $w_2(\cdot)$ is unit-intensity white noise.

To rewrite (21)–(23) in the form (1), (2), in the usual way we take $\mathscr{X} = L_2(0,1)$ endowed with the standard L_2 inner product, let $x(t) = v(t,\cdot)$, $t \geqslant 0$, define A: $\mathrm{Dom}(A) \subset \mathscr{X} \to \mathscr{X}$ by $A\phi = aD^2\phi$ for $\phi \in \mathrm{Dom}A \triangleq H^2(0,1) \cap H_0^1(0,1)$ and define $B \in \mathscr{L}(\mathbb{R}^1, \mathscr{X})$ and $C \in \mathscr{L}(\mathscr{X}, \mathbb{R}^1)$ by $Bu = b(\cdot)u$ for $u \in \mathbb{R}^1$ and $C\phi = \int_0^1 c(\eta)\phi(\eta)\,\mathrm{d}\eta$ for $\phi \in L_2(0,1)$ respectively. Furthermore, let $\hat{\mathscr{X}} \triangleq L_2(0,1) \times \mathbb{R}$, set $w(t) \triangleq (w_1(t,\cdot), w_2(t)) \in \hat{\mathscr{X}}$ and define $H_1 \in \mathscr{L}(\hat{\mathscr{X}}, \mathscr{X})$ and $H_2 \in \mathscr{L}(\hat{\mathscr{X}}, \mathbb{R}^1)$ by $H_1 z = h_1(\cdot)z_1$ and $H_2 z = h_2 z_2$ for $z = (z_1, z_2) \in \hat{\mathscr{X}}$ respectively.

It is well known (see e.g. Reference 23) that A is closed, densely defined and negative definite. Furthermore, A is the infinitesimal generator of a uniformly exponentially stable, analytic (abstract parabolic) Hilbert–Schmidt semigroup $\{T(t)\colon t \geqslant 0\}$ of bounded, self-adjoint linear operators on $\mathscr{X}$.

We consider a linear spline-based Ritz–Galerkin approximation for the open-loop system. For each $N = 2,3,\ldots$ let $\{\phi_j^N\}_{j=1}^{N-1}$ be the linear spline ('hat') functions defined on the interval $[0,1]$ with respect to the uniform partition $\{0, 1/N, 2/N, \ldots, 1\}$, i.e.

$$\phi_j^N(\eta) = \begin{cases} N\eta - j + 1, & \eta \in [(j-1)/N, j/N) \\ j + 1 - N\eta, & \eta \in [j/N, (j+1)/N) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$$

$j = 1,2,\ldots,N-1$. Set $\mathscr{X}^N = \mathrm{span}\{\phi_j^N\}_{j=1}^{N-1}$ and note that $k^N = \dim \mathscr{X}^N = N-1$ and $\mathscr{X}^N \subset H_0^1(0,1)$ for all N. If $\mathscr{P}^N$: $\mathscr{X} \to \mathscr{X}^N$ denotes the orthogonal projection of $\mathscr{X}$ onto $\mathscr{X}^N$, then standard convergence estimates for interpolatory splines[25] can be used to show that $\lim_{N\to\infty} \mathscr{P}^N\phi = \phi$ in $L_2(0,1)$ for $\phi \in L_2(0,1)$.

There are two equivalent ways to obtain an operator representation for the usual Ritz–Galerkin approximation to A. First, A can be extended to a bounded linear operator from $H_0^1(0,1)$ onto its dual, $H^{-1}(0,1)$, via

$$(A\phi)(\psi) = -a\langle D\phi, D\psi\rangle, \quad \phi, \psi \in H_0^1(0,1) \tag{24}$$

Since $\mathscr{X}^N \subset H_0^1(0,1)$ for all $N = 2,3,\ldots$, we define $A^N \in \mathscr{L}(\mathscr{X}^N)$ by $A^N\phi^N = A\phi^N$, $\phi^N \in \mathscr{X}^N$, with $A\phi^N \in H^{-1}(0,1)$ considered to be linear functional on $\mathscr{X}^N$. From the Riesz representation theorem we obtain $A^N\phi^N = \psi^N$ where ψ^N is that element in $\mathscr{X}^N$ which satisfies $(A^N\phi^N)(\chi^N) = -a\langle D\phi^N, D\chi^N\rangle = \langle \psi^N, \chi^N\rangle$.

Alternatively and equivalently, by using the fact that A is self-adjoint, we can define A^N as follows. Let $\mathscr{P}_1^N$: $H_0^1(0,1) \to \mathscr{X}^N$ denote the orthogonal projection of the Hilbert space $H_0^1(0,1)$ onto $\mathscr{X}^N$. Using the definition (24) it is not difficult to show that $-A \in \mathscr{L}(H_0^1(0,1), H^{-1}(0,1))$ is coercive and therefore that A^{-1}: $H^{-1}(0,1) \to H_0^1(0,1)$ exists and is bounded. We then define $A^N \in \mathscr{L}(\mathscr{X}^N)$ to be the inverse of the operator given by $(A^N)^{-1} = \mathscr{P}_1^N A^{-1}|_{\mathscr{X}^N}$.

Using either definition it is easily argued that A^N is well-defined, self-adjoint and is the infinitesimal generator of a uniformly exponentially stable (uniformly in N) semigroup,

$T^N(t) = \exp(tA^N)$, $t \geq 0$, of bounded linear operators on $\mathscr{H}^N$. Also, using the approximation properties of splines it is not difficult to show that $\lim_{N\to\infty}(A^N)^{-1}\mathscr{P}^N\phi = A^{-1}\phi$, $\phi \in \mathscr{H}$. Consequently, the hypotheses of the Trotter–Kato theorem[23] are satisfied and we have $\lim_{N\to\infty} T^N(t)\mathscr{P}^N\phi = T(t)\phi$ and $\lim_{N\to\infty} T^N(t)^*\mathscr{P}^N\phi = T(t)^*\phi$, $\phi \in \mathscr{H}$, uniformly in t for t in bounded intervals. A detailed discussion of the results just outlined can be found in Reference 8.

We define $B^N = \mathscr{P}^N B$ and $C^N = C\mathscr{P}^N$, from which it immediately follows that $\lim_{N\to\infty} B^N = B$ and $\lim_{N\to\infty} C^N = C$ in norm and similarly for their adjoints. For the example we shall consider here, we have chosen $R_1 = r_1 I_{\mathscr{H}}$ and $R_2 = r_2 I_m$, with $r_1, r_2 > 0$. Setting $h_1(\eta) = v_1^{1/2}$, $0 < \eta < 1$, and $h_2 = v_2^{1/2}$, with $v_1, v_2 > 0$, we obtain $V_1 = v_1 I_{\mathscr{H}}$ and $V_2 = v_2$. We then take $R_1^N = \mathscr{P}^N R_1$ and $V_1^N = \mathscr{P}^N V_1$. For the LQG problem the open-loop uniform exponential stability of both the infinite-dimensional system and the approximating systems is sufficient to conclude the strong convergence of the approximating Riccati operators to the unique solutions of the infinite-dimensional Riccati equations, the uniform norm convergence of the approximating controller and observer gains and the strong convergence of the functional gains as $N \to \infty$.[2,7,9]

Since the basis elements $\{\phi_j^N\}_{j=1}^{N-1}$ are piecewise linear with respect to the uniform mesh $\{0, 1/N, 2/N, \ldots, 1\}$ on $[0, 1]$, the equivalent matrix representations for the operators defined above can be computed directly and in closed form. The Gram matrix $\Phi_{ij}^N = \langle \phi_i^N, \phi_j^N \rangle$, $i, j = 1, 2, \ldots, N-1$, is given by $\Phi^N = (1/N)\mathrm{Tridiag}\{\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\}$, and if we define the generalized stiffness matrix Ψ^N by $\Psi_{ij}^N = -a\langle D\phi_i^N, D\phi_j^N \rangle$, $i, j = 1, 2, \ldots, N-1$, then $\Psi^N = aN\,\mathrm{Tridiag}\{1, -2, 1\}$. It follows that $[A^N] = (\Phi^N)^{-1}\Psi^N$, $[B^N] = (\Phi^N)^{-1}b^N$ and $[C^N] = c^N$, with $b_i^N = \langle b, \phi_i^N \rangle = [1/(\beta_2 - \beta_1)] \int_{\beta_1}^{\beta_2} \phi_i^N(\eta)\,d\eta$ and $c_i^N = \langle c, \phi_i^N \rangle = [1/(\gamma_2 - \gamma_1)] \int_{\gamma_1}^{\gamma_2} \phi_i^N(\eta)\,d\eta$, $i = 1, 2, \ldots, N-1$, and that $R_0^N = r_1 \Phi^N$ and $V_0^N = v_1(\Phi^N)^{-1}$.

For our numerical study we set $a = 1, \beta_1 = 0{\cdot}75 - 0{\cdot}03\sqrt{2}$, $\beta_2 = 0{\cdot}75 + 0{\cdot}04\sqrt{2}$, $\gamma_1 = 0{\cdot}25 - 0{\cdot}04\sqrt{2}$, $\gamma_2 = 0{\cdot}25 + 0{\cdot}03\sqrt{2}$, $r_1 = v_1 = 1$, $r_2 = v_2 = 10^{-4}$ and $h_1(\eta) \equiv 1$, and used our technique to compute approximating optimal LQG (i.e. $n_c = N - 1$) and first-order (i.e. $n_c = 1$) compensators for various values of N. The open-loop stability of system (21)–(23) and the approximating systems implies that the finite-dimensional approximating optimal projection equations have a solution. Theorem 2, on the other hand, with $n_u = 0$ and $n_c = 1$ or $n_c = N - 1$, implies that they have at most one solution. Consequently, the system of equations (12)–(15) with the plants Π^N admits a unique solution.

The optimal projection equations (12)–(15) were solved by using the homotopic continuation algorithm described in Reference 19. There it is shown that the operation count for the algorithm is proportional to $p(2n^3 + (m + l)n^2 + (m + l)^3 n_c^3)$, where p is the number of integration steps and n is the dimension of the finite-dimensional plant. This count is competitive with the operation count for the Hamiltonian solution of the standard Riccati equations, which is $O(16n^3)$ for LQG. Also, note that the computational burden for the solution of the optimal projection equations decreases with n_c.

Since $m = l = 1$ in the LQG case, the optimal functional observer and feedback control gains b_c and c_c and the approximating gains b_c^N and c_c^N are all simply L_2 functions with b_c^N and c_c^N elements in $\mathscr{H}^N$. We plot the functions b_c^N and c_c^N we obtained for various values of N in Figures 1 and 2 respectively. The apparent symmetry of the plots given in Figures 1 and 2 is a result of the nearly symmetric placement of the sensor and actuator (i.e. the choice of $\beta_1, \beta_2, \gamma_1$ and γ_2) in this particular example. That convergence is indeed achieved can immediately be observed in the figures. In the fixed-order case with $n_c = 1$, the compensator gains A_c, B_c and C_c are all scalars. Also, for a first-order controller there are only two independent parameters, A_c and $B_c C_c$. In Table 1 we give the values we obtained for A_c^N and

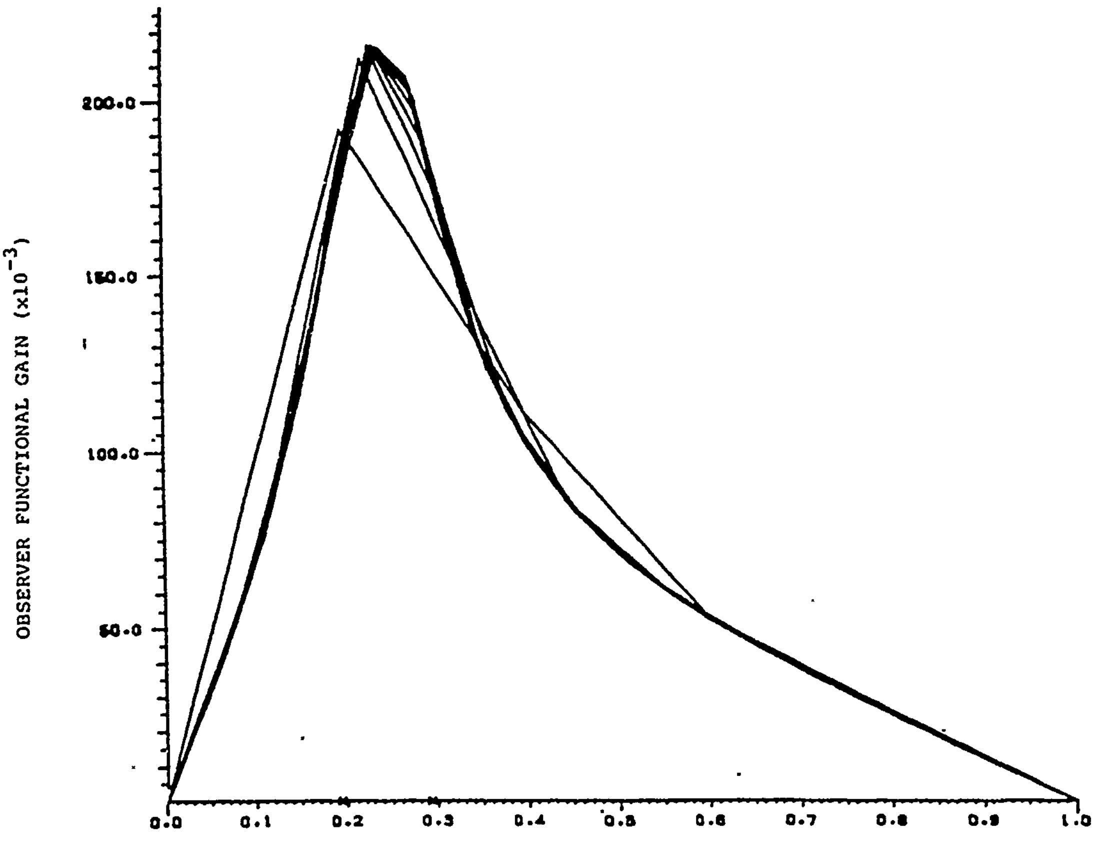


Figure 1. Parabolic system approximating optimal LQG functional observer gains, $N = 4, 8, 12, 16, 20, 24, 28, 32$

$B_c^N C_c^N$ for various values of N. Once again it is clear that the gains are converging as N increases. In addition, in Table 1 we provide the closed-loop costs J_{LQG}^N and J_{FO}^N computed via formula (16) for the LQG and first-order controllers. These closed-loop costs are evaluated using a 64th-order modal approximation to the infinite-dimensional system. For all values of N the performance of the fixed-order compensator was within 2% of the corresponding LQG controller. Thus, for example, the replacement of an approximating optimal LQG controller of any desired order by an approximating optimal first-order controller can yield considerable implementation simplification with only minor performance degradation. Note that for the example we consider here it is possible to compute the open-loop cost for the infinite-dimensional system in closed form. We have

$$\begin{aligned} J_{\mathrm{OL}} &= \operatorname{tr} \int_0^\infty V_1 T^*(t) R_1 T(t)\, \mathrm{d}t = v_1 r_1 \operatorname{tr} \int_0^\infty T(t)^2\, \mathrm{d}t \\ &= v_1 r_1 \sum_{n=1}^\infty \int_0^\infty \exp(-2n^2\pi^2 a t)\, \mathrm{d}t = \frac{v_1 r_1}{2\pi^2 a} \sum_{n=1}^\infty \frac{1}{n^2} \\ &= \frac{v_1 r_1}{12a} = \frac{1}{12} \approx 0.08333 \end{aligned}$$

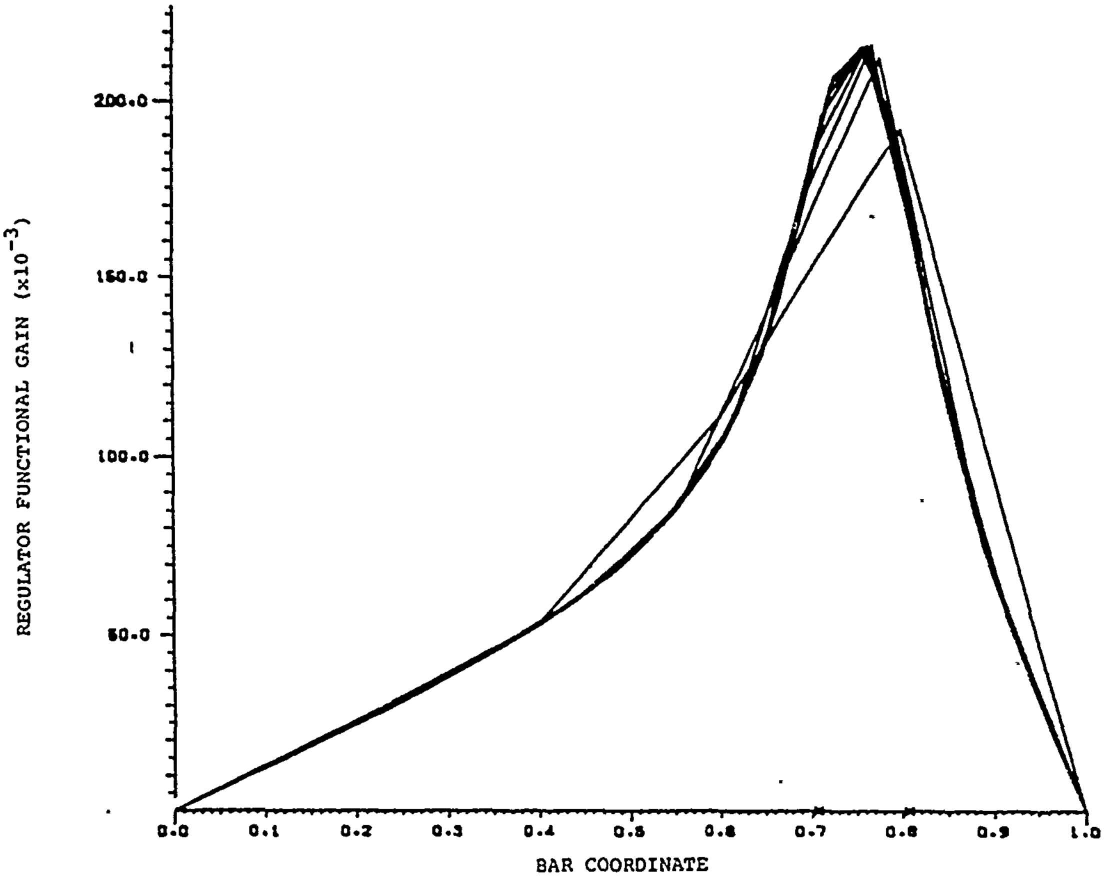


Figure 2. Parabolic system approximating optimal LQG functional control gains; $N = 4, 8, 12, 16, 20, 24, 28, 32$

Table I. Parabolic system approximating optimal first-order compensator gains

N	A_c^N	$B_c^N C_c^N$	J_{LQG}^N	J_{FO}^N
4	−687·6	5470	0·06999	0·07014
8	−720·9	5231	0·06870	0·06993
12	−730·9	5182	0·06872	0·06991
16	−734·3	5145	0·06874	0·06990
20	−738·0	5127	0·06875	0·06990
24	−737·6	5108	0·06876	0·06990
28	−739·8	5109	0·06876	0·06990
32	−738·7	5099	0·06877	0·06990

Finally, for comparison purposes, we tried applying balancing techniques to the LQG controllers to reduce their order. However, with $n_c = 1$, such controllers were found to be destabilizing. On the basis of the results in Reference 13, this was not unexpected. Furthermore, a first-order controller based upon a truncated model consisting of the first mode only was found to yield an unstable closed-loop system for the 64th-order truth model.

As a second example we consider the one-dimensional, single-input/single-output hereditary control system given by

$$\dot{v}(t) = a_0 v(t) + a_1 v(t-\rho) + b_0 u(t) + h_1 w(t), \quad t > 0 \tag{25}$$

$$y(t) = c_0 v(t) + h_2 w(t), \quad t > 0 \tag{26}$$

where $a_0, a_1, b_0, c_0, h_1, h_2, \rho \in \mathbb{R}^1$ with $\rho > 0$, $h_2 \neq 0$, and w is a unit-intensity white noise process. To rewrite (25), (26) in the form (1), (2), we take $\mathscr{X} = \mathbb{R}^2 \times L_2(-\rho, 0)$ endowed with the usual product space inner product, $\langle(\eta, \phi), (\xi, \psi)\rangle = \eta\xi + \int_{-\rho}^{0} \phi\psi$, and let $x(t) = (v(t), v_t)$, $t \geqslant 0$, where for $t \geqslant 0$, $v_t \in L_2(-\rho, 0)$ is given by $v_t(\theta) = v(t+\theta)$, $-\rho \leqslant \theta \leqslant 0$. Define A: $\mathrm{Dom}(A) \subset \mathscr{X} \to \mathscr{X}$ by $A\hat{\phi} = (a_0\phi(0) + a_1\phi(-\rho), D\phi)$ for $\hat{\phi} = (\phi(0), \phi) \in \mathrm{Dom}(A)$ $\triangleq \{(\xi, \psi) \in \mathscr{X}: \psi \in H^1(-\rho, 0), \psi(0) = \xi\}$ and let $B \in \mathscr{L}(\mathbb{R}^1, \mathscr{X})$ and $C \in \mathscr{L}(\mathscr{X}, \mathbb{R}^1)$ be given by $Bu = (b_0 u, 0)$ and $C(\eta, \phi) = c_0\eta$ respectively. Let $\hat{\mathscr{X}} = \mathbb{R}^1$ and define $H_1 \in \mathscr{L}(\hat{\mathscr{X}}, \mathscr{X})$ and $H_2 \in \mathscr{L}(\hat{\mathscr{X}}, \mathbb{R}^1)$ by $H_1 z = (h_1 z, 0)$ and $H_2 z = h_2 z$ respectively for $z \in \mathbb{R}^1$.

The operator A is densely defined and is the infinitesimal generator of a C_0 semigroup $\{T(t): t \geqslant 0\}$ of bounded linear operators on $\mathscr{X}$ with $T(t)(\eta, \phi) = (v(t; \eta, \phi), v_t(\eta, \phi))$, $t \geqslant 0$, where $v(\cdot; \eta, \phi)$ is the unique solution to (25) with $b_0 = h_1 = 0$ and initial conditions $v(0) = \eta$, $v_0 = \phi$. We take $R_1 \in \mathscr{L}(\mathscr{X})$ and $R_2 \in \mathscr{L}(\mathbb{R}^1)$ to be $R_1(\eta, \phi) = (r_1\eta, 0)$ and $R_2 u = r_2 u$ respectively with $r_1, r_2 > 0$. The definitions of H_1 and H_2 given above imply that $V_1 \in \mathscr{L}(\mathscr{X})$ and $V_2 \in \mathscr{L}(\mathbb{R}^1)$ are given by $V_1(\eta, \phi) = (h_1^2\eta, 0)$ and $V_2 z = h_2^2 z$ respectively for $(\eta, \phi) \in \mathscr{X}$ and $z \in \mathbb{R}^1$. Although in this example the open-loop semigroup $\{T(t): t \geqslant 0\}$ is not compact, the operator H_1 is of finite rank and therefore Hilbert–Schmidt. The operator V_1 is thus trace class.

We employ an appropriate scheme recently proposed by Ito and Kappel.[26] We briefly outline it here; a more detailed discussion can be found in Reference.26. For each $N = 1, 2, \dots$ let $\chi_j^N \in L_2(-\rho, 0)$ denote the characteristic function for the interval $[-j\rho/N, -(j-1)\rho/N)$, $j = 1, 2, \dots, N$, and let $\mathscr{X}^N$ be the $(N+1)$-dimensional subspace of $\mathscr{X}$ defined by

$$\mathscr{X}^N = \mathrm{span}\{(1, 0), (0, \chi_1^N), \dots, (0, \chi_N^N)\}$$

Let $\mathscr{P}^N: \mathscr{X} \to \mathscr{X}^N$ denote the orthogonal projection of $\mathscr{X}$ onto $\mathscr{X}^N$. Let $\{\phi_j^N\}_{j=0}^N$ denote the linear B-spline functions defined on the interval $[-\rho, 0]$ with respect to the uniform mesh $\{-\rho, \dots, -\rho/N, 0\}$ and set $\mathscr{X}_1^N = \mathrm{span}\{(\phi_j^N(0), \phi_j^N)\}_{j=0}^N$. Then $\mathscr{X}_1^N$ is an $(N+1)$-dimensional subspace of $\mathrm{Dom}(A)$ and is not difficult to demonstrate that the restriction of $\mathscr{P}^N$ to $\mathscr{X}_1^N$ is a bijection into $\mathscr{X}^N$. Using the fact that A restricted to $\mathscr{X}_1^N$ has range in $\mathscr{X}^N$, we define $A^N \in \mathscr{L}(\mathscr{X}^N)$ by $A^N = A(\mathscr{P}^N)^{-1}$ and set $T^N(t) = \exp(A^N t)$, $t \geqslant 0$. Noting that $R(B) \subset \mathscr{X}^N$, we take $B^N \in \mathscr{L}(\mathbb{R}^1, \mathscr{X}^N)$ to be given by $B^N = B$. Similarly, we take $R_1^N = R_1$ and $V_1^N = V_1$. We set $C^N = C$.

It is shown in Reference 26 that $\mathscr{P}^N(\eta, \phi) \to (\eta, \phi)$, $T^N(t)\mathscr{P}^N(\eta, \phi) \to T(t)(\eta, \phi)$ and $T^N(t)^*\mathscr{P}^N(\eta, \phi) \to T(t)^*(\eta, \phi)$ for $(\eta, \phi) \in \mathscr{X}$ as $N \to \infty$, uniformly in t, for t in bounded subsets of $[0, \infty)$. It then follows that $\lim_{N\to\infty} B^N = B$ and $\lim_{N\to\infty} C^N\mathscr{P}^N = C$ in norm.

For the LQG (full-order) problem the optimal functional observer and feedback control gains b_c and c_c are of the form $b_c = (\beta_0, \beta_1)$ and $c_c = (\gamma_0, \gamma_1)$ respectively with $\beta_0, \gamma_0 \in \mathbb{R}^1$ and $\beta_1, \gamma_1 \in L_2(-\rho, 0)$. The approximating gains are of the form $b_c^N = (\beta_0^N, \beta_1^N)$ and $c_c^N = (\gamma_0^N, \gamma_1^N)$ with $\beta_0^N, \gamma_0^N \in \mathbb{R}^1$ and $\beta_1^N, \gamma_1^N \in \mathrm{span}\{\chi_j^N\}_{j=1}^N$. Since we are treating a one-dimensional example, if $b_0 \neq 0$, the theory in Reference 26 implies that $\beta_0^N \to \beta_0$ and $\gamma_0^N \to \gamma_0$ in $\mathbb{R}^1$ and that $\beta_1^N \to \beta_1$ and $\gamma_1^N \to \gamma_1$ in $L_2(-\rho, 0)$ as $N \to \infty$.

Once again, as in the first example, matrix representations for the operators A^N, B^N, C^N, R_1^N and V_1^N are not difficult to compute in closed form. Indeed, the $(N+1) \times (N+1)$ matrix

representation for the bijection $\mathscr{P}^N\colon \mathscr{Z}_1^N \to \mathscr{Z}_1$ is given by

$$[\mathscr{P}^N] = \begin{bmatrix} 1 & 0 & & & & & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & & & \\ & & & \ddots & & & \\ & & & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ & 0 & & & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Then $[A^N] = [K^N][\mathscr{P}^N]^{-1}$ where

$$[K^N] = \begin{bmatrix} a_0 & 0 & & & 0 & & a_1 \\ N/\rho & -N/\rho & 0 & & & & \\ 0 & N/\rho & -N/\rho & 0 & & & \\ & & & \ddots & & & \\ & & & 0 & N/\rho & -N/\rho & 0 \\ & 0 & & & 0 & N/\rho & -N/\rho \end{bmatrix}$$

We have the $(N+1)\times 1$ matrix $[B^N] = [b_0\ 0 \ldots 0]^{\mathrm{T}}$ and the $1\times(N+1)$ matrix $[C^N] = [c_0\ 0 \ldots 0]$, while $[R_1^N] = r_1[\mathscr{M}^N]$ and $[V_1^N] = h_1^2[\mathscr{M}^N]$ where the $(N+1)\times(N+1)$ matrix $[\mathscr{M}^N]$ is given by

$$[\mathscr{M}^N] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

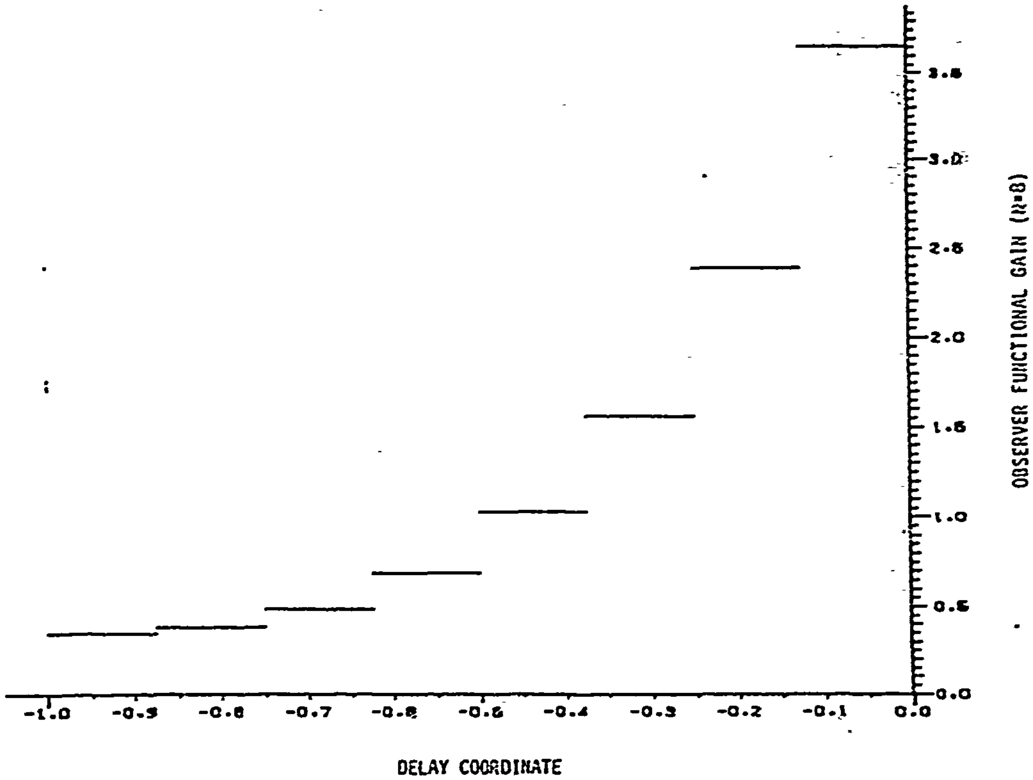
We set $a_0 = a_1 = b_0 = c_0 = r_1 = h_1 = \rho = 1$, $r_2 = 0\cdot 1$ and $h_2 = \sqrt{(0\cdot 1)}$ and computed approximating optimal LQG (i.e. $n_c = N+1$) and first-order (i.e. $n_c = 1$) compensators for $N = 8, 16, 24$ and 32. The optimal LQG observer gains are given in Table 3 and Figure 3; the control gains are given in Table 4 and Figure 4. The symmetry in the observer and control gains

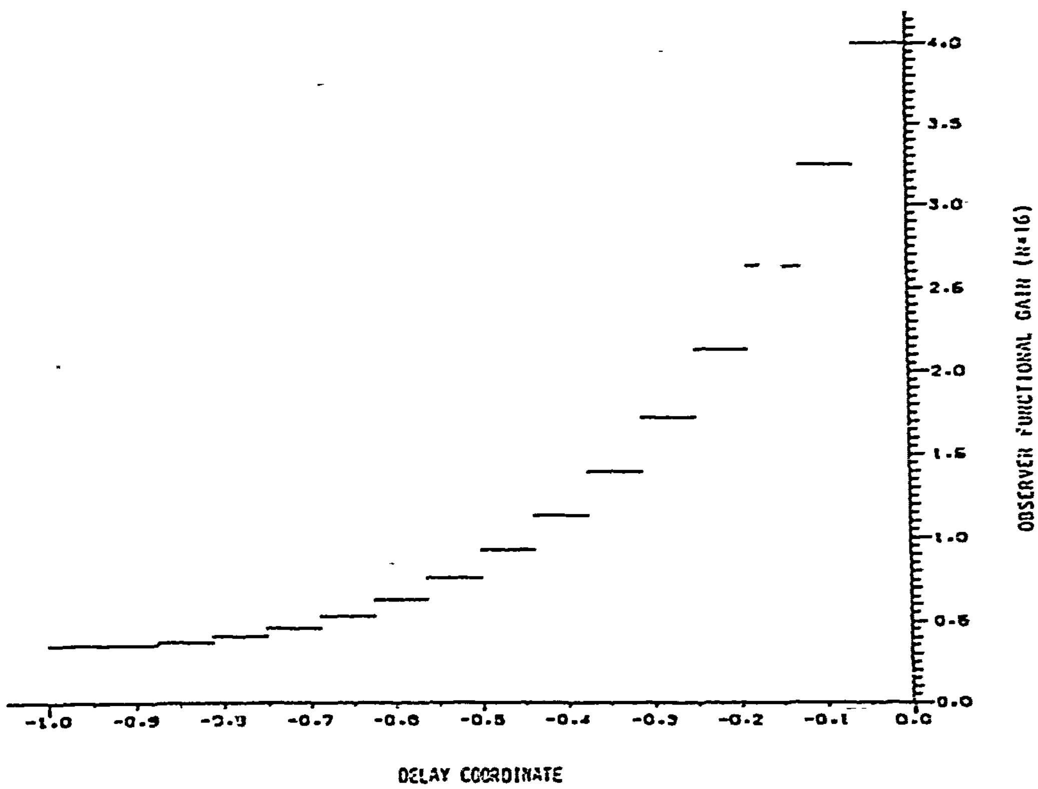
Table II. Hereditary system open-loop poles

$1\cdot 278465$
$-1\cdot 588317 \pm 4\cdot 155305i$
$-2\cdot 417631 \pm 10\cdot 68603i$
$-2\cdot 861502 \pm 17\cdot 05611i$
$-3\cdot 167754 \pm 23\cdot 38558i$
$-3\cdot 401945 \pm 29\cdot 69798i$
$-3\cdot 591627 \pm 36\cdot 00146i$
$-3\cdot 751047 \pm 42\cdot 29965i$
$-3\cdot 888543 \pm 48\cdot 59442i$
$-4\cdot 009422 \pm 54\cdot 88686i$
$-4\cdot 117267 \pm 61\cdot 17761i$
$-4\cdot 214618 \pm 67\cdot 46710i$

Table III. Hereditary system approximating optimal LQG scalar observer gains

N	8	16	24	32
β_0^N	$4\cdot 4213$	$4\cdot 4229$	$4\cdot 4233$	$4\cdot 4234$


OBSERVER FUNCTIONAL GAIN (N=8)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
DELAY COORDINATE


OBSERVER FUNCTIONAL GAIN (N=16)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
DELAY COORDINATE

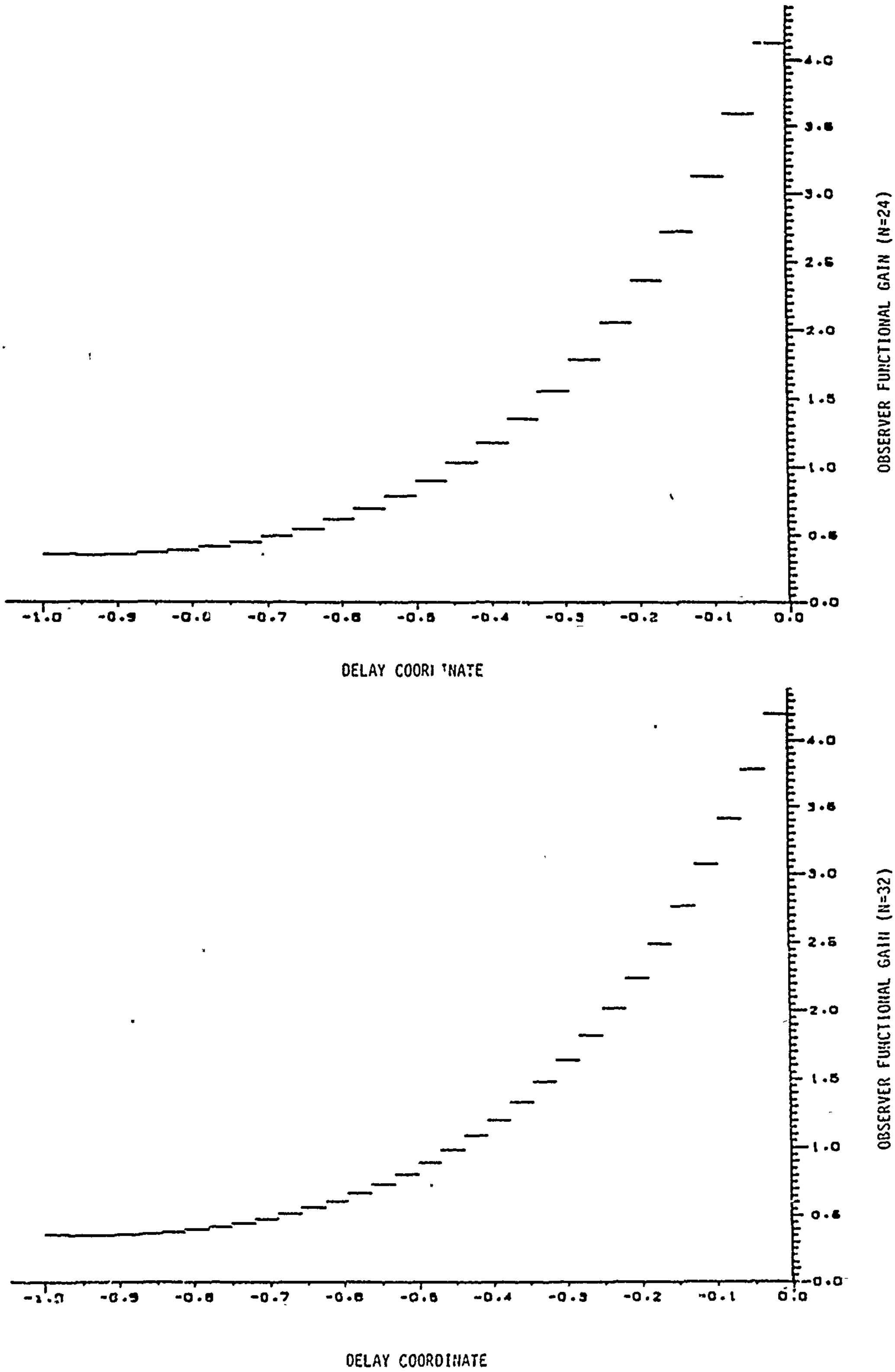
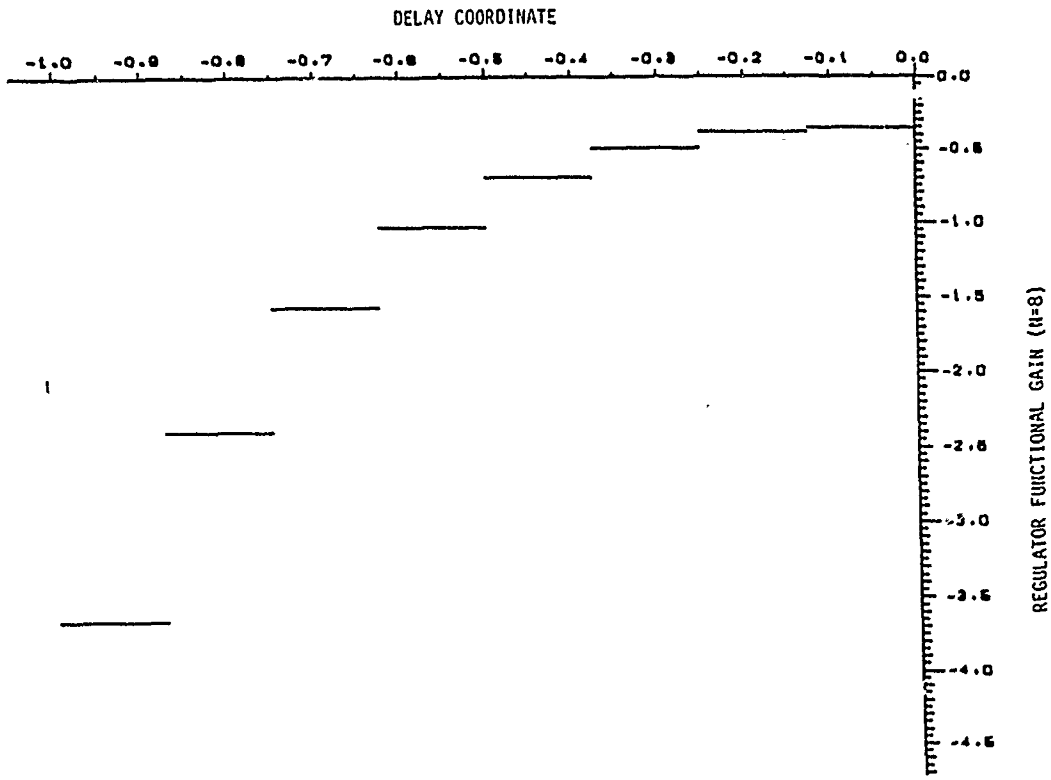
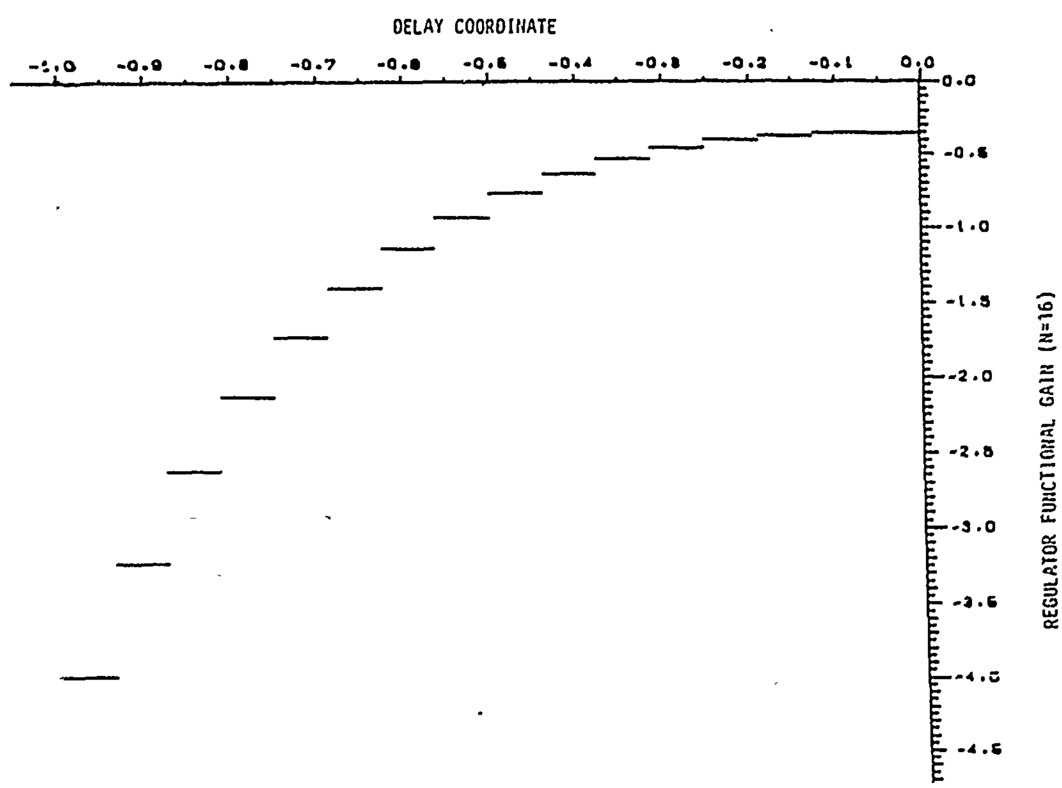
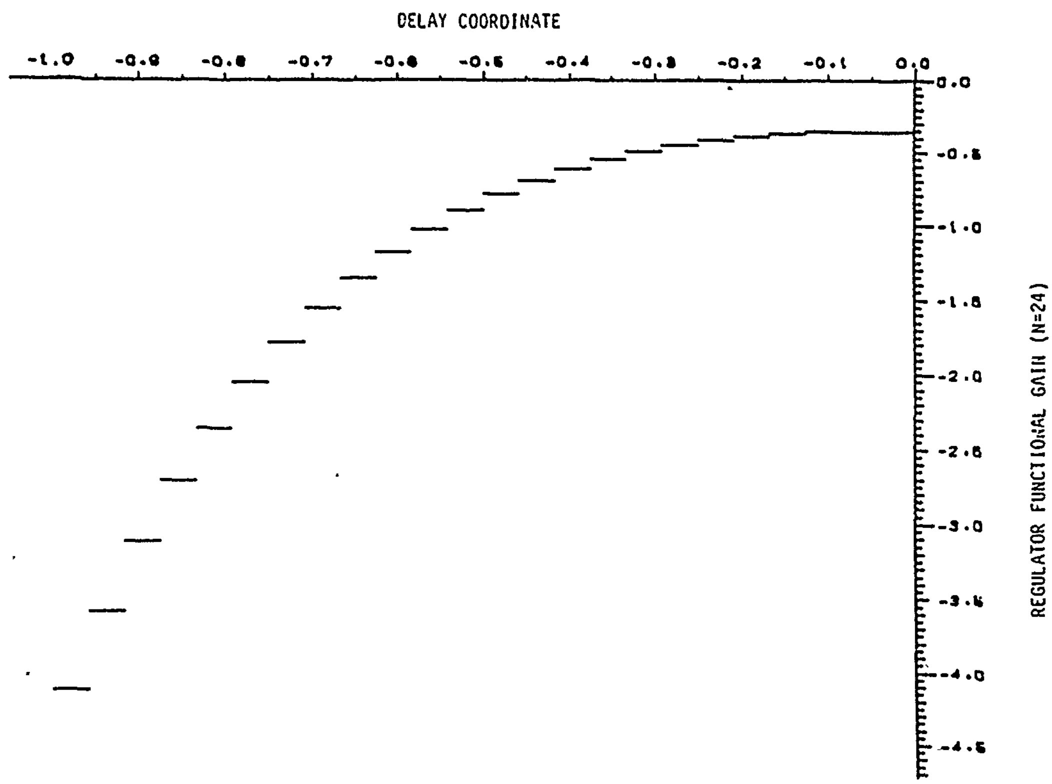


Figure 3. Hereditary system approximating optimal LQG functional observer gains; $N = 8, 16, 24, 32$


DELAY COORDINATE
REGULATOR FUNCTIONAL GAIN (N=8)


DELAY COORDINATE
REGULATOR FUNCTIONAL GAIN (N=16)



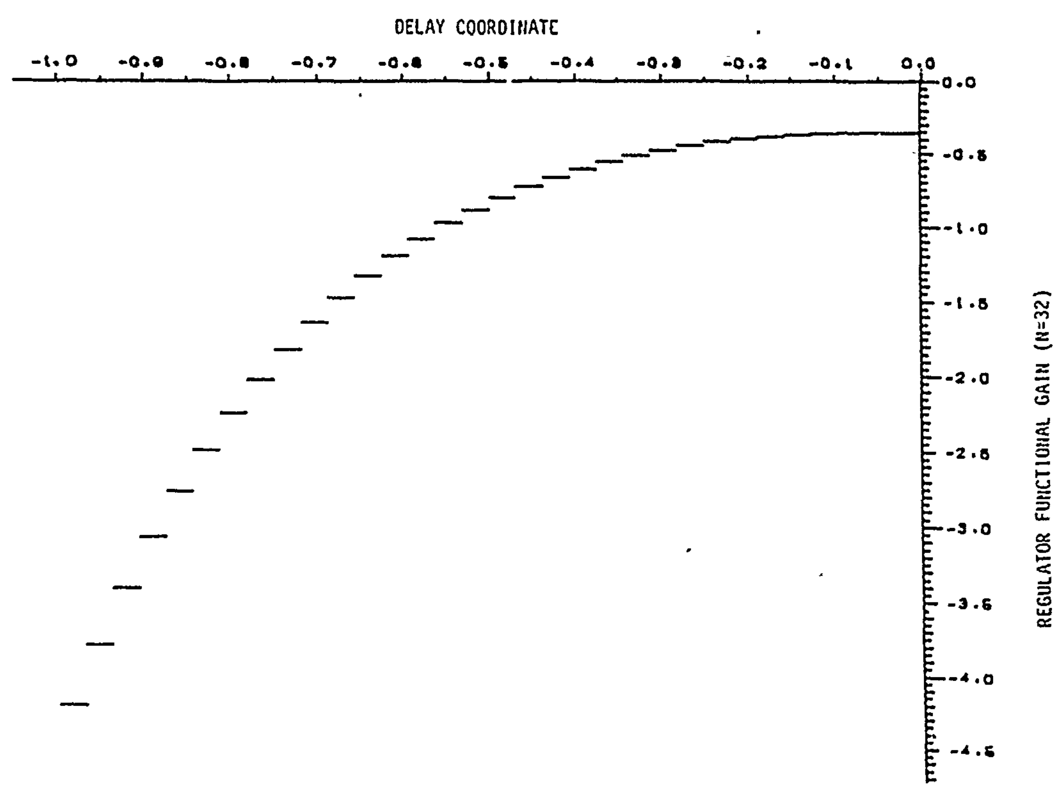


Figure 4. Hereditary system approximating optimal LQG functional control gains; N = 8, 16, 24, 32

Table IV. Hereditary system approximating optimal LQG scalar control gains

N	8	16	24	32
γ_0^N	$-4\cdot4213$	$-4\cdot4229$	$-4\cdot4233$	$-4\cdot4234$

Table V. Hereditary system approximating optimal first-order compensator gains

N	A_c^N	$B_c^N C_c^N$	J_{LQG}^N	J_{FO}^N
8	$-4\cdot835$	$-16\cdot057$	$1\cdot4042$	$1\cdot5221$
16	$-4\cdot936$	$-16\cdot343$	$1\cdot403877$	$1\cdot5298$
24	$-4\cdot959$	$-16\cdot378$	$1\cdot403856$	$1\cdot5309$
32	$-4\cdot962$	$-16\cdot404$	$1\cdot403852$	$1\cdot5317$

is due to the nature of the input and output we have chosen and the usual duality which exists between the optimal regulator and filtering problems. The first 23 open-loop poles of the system[27] are given in Table 2. The approximating first-order compensator gains along with the corresponding and LQG closed-loop costs are given in Table 5. These costs were computed using an evaluation model obtained by setting $N = 64$. Note that the performance of the first-order controllers is within 10% of the performance of the LQG controllers. Once again, on the basis of the numerical results presented here, it appears that the approximating fixed-order compensator gains are converging as $N \rightarrow \infty$.

4. SUMMARY AND CONCLUDING REMARKS

We have proposed an approximation technique for computing optimal fixed-order compensators for distributed parameter systems. Our approach involves using the optimal projection theory for infinite-dimensional systems (which characterizes the optimal fixed-order compensator) developed in Reference 18 in conjunction with finite-dimensional approximation of the infinite-dimensional plant. We demonstrated the feasibility of our approach with two examples wherein we used spline-based Ritz–Galerkin finite element schemes to compute approximating optimal first-order controllers for one-dimensional, single-input/single-output parabolic (heat/diffusion) and hereditary control systems. The numerical studies that we have carried out indicate, at least for the examples that we have considered, that convergence of the compensator gains is achieved and that using the first-order controller would lead to only minimal performance degradation over a standard LQG compensator while simplifying the implementation significantly.

At this point one is led naturally to ask the question of whether or not a satisfactory convergence theory could be developed. We are working on this at present and expect that such a theory would conform closely in form and spirit to the convergence results for LQG approximation found in References 9 and 10 and outlined in Section 2 above. We also intend to consider our approximation ideas in the context of discrete-time or sampled data systems, and for continuous-time systems involving unbounded input and/or output (e.g. boundary control systems) and systems with control or measurement delays.[11,12] Finally, we intend to investigate the application of our approximation framework to other infinite-dimensional

control systems, in particular the vibration control of flexible structures (i.e. second-order systems such as wave, beam or plate equations).

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Appendix F

Minimal Complexity Control Law Synthesis, Part 1: Problem Formulation and Reduction to Optimal Static Output Feedback *

C. N. Nett † D. S. Bernstein ‡ W. M. Haddad §

Abstract

The overall goal in this series of three papers is to make progress towards the development of a control law design methodology which supports the following paradigm: Minimise control law complexity subject to the achievement of a specified accuracy in the face of a specified level of uncertainty. We achieve this goal by developing a general theory of optimal constrained-structure dynamic output feedback compensation. By applying this theory in an iterative fashion, where here the indicated iteration occurs over the choice of the compensator dynamic-structure, the paradigm stated above can in principle be realised.

In this Part 1 of this series of papers the optimal constrained-structure dynamic output feedback problem is formulated in general terms. A method for reducing optimal constrained-structure dynamic output feedback problems to optimal static output feedback problems is developed. This reduction procedure is concretely illustrated for nine special cases of the general optimal constrained-structure dynamic output feedback problem. Taken together, these nine special cases contain most cases of interest in applications. Finally, the utility of these results in applications is described in some detail. Here we consider implementation issues such as operational/physical constraints, operating-point variations, and processor throughput/memory limitations, and describe how anti-windup/bumpless transfer, gain-scheduling, and digital processor implementation can be facilitated by apriori constraining the controller dynamic-structure in an appropriate fashion.

In Part 2 of this series of papers a general theory of optimal static output feedback compensation is developed. This theory addresses both H_2 and H_∞ performance objectives, and in each case provides a Riccati equation characterisation of optimal static output feedback compensation.

In Part 3 of this series of papers numerical methods are developed for solving the systems of coupled Riccati equations which emanate from the general theory of optimal constrained-structure dynamic output feedback compensation developed in Parts 1 and 2.

1 Introduction

In light of i) the increasingly complex nature of the systems requiring controls and ii) the increasingly stringent accuracy required of controlled systems,[1] the predominate considerations in control law design for modern engineering systems have become control law complexity and control law robustness, respectively. Indeed, with i) comes increasing and usually overriding concern with system cost, reliability, and maintainability, and with ii) comes increasingly complex control systems. Since, generally speaking, the more complex the control system, the more it costs, the less reliable it is, and the harder it is to maintain, it follows that i) and ii) conflict with each other through the specification of control system complexity. Similarly, with i) comes increasing levels of system/environmental uncertainty, and with ii) comes control systems which are increasingly robust relative to a fixed level of system/environmental uncertainty. As the maximal achievable level of robustness decreases as the level of system/environmental uncertainty increases [16,17], it follows that i) and ii) are also in conflict with each other through the specification of control system robustness. Correspondingly, control law complexity and control law robustness are, respectively, the predominant considerations in control law design for modern engineering systems.

In light of the above, it seems both natural and appropriate to postulate the following paradigm for control law design for modern engineering systems: Minimise control law complexity subject to the achievement of a specified accuracy in the face of a specified level of uncertainty.[2] Correspondingly, the overall goal in this series of three papers is to make progress towards the development of a control law design methodology which supports this paradigm. We achieve this goal by developing a general theory of optimal constrained-structure dynamic output feedback compensation, where here constrained-structure means that the dynamic-structure (*e.g.,* dynamic-order, pole locations, zero locations, etc.) of the output feedback compensation is apriori constrained in some way. By applying this theory in an iterative fashion, where here the indicated iteration occurs over the choice of the compensator dynamic-structure, the paradigm stated above can in principle be realised.

1.1 Overview

In this Part 1 of this series of papers the optimal constrained-structure dynamic output feedback problem is formulated in general terms. A method for reducing optimal constrained-structure dynamic output feedback problems to optimal static output feedback problems is developed, based on star products, linear fractional transformations, and linear fractional decompositions. This reduction procedure is concretely illustrated for nine special cases of the general optimal constrained-structure dynamic output feedback problem. Taken together, these nine special cases contain most cases of interest in applications. Finally, the utility of these results in applications is described in some detail. Here we consider implementation issues such as operational/physical constraints, operating-point variations, and processor throughput/memory limitations, and describe how anti-windup/bumpless transfer, gain-scheduling, and digital processor implementation can be facilitated by apriori constraining the controller dynamic-structure in an appropriate fashion.

In Part 2 of this series of papers [3] a general theory of optimal static output feedback compensation is developed. This theory addresses both H_2 and H_∞ performance objectives in a general way by considering an H_2 objective subject to an H_∞ constraint. Within this set-up analytical characterisations of optimal static output feedback compensation are given in the form of systems of coupled Riccati equations, and shown

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†Research Engineer, Control Systems Laboratory, GE Corporate Research and Development, Building KW, Room D219, P.O. Box 8, Schenectady, New York 12301, (518) 387-5539.

‡Staff Engineer, Government Aerospace Systems Division, Harris Corporation, Mail Stop 22/4842, Melbourne, Florida 32902, (407) 729-2140.

§Assistant Professor, Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, Florida 32901, (407) 768-8000, Extension 7241.

[1] Here we have in mind advanced tactical fighter aircraft, large flexible space structures, and variable-cycle gas turbine engines, to cite but a few examples.

[2] In layman's terms, this paradigm simply requires that control laws be designed to be "as simple as possible, but no simpler". As such, this paradigm is readily seen to be nothing more than a restatement of the famous maxim of Einstein which has for so long been the creed of successful practicing engineers.

to yield solutions to the nine special cases of the optimal constrained-structure dynamic output feedback problem considered in Part 1.

In Part 3 of this series of papers [1] numerical methods are developed for solving the systems of coupled Riccati equations which emanate from the general theory of optimal constrained-structure dynamic output feedback compensation developed in Parts 1 and 2. Existence of solutions is considered, as is uniqueness of solutions. A homotopy method is proposed for the solution of these systems of coupled Ricatti equations, and the method is concretely illustrated by several numerical examples derived from a gas turbine engine control design problem.

1.2 Key Contributions

There are two particularly noteworthy contributions of this series of papers beyond the specific technical contributions noted above. First, many results on optimal dynamic output feedback compensation recently obtained by other authors (*e.g.*, [5]) are readily shown to be but special cases of the results on optimal static output feedback compensation presented in this series of papers. As such, a significant unification of many known results in optimal control theory is achieved through the results presented in this series of papers. Second, the results presented in this series of papers provide a theoretical [illegible] for the analytical design of optimal "industry standard" controllers, such as proportional-integral (P-I) controllers and lead-lag compensators, to cite but a few examples. Consequently, we feel strongly that the results presented in this series of papers will do much to help bridge the gap that currently exists between control theory and control practice.

1.3 Organization of Part 1

An outline of this paper is as follows: In Section 2 we pour the mathematical foundation upon which the paper is based. In Section 3 the optimal constrained-structure dynamic output feedback problem is formulated, and in Section 4 its reduction to the optimal static output feedback problem is presented. Finally, conclusions are given in Section 5.

2 Mathematical Foundation

In this section we pour the mathematical foundation upon which this paper will be based. We begin by considering the two-input, two-output (TITO) system depicted in Fig. 1. Here T is a partitioned proper real-

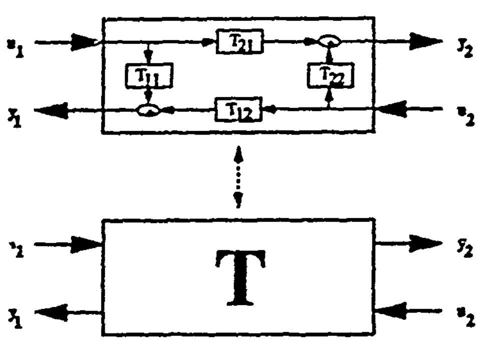

Figure 1: TITO system representation

rational transfer matrix, to which we associate a partitioned standard state-space realisation:

$$T := \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] =: \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (1)$$

In order that the system representation (1) be complately general we will permit the the use of the so-called empty matrix [10]. The empty matrix, denoted [ ], is an entry-less matrix with dimensions 0×0. Proceeding in an obvious way, standard matrix operations are readily extended to encompass the empty matrix. Those extensions required for our purposes here are summarised below:

Definition 2.1 *The transpose of the empty matrix is the empty matrix. Furthermore, given any matrix M, its product with the empty matrix is the empty matrix, and both its sum and its adjoinment with the empty matrix are given by M.*

Our interest in the empty matrix here stems from its rather significant utility relative to representing static and/or SISO systems[3] in terms of (1). To illustrate this utility, consider first the case of a static system M. Using the empty matrix, a state-space realisation of M can be written as follows:

$$M \sim \begin{bmatrix} [\,] & [\,] \\ [\,] & M \end{bmatrix}. \quad (2)$$

Note here that while the empty matrices in (2) can be replaced by zero matrices without invalidating (2), to do so adds uncontrollable and unobservable rigid body modes to the realisation, which is clearly undesirable. Considering now the case of a SISO system M, we use the properties of the empty matrix to write:

$$M = \begin{bmatrix} M & [\,] \\ [\,] & [\,] \end{bmatrix} = \begin{bmatrix} [\,] & [\,] \\ [\,] & M \end{bmatrix}. \quad (3)$$

This construct provides us with two distinct TITO representations of M, corresponding to two different choices for a null input-output pair in Fig. 1. Throughout the sequel the constructs provided in (2) and (3) will be utilised extensively in order to facilitate an integrated development of results for static and/or SISO systems within the context of TITO systems.

As is discussed in some detail in [12], the TITO system can be regarded as the most general operator-theoretic form of system representation. The distinguishing characteristic of this system is, of course, the partitioning of the system inputs and outputs into two vectors each. Because of this partitioning, unutilised outputs and unmanipulated inputs need not be ignored when considering system interconnections; rather, they are explicitly represented. Correspondingly, in considering the general interconnection of two TITO systems, which we shall now do, one is in fact considering the most general form of system interconnection.

2.1 The Star Product

Consider the two TITO systems

$$t := \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} a & b_1 & b_2 \\ \hline c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{array}\right] =: \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (4)$$

and

$$T := \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] =: \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (5)$$

and assume that the dimensions of t_{22} and the transpose of T_{11} are identical. Under this assumption, we may interconnect t and T as indicated in Fig. 2 to yield yet another TITO system, which we denote by $\mathcal{T}$. Recalling Fig. 1, it should be apparent to the reader that standard parallel interconnection, cascade iterconnection, and feedback interconnection are all special cases of the TITO interconnection depicted in Fig. 2. Indeed, with

$$t = \begin{bmatrix} M & I \\ I & 0 \end{bmatrix} \quad , \quad T = \begin{bmatrix} N & [\,] \\ [\,] & [\,] \end{bmatrix}, \quad (6)$$

the interconnection depicted in Fig. 2 describes the standard parallel interconnection of the two systems M and N. Similarly, standard cascade interconnection of M and N is descibed by Fig. 2 when

$$t = \begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix} \quad , \quad T = \begin{bmatrix} N & [\,] \\ [\,] & [\,] \end{bmatrix}, \quad (7)$$

[3] Of course, here SISO means single vector input, single vector output, in contrast to its usual usage.

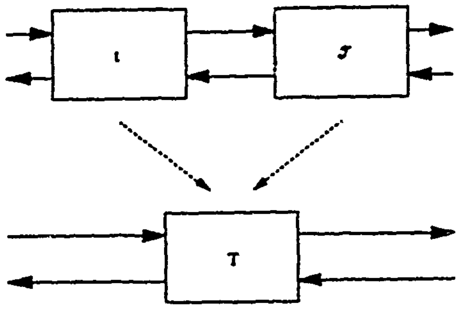

Figure 2: Interconnection of two TITO systems

and standard feedback interconnection of M and N is described by Fig. 2 when

$$t=\begin{bmatrix} 0 & I \\ I & M \end{bmatrix} \quad , \quad \mathcal{T}=\begin{bmatrix} N & [\,] \\ [\,] & [\,] \end{bmatrix}. \tag{8}$$

Since standard parallel, cascade, and feedback interconnections are the basic building blocks of system interconnection, it follows that the TITO interconnection depicted in Fig. 2 is a completely general form of system interconnection.

A transfer matrix formula for TITO interconnection can be derived via a series of straightforward but tedious algebraic manipulations [15,7]. Doing so, we obtain the following expression[4] for the interconnection, T, of t and $\mathcal{T}$ depicted in Fig. 2:

$$T=\begin{bmatrix} t_{11}+t_{12}\mathcal{T}_{11}(I-t_{22}\mathcal{T}_{11})^{-1}t_{21} & t_{12}(I-\mathcal{T}_{11}t_{22})^{-1}\mathcal{T}_{12} \\ \mathcal{T}_{21}(I-t_{22}\mathcal{T}_{11})^{-1}t_{21} & \mathcal{T}_{22}+\mathcal{T}_{21}t_{22}(I-\mathcal{T}_{11}t_{22})^{-1}\mathcal{T}_{12} \end{bmatrix}. \tag{9}$$

It is in terms of this formula that the so-called star product [15] of two TITO systems is defined:

Definition 2.2 *Consider the two TITO systems t and $\mathcal{T}$ described by (4) and (5), respectively. The star product of $\mathcal{T}$ with t, denoted $t\star\mathcal{T}$, is defined as the TITO system T given by (9).*

Implicit in the above definition of the star product of two TITO systems is the definition of the star product of two appropriately partitioned constant (*i.e.*, real, complex) matrices. As such, we may view the star product of two TITO systems as the pointwise star product of their frequency evaluations.

In light of the correspondence between the star product and TITO interconnection, it should come as no surprise to the reader that the star product is associative [15]:

Fact 2.1 *Given three TITO systems t, $\mathcal{T}$, and τ,*

$$(t\star\mathcal{T})\star\tau = t\star(\mathcal{T}\star\tau).$$

A state-space formula for the star-product/TITO interconnection can be derived via a series of straightforward but tedious algebraic manipulations [7]. Doing so, we obtain a formula for a state-space realisation of $T=t\star\mathcal{T}$ in terms of state-space realisations of t and $\mathcal{T}$:

Fact 2.2 *Consider the three TITO systems t, $\mathcal{T}$, and $T=t\star\mathcal{T}$. Suppose that t and $\mathcal{T}$ are described by (4) and (5), respectively, and denote the state-vectors corresponding to the realizations of t and $\mathcal{T}$ given in (4) and (5) by x_t and $x_{\mathcal{T}}$, respectively. Under these conditions,*

$$T\sim\left[\begin{array}{c|c} \begin{bmatrix} a & b_2 \\ c_2 & d_{22} \end{bmatrix}\star\begin{bmatrix} D_{11} & C_1 \\ B_1 & A \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ d_{21} & d_{22} \end{bmatrix}\star\begin{bmatrix} D_{11} & D_{12} \\ B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} c_1 & d_{12} \\ c_2 & d_{22} \end{bmatrix}\star\begin{bmatrix} D_{11} & C_1 \\ D_{21} & C_2 \end{bmatrix} & d\star D \end{array}\right], \tag{10}$$

and the state-vector x_T defined by (10) is the composite of x_t and $x_{\mathcal{T}}$:

$$x_T=\begin{bmatrix} x_t \\ x_{\mathcal{T}} \end{bmatrix}. \tag{11}$$

The state-space formula for the star product/TITO interconnection given in Fact 2.2 is particularly convenient for numerical implementation. Indeed, to implement (10) in MATLAB [10] we would first implement the constant matrix version of (9) as the m-file `star_frq`, which could of course be called on a frequency-by-frequency basis to compute the transfer matrix version of (9). Next, we would implement (10) as the m-file `star_ss`, where here `star_ss` would simply call `star_frq` four times, with appropriate arguments. The resulting implementation of (10) would be comprised of only two MATLAB m-files, and each m-file would contain only a few lines of code.

Recalling that TITO interconnection is a completely general form of system interconnection, we add that the elegant implementation of the star product/TITO interconnection outlined above could in turn be called by other m-files, with arguments as indicated in (6)-(8), to generate more standard system interconnections. Continuing in this vein one could in principle implement a completely general, yet extremely compact computational facility for system interconnection, with only a modest effort. Consequently, we would expect to soon see the TITO system representation and the star product incorporated within existing CACSD packages, such as [10,9,4].

2.2 The Linear Fractional Transformation

In this section we consider the general interconnection of a TITO system with a SISO system as depicted in Fig. 3. As is suggested by the

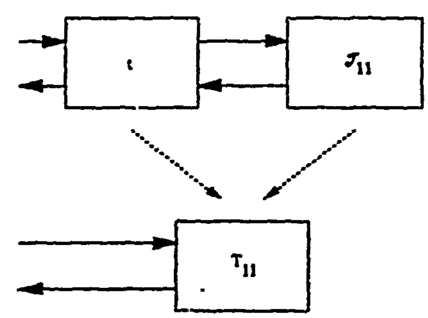

Figure 3: Interconnection of a TITO system with a SISO system

notation, the interconnection depicted in Fig. 3 is but a special case of the general interconnection of two TITO systems depicted in Fig. 2. To give a precise mathematical statement of this correspondence we make use of the properties of the empty matrix and the definition of the star product to write:

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\star\begin{bmatrix} \mathcal{T}_{11} & [\,] \\ [\,] & [\,] \end{bmatrix}= \begin{bmatrix} t_{11}+t_{12}\mathcal{T}_{11}(I-t_{22}\mathcal{T}_{11})^{-1}t_{21} & [\,] \\ [\,] & [\,] \end{bmatrix}=\begin{bmatrix} T_{11} & [\,] \\ [\,] & [\,] \end{bmatrix}. \tag{12}$$

Though (12) completely describes the interconnection depicted in Fig. 3, for notational simplicity it is desirable to introduce an abbreviated notation to describe the correspondence between t, $\mathcal{T}_{11}$, and T

[4] Throughout, any indicated matrix expression is assumed to involve matrices of compatible dimensions, and any indicated matrix inverse is assumed to exist.

implicit in (12). This leads us to define the so-called linear fractional transformation [15] of a SISO system under a TITO system as follows:

Definition 2.3 *Consider the TITO system t described by (4) and the SISO system $\mathcal{T}_{11}$. The linear fractional transformation of $\mathcal{T}_{11}$ under t, denoted $t \circ \mathcal{T}_{11}$, is defined as the SISO system T_{11} given by:*

$$T_{11} = t_{11} + t_{12}\mathcal{T}_{11}(I - t_{22}\mathcal{T}_{11})^{-1}t_{21}. \tag{13}$$

The utility of the linear fractional transformation lies with the fact that it enables (12) to be compactly restated as (13). As interconnections of the form depicted in Fig. 3 play a key role in the sequel, this notational simplication does much to streamline the remainder of this paper.

For precision in what follows, a formal statement of the correspondence between the star product and the linear fractional transformation is given below. The proof of this result is immediate using the definitions of the star product and the linear fractional transformation:

Fact 2.3 *Consider the TITO system t and the two SISO systems T_{11} and $\mathcal{T}_{11}$. Under these conditions, the following three statements are equivalent:*

1. *The relation $T_{11} = t \circ \mathcal{T}_{11}$ holds.*
2. *There exist SISO systems $\mathcal{T}_{12}$, $\mathcal{T}_{21}$, and $\mathcal{T}_{22}$ for which*

$$T_{11} = \begin{bmatrix} I & 0 \end{bmatrix} (t \star \mathcal{T}) \begin{bmatrix} I \\ 0 \end{bmatrix},$$

where here $\mathcal{T}$ is defined in the obvious fashion.

3. *The relation*

$$T_{11} = \begin{bmatrix} I & 0 \end{bmatrix} (t \star \mathcal{T}) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

holds for all SISO systems $\mathcal{T}_{12}$, $\mathcal{T}_{21}$, and $\mathcal{T}_{22}$, where here $\mathcal{T}$ is defined in the obvious fashion.

The following property of the linear fractional transformation plays a key role in the sequel. Its proof is immediate using the definition of the the linear fractional transformation:

Fact 2.4 *Consider the TITO system t described by (4) and define the following function:*

$$\mathbf{A}(\cdot) : \mathcal{T}_{11} \mapsto t \circ \mathcal{T}_{11},$$

where here $\mathcal{T}_{11}$ is an arbitrary SISO system. Under these conditions, $\mathbf{A}(\cdot)$ is affine if and only if $t_{22} = 0$, in which case $t_{11} = \mathbf{A}(0)$. Furthermore, $\mathbf{A}(\cdot)$ is linear if and only if $t_{22} = 0$ and $t_{11} = 0$.

A state-space formula for the linear fractional transformation follows immediately from the corresponding formula (10) for the star product via the correspondence between the star product and the linear fractional transformation given in Fact 2.3. Indeed, writing:

$$\begin{bmatrix} \mathcal{T}_{11} & [\,] \\ [\,] & [\,] \end{bmatrix} \sim \left[\begin{array}{c|cc} A & B_1 & [\,] \\ \hline C_1 & D_{11} & [\,] \\ [\,] & [\,] & [\,] \end{array}\right], \tag{14}$$

where here

$$\mathcal{T}_{11} \sim \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \tag{15}$$

we need only substitute (14) into (10) to obtain the following corollary to Fact 2.2:

Fact 2.5 *Consider the TITO system t and the two SISO systems T_{11} and $T_{11} = t \circ \mathcal{T}_{11}$. Suppose that t and $\mathcal{T}_{11}$ are described by (4) and (15), respectively, and denote the state-vectors corresponding to the realisations of t and $\mathcal{T}_{11}$ given in (4) and (15) by x_t and $x_{\mathcal{T}_{11}}$, respectively. Under these conditions,*

$$T_{11} \sim \left[\begin{array}{c|c} \begin{bmatrix} a & b_2 \\ c_2 & d_{22} \end{bmatrix} \star \begin{bmatrix} D_{11} & C_1 \\ B_1 & A \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ d_{21} & d_{22} \end{bmatrix} \star \begin{bmatrix} D_{11} & [\,] \\ B_1 & [\,] \end{bmatrix} \\ \hline \begin{bmatrix} c_1 & d_{12} \\ c_2 & d_{22} \end{bmatrix} \star \begin{bmatrix} D_{11} & C_1 \\ [\,] & [\,] \end{bmatrix} & d \circ D_{11} \end{array}\right], \tag{16}$$

and the state-vector $x_{T_{11}}$ defined by (16) is the composite of x_t and $x_{\mathcal{T}_{11}}$:

$$x_{T_{11}} = \begin{bmatrix} x_t \\ x_{\mathcal{T}_{11}} \end{bmatrix}. \tag{17}$$

2.3 Specializations for Static Right Operands

In this section we specialise the state-space formulas for the star product and the linear fractional transformation given by (10) and (16), respectively, to the case where the right operand in the respective operations is a static system. To do this, we begin by assuming that $\mathcal{T}$ and $\mathcal{T}_{11}$ as described in Facts 2.2 and 2.5, respectively, are static systems. Next we apply the construct for static systems given by (2) to $\mathcal{T}$ and $\mathcal{T}_{11}$, respectively:

$$\mathcal{T} \sim \left[\begin{array}{c|cc} [\,] & [\,] & [\,] \\ \hline [\,] & \mathcal{T}_{11} & \mathcal{T}_{12} \\ [\,] & \mathcal{T}_{21} & \mathcal{T}_{12} \end{array}\right], \tag{18}$$

$$\mathcal{T}_{11} \sim \begin{bmatrix} [\,] & [\,] \\ [\,] & \mathcal{T}_{11} \end{bmatrix}. \tag{19}$$

Substituting now from (18) into (10) and simplifying the resulting expressions using the properties of the empty matrix and the definition of the star product, we obtain the following specialisation of Fact 2.2:

Theorem 2.1 *Consider the three TITO systems t, $\mathcal{T}$, and $T = t \star \mathcal{T}$. Suppose that t is described by (4), and denote the state-vector corresponding to the realisation of t given in (4) by x_t. Finally, assume that $\mathcal{T}$ is a static system. Under these conditions,*

$$T \sim \left[\begin{array}{c|c|c} a & b_1 & b_2 \\ \hline c_1 & d_{11} & d_{12} \\ \hline c_2 & d_{21} & d_{22} \end{array}\right] \star \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{bmatrix}, \tag{20}$$

and the state-vector x_T defined by (20) is x_t.

Similarly, substituting from (19) into (16) and simplifying the resulting expressions using the properties of the empty matrix and the definition of the linear fractional transformation, we obtain the following specialisation of Fact 2.5:

Theorem 2.2 *Consider the TITO systems t and the two SISO systems $\mathcal{T}_{11}$ and $T_{11} = t \circ \mathcal{T}_{11}$. Suppose that t is described by (4), and denote the state-vector corresponding to the realisation of t given in (4) by x_t. Finally, assume that $\mathcal{T}_{11}$ is a static system. Under these conditions,*

$$T_{11} \sim \left[\begin{array}{c|c|c} a & b_1 & b_2 \\ \hline c_1 & d_{11} & d_{12} \\ \hline c_2 & d_{21} & d_{22} \end{array}\right] \circ \mathcal{T}_{11}, \tag{21}$$

and the state-vector $x_{T_{11}}$ defined by (21) is x_t.

The reader should make careful note of how the left operand on the right-hand side of both (20) and (21) is partitioned. This is not a typographical error; in fact, the right-hand sides of both (20) and (21) would be undefined if the partitioning of this matrix was the "standard" partitioning indicated in (4).

Interestingly, Theorems 2.1 and 2.2 show that the operation of state-space realisation "distributes" over the operations of star product and linear fractional transformation when the right operand is a static system. Indeed, for static right operands the star product operating on transfer matrices corresponds to the star product operating on the corresponding state-space realisations, appropriately partitioned, and similarly for the linear fractional transformation. In the language of abstract

algebra, state-space realisation is thus seen to be a morphism from the set of transfer matrices to the set of state-space realisations under the operations of star product and linear fractional transformation with a static right operand. Though we believe it likely that these observations belie a deeper significance of Theorems 2.1 and 2.2, no further development along these lines will be attempted here as it would take us too far afield.

Theorems 2.1 and 2.2 play a key role in the sequel. The reason for this is, of course, that they describe the situation of static output feedback. Indeed, Thm. 2.2 describes the situation of static output feedback compensation, $\mathcal{T}_{11}$, applied to the dynamic system t, and similarly for Thm. 2.1.

3 Problem Formulation

In this section the optimal constrained-structure dynamic output feedback problem is formulated in general terms. Starting with the general formulation of the optimal (unconstrained) dynamic output feedback problem utilised in [6], we extend this set-up to encompass the optimal constrained-structure dynamic output feedback problem. The resulting set-up is as indicated in Fig. 4, where here g is the given dynamic system

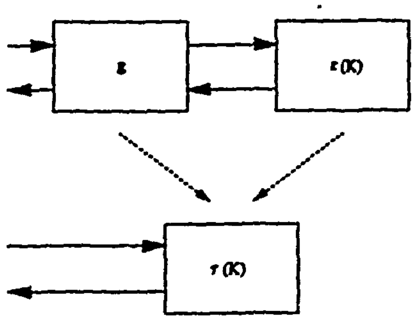

Figure 4: Constrained-structure dynamic output feedback

to be controlled:

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} a & b_1 & b_2 \\ \hline c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{array}\right] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{22}$$

and $\kappa(K)$ is the dynamic controller to be designed:

$$\kappa(K) \sim \begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix}. \tag{23}$$

The essential distinction between the set-up depicted in Fig. 4 and the set-up utilised in [6] is the functional dependence of κ and τ on K. Indeed, in the set-up utilised in [6] the functional dependence of κ on K is absent, and τ is written as a function of κ rather than K.

The significance of the functional dependence on K in (23) is that it permits the dynamic-structure of the controller to be constrained apriori in the problem formulation. To do this we take K to be an arbitrary real matrix of fixed dimensions, and regard K as the "design parameters" in a constrained-structure dynamic controller. Correspondingly, the function $\kappa(\cdot)$ is viewed as specifying the fixed dynamic-structure of the constrained-structure controller.

In order to provide a concrete illustration of the concepts set forth above we consider a simple, yet important example of constrained-structure dynamic output feedback; namely, proportional-integral (P-I) control. A controller of this form can be described as follows:

$$K_p + \frac{K_I}{s}, \tag{24}$$

where here K_p and K_I are the proportional and integral gain matrices of the controller, respectively. Based on (24), we now set

$$\kappa(K) = K_2 + \frac{K_1}{s} \tag{25}$$

in order to constrain the dynamic-structure of the controller to a P-I form, where here K can be taken as either

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \tag{26}$$

or

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}. \tag{27}$$

Equivalently, we could set either

$$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix} = \begin{bmatrix} 0 & I \\ K_1 & K_2 \end{bmatrix} \tag{28}$$

or

$$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix} = \begin{bmatrix} 0 & K_1 \\ I & K_2 \end{bmatrix} \tag{29}$$

in order to constrain the dynamic-structure of the controller to a P-I form,[5] where here K is taken as in (26) in the case of (28) and (27) in the case of (29).

From a computational standpoint, the state-space specifications (28) and (29) are typically more convenient to work with than the transfer function specification (25). Which of these two state-space representations is most appropriate depends on the dimensions of the P-I controller in (25). Generically, (28) is minimal if and only if the controller has at least as many outputs as inputs, and (29) is minimal if and only if the controller has at least as many inputs as outputs [8]. As such, (28) must be utilised when the controller has fewer inputs than outputs, and (29) must be utilised when the controller has fewer outputs than inputs.

In (28) the dynamics and controls in the controller realisation are fixed apriori to 0 and I, respectively, in order to constrain the dynamic-structure of the controller to a P-I form. A natural extension of this set-up is to allow other choices of fixed dynamics and controls to be made, beyond the specific choices $\alpha(K) = 0$ and $\beta(K) = I$. Pursuing this extension results in the formulation of the the so-called fixed dynamics/controls problem (FDCP). Likewise, a natural extension of (29) is the so-called fixed dynamics/observations problem (FDOP). Each of these two problems is described in Table 1, along with seven other variations on this theme. Taken together, these nine special cases of the general set-up suffice to handle most constrained-structure dynamic output feedback problems of interest in applications. Nonetheless, we are quick to add that the problems cataloged in Table 1 represent only a small fraction of the total capabilities of the general problem formulation.[6]

3.1 Optimality Criterion

Having formulated the constrained-structure dynamic output feedback problem in general terms, we now adjoin to this set-up an abstract notion of optimality so as to be able to discuss the optimal constrained-structure dynamic output feedback problem in general terms. For this purpose we postulate the following design objective corresponding to Fig. 4:

Design K to "optimise" the closed-loop system:

$$\tau(K) := g \circ \kappa(K), \tag{30}$$

subject to the constraint of internal stability.[7]

[5] The realisations given in (28) and (29) are so-called minimal modal canonical form realisations [13]. As is discussed in [13] and elaborated upon in the sequel, these canonical forms can be used to facilitate an integrated design/implementation of nonlinear controllers.

[6] Omitted from Table 1 are the so-called three block problems, certain "mixed" 2-block problems, and decentralised control problems; however, their formulations are readily generated. They are omitted here because they give rise to problems where the structure of K is constrained, and we wish to postpone a discussion of this case until Sec. 5.

[7] The notion of internal stability here is the usual one, defined in terms of the realisations of g and $\kappa(K)$ given in (22) and (23), respectively. Precisely, denoting the state vectors corresponding to these realisations by x_g and $x_{\kappa(K)}$, respectively, internal stability here means that the composite state vector for the closed-loop system formed from x_g and $x_{\kappa(K)}$ decays to zero from all initial conditions under zero input conditions.

In the above the data are the plant g described by (22) and the dynamic-structure of the controller. As previously discussed, the latter is specified in terms of the function $\kappa(\cdot)$ or, equivalently, in terms of the function

$$\begin{bmatrix} \alpha(\cdot) & \beta(\cdot) \\ \gamma(\cdot) & \delta(\cdot) \end{bmatrix}, \tag{31}$$

corresponding to a realisation of $\kappa(\cdot)$. The design objective is then to choose the "free parameters" K in this fixed dynamic-structure to "optimise" the closed-loop system $\tau(K)$, subject to the constraint of internal stability.

The specification of optimality in the above is completely arbitrary, subject to the requirement that only $\tau(\cdot)$ be required for its specification. For example, in Part 2 of this paper [3] the output of $\tau(K)$ in Fig. 4 is partitioned into two components, corresponding to the following row-partition of $\tau(K)$:

$$\tau(K) = \begin{bmatrix} \tau_1(K) \\ \tau_2(K) \end{bmatrix}. \tag{32}$$

In terms of this partition, the following optimality criterion is imposed [2]:

$$\textit{minimise} \parallel \tau_1(K) \parallel_2 \textit{subject to} \parallel \tau_2(K) \parallel_\infty < 1, \tag{33}$$

where here $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are the H_2 and H_∞ norms, respectively [6]. As this notion of optimality requires only $\tau(\cdot)$ for its specification, the requirement imposed above is satisfied.

As discussed in Part 2, the notion of optimality specified by (32) and (33) addresses both H_2 and/or H_∞ performance criteria in a general and flexible way. As is well-known, many accuracy and robustness concerns may be addressed by way of performance criteria of this type [11].

4 Reduction Procedure

In this section we develop a general method for reducing optimal constrained-structure dynamic output feedback problems to optimal static output feedback problems. The essence of this reduction procedure is illustrated in Fig. 5. In short, $\kappa(K)$ is written as a linear

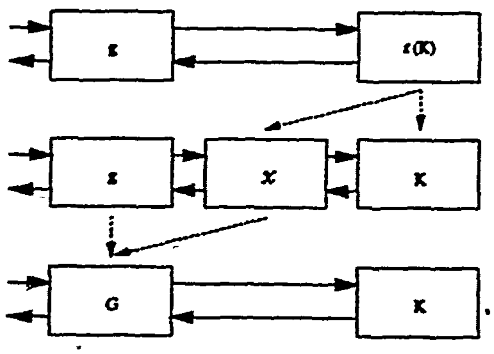

Figure 5: Reduction to static output feedback

fractional transformation of K under

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{34}$$

where here K is independent of K. The plant g is then augmented with K to yield the augmented plant

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{35}$$

The result is the static output feedback problem of designing the static K for the augmented plant G. By solving this problem we obtain a solution to the optimal constrained-structure dynamic output feedback problem of designing $\kappa(K)$ for the original plant g. As such, the original optimal constrained-structure dynamic output feedback problem is effectively reduced to an optimal static output feedback problem.

For added precision in what follows, the reduction procedure described above is restated in mathematical terms as the following algorithm:

A1 (Decomposition): Find K such that

$$\kappa(K) = K \circ K \tag{36}$$

for all K, with K independent of K.

A2 (Augmentation): Compute

$$G := g \star K. \tag{37}$$

A3 (Optimisation): Find K^* which "optimises" the closed-loop system

$$\tau_s(K) := G \circ K \tag{38}$$

defined by G and K, subject to the constraint of internal stability.

A4 (Implementation): Compute $\kappa(K^*)$.

There are two primary questions which must be answered in order to place the above algorithm on a firm analytical foundation: First, under what conditions is the algorithm valid, in the sense that the solution to the optimal static output feedack problem yields a solution to the optimal constrained-structure dynamic output feedback problem? Second, under what conditions can the algorithm be successfully executed, and how can the various steps in the algorithm be systematically executed under these conditions? From these two general questions arise the following four specific technical questions:

Q1 (Decomposition): Under what conditions does the decomposition (36) exist, how is it determined, and to what extent is it uniquely determined?

Q2 (Augmentation): How is G in (37) computed, and to what extent is G uniquely determined?

Q3 (Optimisation): Under what conditions does there exist a solution to the static output feedback problem, and how is this solution determined?

Q4 (Implementation): Under what conditions does K^* yielded by the algorithm "optimise" the closed-loop system $\tau(K)$ defined in (30)?

Q5 (Implementation): Under what conditions does $\kappa(K^*)$ yielded by the algorithm internally stabilise g?

With the exception of Q3, each of the questions posed above are answered in the remainder of this section. In doing so, the validity of the reduction procedure is established, and systematic methods for its execution are presented. We proceed in order through the questions in our development, establishing a separate section for each question as we proceed. Throughout, we will make free use of the notation established in (22), (23), (30), (34), (35), and (38). However, initially only g, $\kappa(\cdot)$, and their realisations are assumed given. Finally, we will denote by x_g, $x_{\kappa(K)}$, x_K, and x_G the state-vectors corresponding to the realisations of g, $\kappa(K)$, K, and G.

The resolution of Q3 is deferred to Parts 2 and 3 of this paper [3,1], wherein a reasonably complete answer to this question is given.

4.1 Decomposition

In this section Q1 is answered. We begin with a simple extension of Thm. 2.2:

Theorem 4.1 *There exists K which satisfies (36) if there exists a solution to the following real matrix equation:*

$$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix} = \left[\begin{array}{cc|c} \alpha(0) & \beta(0) & B_2 \\ \gamma(0) & \delta(0) & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] \circ K, \tag{39}$$

in which case

$$K \sim \left[\begin{array}{c|cc} \alpha(0) & \beta(0) & B_2 \\ \hline \gamma(0) & \delta(0) & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{40}$$

solves (36).

The significance of Thm. 4.1 is that it reduces the question of static linear fractional decomposition of the dynamic system $s(K)$ to the question of static linear fractional decomposition of the static system

$$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix}, \tag{41}$$

which is simply a real matrix problem. However, Thm. 4.1 is not completely satisfactory for two important reasons: First, the conditions of the theorem are only sufficient, and not necessary. Second, the real matrix equation (39) is nonlinear, and hence difficult to solve in general. Though we are unable to alleviate these deficiencies in the general case, in an important special case we have been able to alleviate them completely.

Theorem 4.2 *Suppose (41) is affine in K. Under this condition, there exists K which satisfies (36) if and only if there exists a solution to the following real matrix equation:*

$$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix} - \begin{bmatrix} \alpha(0) & \beta(0) \\ \gamma(0) & \delta(0) \end{bmatrix} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K \begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \tag{42}$$

in which case

$$K \sim \left[\begin{array}{c|cc} \alpha(0) & \beta(0) & B_2 \\ \hline \gamma(0) & \delta(0) & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{43}$$

solves (36).

The proof of Thm. 4.2 makes use of Thm. 4.1 and Fact. 2.4 in a straightforward way. Note that the conditions of Thm. 4.2 are both necessary and sufficient, and the real matrix equation (42) is linear. This improvement over Thm. 4.1 has come at the expense of the assumption that (41) is affine. However, this assumption is satisfied in most applications of interest. As evidence to support this claim we note that each of the constrained-structure dynamic output feedback problems cataloged in Table 1 satisfies this assumption.

In order to illustrate the significant utility of Thm. 4.2 in executing A1, consider the FDCP cataloged in Table 1. As noted above, (41) is affine in this problem, so we may apply Thm. 4.2 to immediately reduce the computation of K in A1 to the solution of (42), which in this case reduces to:

$$\begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \begin{bmatrix} K_1 & K_3 \end{bmatrix} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}. \tag{44}$$

By inspection,

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad , \quad \begin{bmatrix} C_2 & D_{21} \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ 0 & I \end{array}\right] \tag{45}$$

solves (44). Substituting (45) into (43) we obtain:

$$K \sim \left[\begin{array}{c|cc} \alpha_o & \beta_o & 0 \\ \hline 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{array}\right], \tag{46}$$

thus completing the execution of A1.

The K computed above for the FDCP is cataloged in Table 2, along with K corresponding to each of the other constrained-structure dynamic output feedback problems cataloged in Table 1. For each of these other problems A1 was executed as outlined above using Thm. 4.2 in order to obtain K.

In order to address the issue of the uniqueness of K and thereby complete our study of Q1 we present the following simple result:

Theorem 4.3 *Any solution of (42) satisfies the following relation:*

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \begin{bmatrix} C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} \alpha(I) & \beta(I) \\ \gamma(I) & \delta(I) \end{bmatrix} - \begin{bmatrix} \alpha(0) & \beta(0) \\ \gamma(0) & \delta(0) \end{bmatrix}. \tag{47}$$

As will be shown in the next section, Thm. 4.3 is all that is required to completely characterise the extent to which G in (37) is uniquely determined.

4.2 Augmentation

In this section Q2 is answered. Throughout, we assume that (41) is affine and a realisation of a K satisfying (36) has been computed via Thm. 4.2. As indicated in the last section, little is lost by imposing these two assumptions. For convenience, we also assume that $d_{22} = 0$. Little is lost by imposing this assumption, too, as $d_{22} = 0$ whenever sensors and actuators are modelled as bandlimited devices.

We begin by giving an explicit formula for G defined in (37) based on Fact 2.2:

Theorem 4.4 *Suppose (37) holds. Under this condition,*

$$G \sim \left[\begin{array}{c|c} \begin{bmatrix} a & b_2 \\ c_2 & 0 \end{bmatrix} \star \begin{bmatrix} \delta(0) & \gamma(0) \\ \beta(0) & \alpha(0) \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ d_{21} & 0 \end{bmatrix} \star \begin{bmatrix} \delta(0) & D_{12} \\ \beta(0) & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} c_1 & d_{12} \\ c_2 & 0 \end{bmatrix} \star \begin{bmatrix} \delta(0) & \gamma(0) \\ D_{21} & C_2 \end{bmatrix} & \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & 0 \end{bmatrix} \star \begin{bmatrix} \delta(0) & D_{12} \\ D_{21} & 0 \end{bmatrix} \end{array}\right]. \tag{48}$$

It is a simple matter to compute realisations of G corresponding to each of the constrained-structure dynamic output feedback problems cataloged in Table 1 using Thm. 4.4. To do this we simply substitute the data for K in Table 2 into (48). For the convenience of the reader the results of this simple exercise are cataloged in Table 3.

In order to address the issue of the uniqueness of G and thereby complete our study of Q2 we derive the following result from Thm. 4.3:

Theorem 4.5 *Denote the realisation of G given in (48) as in (35). Under this condition, only B_2, C_2, D_{12}, and D_{21} in this realisation are not uniquely specified, and these matrices satisfy the following relations:*

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \begin{bmatrix} C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} 0 & b_2 \\ I & 0 \end{bmatrix} \left\{ \begin{bmatrix} \alpha(I) & \beta(I) \\ \gamma(I) & \delta(I) \end{bmatrix} - \begin{bmatrix} \alpha(0) & \beta(0) \\ \gamma(0) & \delta(0) \end{bmatrix} \right\} \begin{bmatrix} 0 & I \\ c_2 & 0 \end{bmatrix} \tag{49}$$

$$D_{12} D_{21} = d_{12} [\delta(I) - \delta(0)] d_{21}. \tag{50}$$

The significance of Thm. 4.5 is that it shows that the singularity structure of G relative to the optimal static output feedback problem is essentially uniquely determined, and equivalent to the singularity structure of g relative to the optimal constrained-structure dynamic output feedback problem. As such, we conclude that the "design opportunity" that exists in choosing K in A1 can neither be exploited or misused to fundamentally effect A3 or alter the answer to Q2. For all practical purposes then, in a given problem all possible manifestations of the reduction procedure are essentially equivalent.

4.3 Implementation

In this section Q4 and Q5 are answered. We begin by answering Q4:

Theorem 4.6 *Suppose (36) and (37) hold. Under this condition,*

$$\tau(\cdot) = \tau_s(\cdot). \tag{51}$$

Proof: To prove the theorem we write:

$$\begin{aligned}
\tau(K) = g \circ \kappa(K) &= g \circ (K \circ K) \\
&= \begin{bmatrix} I & 0 \end{bmatrix} \left\{ g \star \begin{bmatrix} K \circ K & [\,] \\ [\,] & [\,] \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} && (52) \\
&= \begin{bmatrix} I & 0 \end{bmatrix} \left\{ g \star \left(K \star \begin{bmatrix} K & [\,] \\ [\,] & [\,] \end{bmatrix} \right) \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} && (53) \\
&= \begin{bmatrix} I & 0 \end{bmatrix} \left\{ (g \star K) \star \begin{bmatrix} K & [\,] \\ [\,] & [\,] \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} && (54) \\
&= (g \star K) \circ K && (55) \\
&= G \circ K = \tau_s(K),
\end{aligned}$$

using Fact 2.3 to write (52) and (55), the definition of the star product and the properties of the empty matrix to write (53), and Fact 2.1 to write (54)...

Note that in the proof of Thm. 4.6 we have actually shown the more general result that the composition of two linear fractional transformations can be rewritten as a single linear fractional transformation through the use of the star product [15].

Having answered Q4, we now answer Q5.

Theorem 4.7 *Suppose K is given, and define internal stability of the interconnection of K with G in terms of the state-space realization of G given in (48).*[8] *Under these conditions, $\kappa(K)$ is internally stabilizing for g if and only if K is internally stabilizing for G.*

Proof: To prove the theorem we simply use Facts 2.2 and 21 to write:

$$\begin{bmatrix} x_g \\ x_{\kappa(K)} \end{bmatrix} = \begin{bmatrix} x_g \\ x_K \end{bmatrix} = \begin{bmatrix} x_G \end{bmatrix}, \tag{56}$$

in which case the theorem clearly holds...

Having answered Q1, Q2, Q4, and Q5, we have established the validity of the reduction procedure and also developed systematic methods for its execution. As such, the stage is now set for a thorough study of Q3, and this study will commence in Part 2 of this paper [3].

5 Conclusions

In this paper the optimal constrained-structure dynamic output feedback problem has been formulated in general terms, and a general method developed for reducing optimal constrained-structure dynamic output feedback problems to optimal static output feedback problems. This reduction procedure was concretely illustrated for nine special cases of the general optimal constrained-structure dynamic output feedback problem. Furthermore, it was shown in some detail how one of these special cases could be used to constrain the dynamic-structure of the controller to a P-I form.

We expect that the reader will have little difficulty formulating his own particular problems of interest (*e.g.*, lead/lag compensator design problems) in terms of one of the nine special cases of the general optimal constrained-structure dynamic output feedback problem considered here. We also expect that the reader will have little difficulty formulating special cases tailored to suit his tastes if the need arises to do so.

Of course, in this paper we have in no way solved the optimal constrained-structure dynamic output feedback problem; rather, we have only shown how to reduce this problem to the optimal static output feedback problem. However, in Part 2 of this series of papers a general theory of optimal static output feedback compensation is developed which, when combined with the numerical methods presented in Part 3 of this series of papers, should allow optimal constrained-structure dynamic output feedback compensation problems to be systematically solved in applications.

We feel strongly that the availability of a systematic procedure for solving optimal constrained-structure dynamic output feedback problems will do much to bridge the gap that currently exists between control theory and control practice. In particular, we expect that a methodology of this sort will enable a significant integration of the control law design and implementation processes [13]. To give but one piece of evidence in support of this claim, consider the problem of implementing a full-envelope control law for a nonlinear process [14]. Typically, linear controllers are designed at various operating points and then gain-scheduled in order to achieve full-envelope operation. In this process the dynamics of the linear controllers designed at each operating point are usually constrained apriori to a fixed form (*e.g.*, a P-I structure), and then the "gains" of the resulting linear controllers are appropriately scheduled (*i.e.*, proportional and integral gain schedules). The process of designing the constrained-structure control laws at each operating point would, of course, be facilitated by linear control law design methodologies of the type described above. Indeed, in the notation of Sec. 3, the controller "gain" would be nothing more than K, and the gain-schedule would amount to nothing more than a schedule of K! This situation should be contrasted to the situation for most currently available linear control law design methodologies, wherein κ is designed, instead of K, subject to no constraints on its dynamic structure. In this case it is often impossible to formulate a gain-schedule, as the dynamic structures of controllers designed at different operating points can vary significantly; instead, one has no choice but to formulate a "κ-schedule", and this is at best a difficult proposition for rather obvious reasons.

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Problem	Data	$\begin{bmatrix} \alpha(K) & \beta(K) \\ \gamma(K) & \delta(K) \end{bmatrix}$	K	Problem Type
Fixed-Order Problem (FoP)	$\dim(\alpha(K))$	$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$	$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$	4-Block
Fixed Dynamics/Controls Problem (FDCP)	$\begin{bmatrix} \alpha_o & \beta_o \end{bmatrix}$	$\begin{bmatrix} \alpha_o & \beta_o \\ K_1 & K_2 \end{bmatrix}$	$\begin{bmatrix} K_1 & K_2 \end{bmatrix}$	2-Block, Unmixed
Fixed Observations/Feedthroughs Problem (FOFP)	$\begin{bmatrix} \gamma_o & \delta_o \end{bmatrix}$	$\begin{bmatrix} K_1 & K_2 \\ \gamma_o & \delta_o \end{bmatrix}$		
Fixed Dynamics/Observations Problem (FDOP)	$\begin{bmatrix} \alpha_o \\ \gamma_o \end{bmatrix}$	$\begin{bmatrix} \alpha_o & K_1 \\ \gamma_o & K_2 \end{bmatrix}$	$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$	
Fixed Controls/Feedthroughs Problem (FCFP)	$\begin{bmatrix} \beta_o \\ \delta_o \end{bmatrix}$	$\begin{bmatrix} K_1 & \beta_o \\ K_2 & \delta_o \end{bmatrix}$		
Free Dynamics Problem (fDP)	$\begin{bmatrix} - & \beta_o \\ \gamma_o & \delta_o \end{bmatrix}$	$\begin{bmatrix} K & \beta_o \\ \gamma_o & \delta_o \end{bmatrix}$	K	1-Block
Free Controls Problem (fCP)	$\begin{bmatrix} \alpha_o & - \\ \gamma_o & \delta_o \end{bmatrix}$	$\begin{bmatrix} \alpha_o & K \\ \gamma_o & \delta_o \end{bmatrix}$		
Free Observations Problem (fOP)	$\begin{bmatrix} \alpha_o & \beta_o \\ - & \delta_o \end{bmatrix}$	$\begin{bmatrix} \alpha_o & \beta_o \\ K & \delta_o \end{bmatrix}$		
Free Feedthroughs Problem (fFP)	$\begin{bmatrix} \alpha_o & \beta_o \\ \gamma_o & - \end{bmatrix}$	$\begin{bmatrix} \alpha_o & \beta_o \\ \gamma_o & K \end{bmatrix}$		

Table 1: Constrained-structure dynamic output feedback examples

Problem	$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$
FoP	$\left[\begin{array}{c\|c\|cc} 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]$
FDCP	$\left[\begin{array}{ccc} \alpha_o & \beta_o & 0 \\ \hline 0 & 0 & I \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right]$
FOFP	$\left[\begin{array}{ccc} 0 & 0 & I \\ \hline \gamma_o & \delta_o & 0 \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right]$
FDOP	$\left[\begin{array}{c\|c\|cc} \alpha_o & 0 & I & 0 \\ \gamma_o & 0 & 0 & I \\ 0 & I & 0 & 0 \end{array}\right]$

Problem	$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$
FCFP	$\left[\begin{array}{c\|c\|cc} 0 & \beta_o & I & 0 \\ 0 & \delta_o & 0 & I \\ I & 0 & 0 & 0 \end{array}\right]$
fDP	$\begin{bmatrix} 0 & \beta_o & I \\ \gamma_o & \delta_o & 0 \\ I & 0 & 0 \end{bmatrix}$
fCP	$\begin{bmatrix} \alpha_o & 0 & I \\ \gamma_o & \delta_o & 0 \\ 0 & I & 0 \end{bmatrix}$
fOP	$\begin{bmatrix} \alpha_o & \beta_o & 0 \\ 0 & \delta_o & I \\ I & 0 & 0 \end{bmatrix}$
fFP	$\begin{bmatrix} \alpha_o & \beta_o & 0 \\ \gamma_o & 0 & I \\ 0 & I & 0 \end{bmatrix}$

Table 2: Static linear fractional decomposition examples

Problem	$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$ when $d_{22} = 0$
FoP	$\left[\begin{array}{cc\|c\|cc} a & 0 & b_1 & 0 & b_2 \\ 0 & 0 & 0 & I & 0 \\ \hline c_1 & 0 & d_{11} & 0 & d_{12} \\ \hline 0 & I & 0 & 0 & 0 \\ c_2 & 0 & d_{21} & 0 & 0 \end{array}\right]$
FDCP	$\left[\begin{array}{cc\|c\|c} a & 0 & b_1 & b_2 \\ \beta_o c_2 & \alpha_o & \beta_o d_{21} & 0 \\ \hline c_1 & 0 & d_{11} & d_{12} \\ \hline 0 & I & 0 & 0 \\ c_2 & 0 & d_{21} & 0 \end{array}\right]$
FOFP	$\left[\begin{array}{cc\|c\|c} a + b_2\delta_o c_2 & b_2\gamma_o & b_1 + b_2\delta_o d_{21} & 0 \\ 0 & 0 & 0 & I \\ \hline c_1 + d_{12}\delta_o c_2 & d_{12}\gamma_o & d_{11} + d_{12}\delta_o d_{21} & 0 \\ \hline 0 & I & 0 & 0 \\ c_2 & 0 & d_{21} & 0 \end{array}\right]$
FDOP	$\left[\begin{array}{cc\|c\|cc} a & b_2\gamma_o & b_1 & 0 & b_2 \\ 0 & \alpha_o & 0 & I & 0 \\ \hline c_1 & d_{12}\gamma_o & d_{11} & 0 & d_{12} \\ \hline c_2 & 0 & d_{21} & 0 & 0 \end{array}\right]$
FCFP	$\left[\begin{array}{cc\|c\|cc} a + b_2\delta_o c_2 & 0 & b_1 + b_2\delta_o d_{21} & 0 & b_2 \\ \beta_o c_2 & 0 & \beta_o d_{21} & I & 0 \\ \hline c_1 + d_{12}\delta_o c_2 & 0 & d_{11} + d_{12}\delta_o d_{21} & 0 & d_{12} \\ \hline 0 & I & 0 & 0 & 0 \end{array}\right]$
fDP	$\left[\begin{array}{cc\|c\|c} a + b_2\delta_o c_2 & b_2\gamma_o & b_1 + b_2\delta_o d_{21} & 0 \\ c_2\beta_o & 0 & \beta_o d_{21} & I \\ \hline c_1 + d_{12}\delta_o c_2 & d_{12}\gamma_o & d_{11} + d_{12}\delta_o d_{21} & 0 \\ \hline 0 & I & 0 & 0 \end{array}\right]$
fCP	$\left[\begin{array}{cc\|c\|c} a + b_2\delta_o c_2 & b_2\gamma_o & b_1 + b_2\delta_o d_{21} & 0 \\ 0 & \alpha_o & 0 & I \\ \hline c_1 + d_{12}\delta_o c_2 & d_{12}\gamma_o & d_{11} + d_{12}\delta_o d_{21} & 0 \\ \hline c_2 & 0 & d_{21} & 0 \end{array}\right]$
fOP	$\left[\begin{array}{cc\|c\|c} a + b_2\delta_o c_2 & 0 & b_1 + b_2\delta_o d_{21} & b_2 \\ \beta_o c_2 & \alpha_o & \beta_o d_{21} & 0 \\ \hline c_1 + d_{12}\delta_o c_2 & 0 & d_{11} + d_{12}\delta_o d_{21} & d_{12} \\ \hline 0 & I & 0 & 0 \end{array}\right]$
fFP	$\left[\begin{array}{cc\|c\|c} a & b_2\gamma_o & b_1 & b_2 \\ \beta_o c_2 & \alpha_o & \beta_o d_{21} & 0 \\ \hline c_1 & d_{12}\gamma_o & d_{11} & d_{12} \\ \hline c_2 & 0 & d_{21} & 0 \end{array}\right]$

Table 3: Static output feedback examples

Minimal Complexity Control Law Synthesis, Part 2: Problem Solution via H_2/H_∞ Optimal Static Output Feedback*

D.S. Bernstein† W.M. Haddad‡ C.N. Nett§

Abstract

In part 1 of this two-part paper [1] it was shown that a large class of fixed-structure control laws can be recast as static output feedback controllers for a suitably modified plant. Accordingly, we develop here a comprehensive theory for designing static output feedback controllers. Our results go beyond earlier work by addressing both $\mathbf{H}_2$ and $\mathbf{H}_\infty$ performance criteria and by accounting fully for all of the singularities in the problem formulation. The results are applied to the fixed-order problem (FoP) [1] to obtain a major unification of prior results, namely: the Bernstein-Haddad $\mathbf{H}_2/\mathbf{H}_\infty$ fixed-order dynamic compensator theory, the Glover-Doyle full-order $\mathbf{H}_\infty$ dynamic compensator theory, the Hyland-Bernstein $\mathbf{H}_2$ fixed-order dynamic compensator (optimal projection) theory, and the classical LQG theory.

1. Introduction

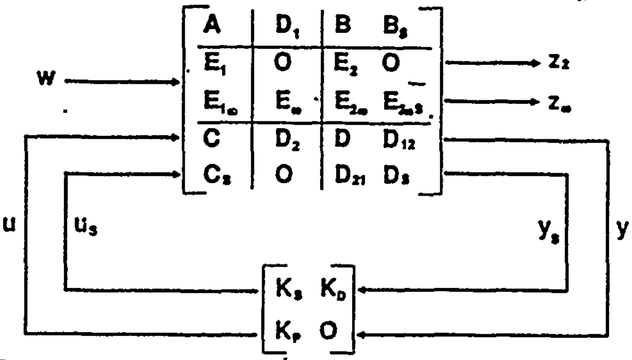
As discussed in part 1 of this two-part paper [1], it is often desirable in practice to design controllers of minimal compexity while meeting all other design specificatons. The principal contribution of [1] was the development of a systematic methodology for recasting a large class of constrained structure control laws as static output feedback design problems. This formulation includes such practical problems as the fixed-order problem in which only the compensator order is prespecified [2,3] and the fixed dynamics/controls problem (FDCP) in which the compensator poles are specified a priori.

The purpose of the present paper is to construct a general theory of static output feedback controller design which operates within the framework developed in [1]. The theory of static output feedback developed herein fully encompasses the prior literature [4–12] on static output feedback. However, we go beyond this literature in two principal ways. First, we consider not only the usual $\mathbf{H}_2$ performance measure but also an $\mathbf{H}_\infty$ criterion to permit simultaneous consideration of both $\mathbf{H}_2$ and $\mathbf{H}_\infty$ design aspects. This formulation thus permits the treatment of loop shaping and unstructured uncertainty as well as rms specifications. The $\mathbf{H}_\infty$ results are in the vein of recent Riccati equation solutions [13–17], while the combined $\mathbf{H}_2/\mathbf{H}_\infty$ theory was developed in [18,19].

The second extension of static output feedback theory developed herein involves a complete treatment of all singularities arising within the various feedback paths. In the usual setting [4–11], the measurements are assumed to be noise free while the controls are weighted. Hence this feedback path is partially singular. The dual path involving noisy measurements and unweighted controls can be treated in a similar manner (see [12]). The totally singular path involving nonnoisy measurements and unweighted controls is also included in this paper. As will be seen, this path is the key feature which allows us to address several of the problems defined in [1] including the fixed-order problem. The inclusion of the totally singular path also allows us to make connections with the literature on singular control theory [20–36] as well as singular estimation theory [37–41]. The authors are particularly indebted to Professor Y. Halevi whose expertise in singular problems [34,36,40,41] had a direct impact on the present work. Further exploration of the singular aspects of the static output feedback problem will be explored in future papers.

To demonstrate the utility of our approach we apply it to one of the problems considered in [1], namely the fixed-order problem, to obtain several results on dynamic compensator design. In a series of specialisations, we obtain the Bernstein-Haddad $\mathbf{H}_2/\mathbf{H}_\infty$ fixed-order dynamic compensator theory [18,19], the Glover-Doyle full-order $\mathbf{H}_\infty$ dynamic compensator theory [15], the Hyland-Bernstein $\mathbf{H}_2$ fixed-order dynamic compensator (optimal projection) theory [3], and the classical LQG theory.

One important ramification of the reduction to static output feedback developed in [1] is that it refocuses attention on the problem of stabilisation via static output feedback [42–49]. Although a complete solution to this longstanding problem is not yet available, it is clear from the results in [1] that a solution to the static output feedback problem would effectively provide a solution to a broad class of constrained structure stabilisation problems such as stabilisation via reduced-order dynamic compensation ([50,51]).



Notation.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$
I_r, $(\ )^{\mathrm{T}}$, tr	$r\times r$ identity, transpose, trace
$\otimes$, $\oplus$	Kronecker product, Kronecker sum
$n, m, m_S, \ell, \ell_S, d, q, q_\infty$	positive integers
$x, u, u_S, y, y_S, z_2, z_\infty$	$n, m, m_S, \ell, \ell_S, q, q_\infty$-dimensional vectors
$w(\cdot)$	d-dimensional disturbance signal
$A, B, B_S, C, C_S, K_P, K_D, K_S$	$n\times n, n\times m, n\times m_S, \ell\times n, \ell_S\times n, m\times\ell_S, m_S\times\ell, m_S\times\ell_S$ matrices
D_1, D_2, E_1, E_2	$n\times d, \ell_S\times d, q\times n, q\times m$ matrices
V_1, V_{12}, V_2	$D_1D_1^{\mathrm{T}},\ D_1D_2^{\mathrm{T}},\ D_2D_2^{\mathrm{T}},\ V_2>0$
R_1, R_{12}, R_2	$E_1^{\mathrm{T}}E_1,\ E_1^{\mathrm{T}}E_2,\ E_2^{\mathrm{T}}E_2,\ R_2\ge 0$
$\tilde{A}_0$	$A+BK_SPC_S+B_SK_SC+B_SK_DDK_PC$
$\tilde{A}, \hat{A}$	$\tilde{A}_0+B_SK_SC_S,\ A+BKC$
$\tilde{D},\ \tilde{E}$	$D_1+B_SK_DD_2,\ E_1+E_2KC_S$
$\tilde{R}$	$R_1+R_{12}KC_S+(KC_S)^{\mathrm{T}}R_{12}^{\mathrm{T}}+(KC_S)^{\mathrm{T}}R_2KC_S$
$\tilde{V}$	$V_1+V_{12}(B_SK_D)^{\mathrm{T}}+B_SK_DV_{12}^{\mathrm{T}}+B_SK_DV_2(B_SK_D)^{\mathrm{T}}$
$E_{1\infty}, E_{2\infty}, E_\infty, E_{2\infty S}$	$q_\infty\times n, q_\infty\times m, q_\infty\times d, q_\infty\times m_S$ matrices
$M,\ N$	$I_{q_\infty}-\gamma^{-2}E_\infty E_\infty^{\mathrm{T}},\ I_d-\gamma^{-2}E_\infty^{\mathrm{T}}E_\infty$
$V_{1\infty},\ V_{12\infty},\ V_{2\infty}$	$D_1N^{-1}D_1^{\mathrm{T}},\ D_1N^{-1}D_2^{\mathrm{T}},\ D_2N^{-1}D_2^{\mathrm{T}}$
$R_{01\infty}, R_{02\infty}, R_{1\infty}, R_{12\infty}, R_{2\infty}$	$E_\infty^{\mathrm{T}}M^{-1}E_{1\infty}, E_\infty^{\mathrm{T}}M^{-1}E_{2\infty}, E_{1\infty}^{\mathrm{T}}M^{-1}E_{1\infty}, E_{1\infty}^{\mathrm{T}}M^{-1}E_{2\infty}, E_{2\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$
$\tilde{V}_\infty$	$V_{1\infty}+V_{12\infty}(B_SK_D)^{\mathrm{T}}+B_SK_DV_{12\infty}^{\mathrm{T}}+B_SK_DV_{2\infty}(B_SK_D)^{\mathrm{T}}$
$\tilde{E}_\infty$	$E_{1\infty}+E_{2\infty}KC_S$

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† Staff Engineer, Government Aerospace Systems Division, Harris Corporation, Mail Stop 22/4842, Melbourne, Florida 32902, (407) 729-2140.

‡ Assistant Professor, Department of Mechanical and Aerospace Engineering, Florida Institute of Technology, Melbourne, Florida 32901, (407) 768-8000, Extension 7241.

§ Research Engineer, Control Systems Laboratory, GE Corporate Research and Development, Building KW, Room D219, P.O. Box 8, Schenectady, New York 12301, (518) 387-5539.

$\tilde{R}_{01\infty}$ $\quad E_\infty^{\mathrm{T}} M^{-1}\tilde{E}_\infty = R_{01\infty} + R_{02\infty}KC_S$

$\tilde{R}_\infty$ $\quad \tilde{E}_\infty^{\mathrm{T}} M^{-1}\tilde{E}_\infty = R_{1\infty} + R_{12\infty}KC_S + (KC_S)^{\mathrm{T}}R_{12\infty}^{\mathrm{T}} + (KC_S)^{\mathrm{T}}R_{2\infty}KC_S$

2. Underlying Formulation: Optimal $\mathbf{H}_2/\mathbf{H}_\infty$ Static Output Feedback

In this section we present the $\mathbf{H}_2/\mathbf{H}_\infty$ Static Output Feedback Problem which forms the basis for addressing all of the constrained structure control laws considered in [1].

$\mathbf{H}_2/\mathbf{H}_\infty$ Static Output Feedback Problem. Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_S u_S(t) + D_1 w(t), \quad t \in [0,\infty), \tag{2.1}$$
$$y(t) = Cx(t) + Du(t) + D_2 w(t), \tag{2.2}$$
$$y_S(t) = C_S x(t) \tag{2.3}$$

with feedback law

$$u_S(t) = K_S y_S(t) + K_D y(t), \tag{2.4}$$
$$u(t) = K_P y_S(t), \tag{2.5}$$

and performance variables

$$z_2(t) = E_1 x(t) + E_2 u(t), \tag{2.6}$$
$$z_\infty(t) = E_{1\infty} x(t) + E_{2\infty} u(t) + E_\infty w(t). \tag{2.7}$$

Then determine gains (K_S, K_D, K_P) satisfying the following design criteria:

i) the closed-loop system (2.1)–(2.5) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;

ii) for specified $\gamma > 0$, the $q_\infty \times d$ transfer function

$$H(s) \triangleq \tilde{E}_\infty(sI_n - \tilde{A})^{-1}\tilde{D} + E_\infty \tag{2.8}$$

from $w(\cdot)$ to $z_\infty(\cdot)$ satisfies the $\mathbf{H}_\infty$ norm constraint

$$\|H(s)\|_\infty \le \gamma; \tag{2.9}$$

iii) the $\mathbf{H}_2$ norm of the $q \times d$ transfer function

$$G(s) \triangleq \tilde{E}(sI_n - \tilde{A})^{-1}\tilde{D} \tag{2.10}$$

is minimised, i.e., minimise

$$J(K_S, K_D, K_P) \triangleq \|G(s)\|_2^2. \tag{2.11}$$

Remark 2.1. Using Parseval's theorem it is easy to show that the $\mathbf{H}_2$ performance criterion (2.11) is equivalent to the more familiar expression involving an averaged integral, i.e.,

$$J(K_S, K_D, K_P) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\{\int_0^t x^{\mathrm{T}}(s)R_1x(s) + 2x^{\mathrm{T}}(s)R_{12}u(s) + u^{\mathrm{T}}(s)R_2u(s)\, ds\}. \tag{2.12}$$

with $w(\cdot)$ interpreted as standard white noise.

Remark 2.2. Note that the controller architecture is quite general in that it includes three distinctly different types of loops. The first type involves feeding back nonnoisy measurements to weighted controls. This is the standard setting in the optimal output-feedback literature ([4–12]). In addition, we include the dual situation which involves feeding back noisy measurements to unweighted controls (see [12]). The third feedback loop involves feeding back nonnoisy measurements to unweighted controls which corresponds to a singular static output feedback control problem.

Several remarks are in order to clarify the problem statement and Figure 1. First note that the direct transmission between disturbances $w(\cdot)$ and $\mathbf{H}_2$ performance variables $z_2(\cdot)$ is assumed to be zero. This assumption entails no loss of generality since if this term were nonzero, then the transfer function between $w(\cdot)$ and $z_2(\cdot)$ would be nonstrictly proper yielding an infinite $\mathbf{H}_2$ cost.

Next note that the input and measurement matrices of the plant are partitioned as $[B \quad B_S]$ and $\begin{bmatrix} C \\ C_S \end{bmatrix}$, respectively. The partitioning of the measurements emphasises that the measurements $y(\cdot)$ are noisy (since $D_2D_2^{\mathrm{T}} > 0$) while the measurements $y_S(\cdot)$ are nonnoisy. The partitioning of the controls reflects the fact that the controls $u_S(\cdot)$ are unweighted in the $\mathbf{H}_2$ cost while $u(\cdot)$ may be weighted. Note that we only assume $E_2^{\mathrm{T}}E_2 \ge 0$ since, as will be seen, the $\mathbf{H}_\infty$ weighting $E_{2\infty}$ can compensate for the singularity in $E_2^{\mathrm{T}}E_2$. The controls $u(\cdot)$ can be distinguished as those controls to which the noisy measurements are not fed back. Because of the control and measurement partitioning, there are four possible feedback paths. However, in order for the $\mathbf{H}_2$ cost to be finite we require that the direct transmission in the controller between $y(\cdot)$ and $u(\cdot)$ be zero (see the (2,2) block of the feedback gain matrix). Finally, there are several elements in Figure 1 which will not be considered in the present paper. Specifically, $E_{2\infty S}$ entails a direct transmission path between measurement noise and the $\mathbf{H}_\infty$ performance variables $z_\infty(\cdot)$. This path, however, leads to complexities in the optimisation procedure which are deferred to a later treatment. Hence we set $E_{2\infty S} = 0$. Note that we do allow a direct transmission term E_∞ between disturbances and $\mathbf{H}_\infty$ performance variables. In addition, we set $D_{12} = 0, D_{21} = 0$, and $D_S = 0$ since these lead to algebraic loops (except when $D = 0$ and K_S is omitted, in which case D_S does not entail an algebraic loop). In the present paper, however, we shall only consider the direct feedthrough term D.

Although the $\mathbf{H}_2/\mathbf{H}_\infty$ Static Output Feedback Problem illustrated in Figure 1 is quite general, special cases can readily be discerned. For example, if either y, y_S, u, or u_S is absent, then corresponding feedback gains can also be assumed to be absent. In most cases the optimality conditions given below specialise in an obvious way. The only case requiring some caution is that in which y, y_S, u, and u_S are all present so that K_P and K_D are required but K_S is not utilised. Specialisations to this case must thus be noted separately.

Note that the closed-loop system (2.1)–(2.5) can be written as

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{D}w(t) \tag{2.13}$$

and that (2.11), or equivalently (2.12), becomes

$$J(K_S, K_D, K_P) = \lim_{t\to\infty} \mathbb{E}\left[x^{\mathrm{T}}(t)\tilde{R}x(t)\right]. \tag{2.14}$$

Note that the problem statement involves both $\mathbf{H}_2$ and $\mathbf{H}_\infty$ performance weights. In particular, the matrices R_1 and R_2 are the $\mathbf{H}_2$ weights for the state and control variables. Using the $\mathbf{H}_2$ performance variables (2.6) the cost (2.11) can be written as

$$J(K_S, K_D, K_P) = \lim_{t\to\infty} \mathbb{E}[z_2^{\mathrm{T}}(t)z_2(t)]. \tag{2.15}$$

For convenience we thus define $R_1 \triangleq E_1^{\mathrm{T}}E_1$ and $R_2 \triangleq E_2^{\mathrm{T}}E_2$ which appear in subsequent expressions. Note that $R_{12} \triangleq E_1^{\mathrm{T}}E_2$ is an $\mathbf{H}_2$ cross-weighting term which is included for greater design flexibility.

For the $\mathbf{H}_\infty$ performance constraint, the transfer function (2.8) involves weighting matrices $E_{1\infty}, E_{2\infty}$, and E_∞ for the state, control, and disturbance variables. The matrices $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$ are thus the $\mathbf{H}_\infty$ counterparts of the $\mathbf{H}_2$ weights R_1 and R_2. Here $M \triangleq I_{q_\infty} - \gamma^{-2}E_\infty E_\infty^{\mathrm{T}}$ arises due to the feedthrough term to the $\mathbf{H}_\infty$ performance criterion. As in the $\mathbf{H}_2$ case we allow an $\mathbf{H}_\infty$ cross-weighting term $R_{12\infty} \triangleq E_{1\infty}^{\mathrm{T}}M^{-1}E_{2\infty}$. Finally, the dual design feature of plant disturbance and sensor noise correlation is also permitted. As in [18], the disturbances $w(\cdot)$ are interpreted as white noise signals within the context of $\mathbf{H}_2$ optimality while, for the purpose of $\mathbf{H}_\infty$ attenuation, the very same disturbance signals have the alternative interpretation of deterministic L_2 unit norm functions.

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given controller (K_S, K_D, K_P) then the $\mathbf{H}_2$ performance (2.11) is given by

$$J(K_S, K_D, K_P) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{2.16}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}\left[x(t)x^{\mathrm{T}}(t)\right] \tag{2.17}$$

satisfies the $n \times n$ algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{2.18}$$

Remark 2.3. Using (2.16) and (2.18) it can be shown that the $\mathbf{H}_2$ cost criterion (2.11) can be written in terms of the L_2 norm of the impulse response of the closed-loop system. Specifically, writing $\tilde{Q}$ satisfying (2.18) as

$$\tilde{Q} = \int_0^\infty e^{\tilde{A}t}\tilde{V}e^{\tilde{A}^{\mathrm{T}}t}dt,$$

(2.16) becomes

$$J(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}) = \int_0^\infty \|\tilde{E}e^{\tilde{A}^{\mathrm{T}}t}\tilde{D}\|_{\mathrm{F}}^2 dt,$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. This stochastic performance measure can thus also be given a deterministic interpretation by letting $w(t)$ denote impulses at $t=0$.

The key step in enforcing the disturbance attenuation constraint (2.9) is to replace the algebraic Lyapunov equation (2.18) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance.

Lemma 2.1. Let $(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}})$ be given and assume there exists an $n \times n$ nonnegative-definite matrix Q satisfying

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \gamma^{-2}(\tilde{D}E_\infty^{\mathrm{T}} + Q\tilde{E}_\infty^{\mathrm{T}})M^{-1}(\tilde{D}E_\infty^{\mathrm{T}} + Q\tilde{E}_\infty^{\mathrm{T}})^{\mathrm{T}} + \tilde{V}. \tag{2.19}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilisable} \tag{2.20}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.21}$$

In this case,

$$\|H(s)\|_\infty \le \gamma \tag{2.22}$$

and

$$\tilde{Q} \le Q. \tag{2.23}$$

Consequently,

$$J(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}) \le \mathcal{J}(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q), \tag{2.24}$$

where

$$\mathcal{J}(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q) \triangleq \operatorname{tr} Q\tilde{R}. \tag{2.25}$$

Proof. See [18]. □

Remark 2.4. An equivalent form of (2.19) is given by

$$0 = (\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})Q + Q(\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})^{\mathrm{T}} + \gamma^{-2}Q\tilde{R}_\infty Q + \tilde{V}_\infty. \tag{2.26}$$

Lemma 2.1 shows that replacing (2.18) by (2.19) enforces the $\mathbf{H}_\infty$ disturbance attenuation constraint and yields an upper bound for the $\mathbf{H}_2$ performance criterion. That is, given a controller $K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}$ for which there exists a nonnegative-definite solution to (2.19), the actual $\mathbf{H}_2$ performance $J(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}})$ of the controller is guaranteed to be no worse then the bound given by $\mathcal{J}(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q)$. Hence, as in [18] we interpret $\mathcal{J}(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q)$ as an auxiliary cost and determine $(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q)$ which minimises $\mathcal{J}(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}}, Q)$, and thus provides an optimised bound for the actual $\mathbf{H}_2$ performance $J(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}})$ while enforcing the disturbance attenuation constraint (2.22).

3. Sufficient Conditions for $\mathbf{H}_\infty$ Disturbance Attenuation

In this section we state sufficient conditions for characterising static output feedback controllers guaranteeing closed-loop stability, constrained $\mathbf{H}_\infty$ disturbance attenuation, and an optimised $\mathbf{H}_2$ performance bound.

Theorem 3.1. Suppose there exist $n \times n$ nonnegative definite matrices Q, P satisfying

$$0 = (\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})Q + Q(\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})^{\mathrm{T}} + \gamma^{-2}Q\tilde{R}_\infty Q + \tilde{V}_\infty, \tag{3.1}$$

$$0 = (\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty} + \gamma^{-2}Q\tilde{R}_\infty)^{\mathrm{T}}P + P(\tilde{A} + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty} + \gamma^{-2}Q\tilde{R}_\infty) + \tilde{R}, \tag{3.2}$$

and

$$R_2 \otimes C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}} + \gamma^{-2}R_{2\infty} \otimes C_{\mathrm{S}}QPQC_{\mathrm{S}}^{\mathrm{T}} > 0, \tag{3.3}$$

$$C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}} > 0, \quad B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}} > 0, \qquad (3.4), (3.5)$$

where $(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}})$ are given by

$$K_{\mathrm{S}} = -(B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}})^{-1}B_{\mathrm{S}}^{\mathrm{T}}P[(\tilde{A}_0 + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})Q + Q(\tilde{A}_0 + \gamma^{-2}\tilde{D}\tilde{R}_{01\infty})^{\mathrm{T}} + \gamma^{-2}Q\tilde{R}_\infty Q + \tilde{V}_\infty]C_{\mathrm{S}}^{\mathrm{T}}(C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}})^{-1}, \tag{3.6}$$

$$K_{\mathrm{D}} = -(B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}})^{-1}B_{\mathrm{S}}^{\mathrm{T}}P[QC^{\mathrm{T}} + V_{12\infty} + \gamma^{-2}QR_{01\infty}^{\mathrm{T}}D_2^{\mathrm{T}}]V_{2\infty}^{-1}, \tag{3.7}$$

$$R_2K_{\mathrm{P}}C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}} + \gamma^{-2}R_{2\infty}K_{\mathrm{P}}C_{\mathrm{S}}QPQC_{\mathrm{S}}^{\mathrm{T}} = -[B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \gamma^{-2}R_{12\infty}^{\mathrm{T}}QP + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}}P]QC_{\mathrm{S}}^{\mathrm{T}} \tag{3.8}$$

Then $(\tilde{A}, \tilde{D})$ is stabilisable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the $\mathbf{H}_\infty$ disturbance attenuation constraint

$$\|H(s)\|_\infty \le \gamma, \tag{3.9}$$

and the $\mathbf{H}_2$ performance criterion $J(K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{P}})$ satisfies the bound

$$\|G(s)\|_2^2 \le \operatorname{tr} Q\tilde{R}. \tag{3.10}$$

Proof. See the journal version of this paper. □

Theorem 3.1 presents sufficient conditions for designing static output feedback controllers with a prespecified constraint on the $\mathbf{H}_\infty$ norm of the closed-loop transfer function of a given state-space model. These sufficient conditions comprise a system of one Riccati equation and one Lyapunov equation.

Remark 3.1. Note that in order to obtain explicit expressions for the gains K_{D} and K_{S} we require that $C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}}$ and $B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}}$ be positive definite. Unfortunately these conditions cannot be confirmed until Q and P are available. However, it can be shown that if $C_{\mathrm{S}}V_1C_{\mathrm{S}}^{\mathrm{T}}$ is positive definite then so is $C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}}$, and if $B_{\mathrm{S}}^{\mathrm{T}}R_1B_{\mathrm{S}}$ is positive definite then so is $B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}}$. These results follow from an analysis of the integral formulas for the Gramians Q and P. The interesting aspect of these conditions is that they correspond to first-order singularity hypotheses arising in the singular estimation and control literature [20–41].

Remark 3.2. It can be shown that the optimality condition corresponding to the gain K_{S} is given by

$$0 = C_{\mathrm{S}}QPB_{\mathrm{S}}. \tag{3.11}$$

Condition (3.11) provides the key to showing the relationship between the static output feedback problem and the fixed-order dynamic compensation problem treated in Section 5.

Remark 3.3. If the singular path is absent, i.e., the gain K_{S} is not utilised, then the expressions (3.7) and (3.8) for K_{S} and K_{P} should be replaced by

$$K_{\mathrm{D}} = (B_{\mathrm{S}}^{\mathrm{T}}PB_{\mathrm{S}})^{-1}B_{\mathrm{S}}^{\mathrm{T}}P[QC^{\mathrm{T}} + V_{12\infty} + \gamma^{-2}QR_{01\infty}^{\mathrm{T}}D_2 + \gamma^{-2}QC^{\mathrm{T}}K_{\mathrm{P}}^{\mathrm{T}}D_2^{\mathrm{T}}]V_{2\infty}^{-1}, \tag{3.12}$$

$$R_2K_{\mathrm{P}}C_{\mathrm{S}}QC_{\mathrm{S}}^{\mathrm{T}} + \gamma^{-2}R_{2\infty}K_{\mathrm{P}}C_{\mathrm{S}}QPQC_{\mathrm{S}}^{\mathrm{T}} = -[B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \gamma^{-2}R_{12\infty}^{\mathrm{T}}QP + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_1^{\mathrm{T}}P + \gamma^{-2}R_{02\infty}^{\mathrm{T}}D_2^{\mathrm{T}}K_{\mathrm{D}}^{\mathrm{T}}B_{\mathrm{S}}^{\mathrm{T}}P]QC_{\mathrm{S}}^{\mathrm{T}}, \tag{3.13}$$

where Q, P satisfy (3.1) and (3.2) respectively with $\tilde{A}$ replaced by $\tilde{A}_0$.

Remark 3.4. Note that if $R_2 \triangleq \alpha^2\hat{R}_2$ and $R_{2\infty} \triangleq \beta^2\hat{R}_2$, where α and β are real numbers such that $\alpha^2 + \beta^2 > 0$ and $\hat{R}_2 \in$

$\mathbb{R}^{m\times m}$ is positive definite, the gain expression (3.8) for K_P simplifies to

$$K_P = -\hat{R}_2^{-1}P_{a\infty}QC_S^T(\alpha^2 C_S Q C_S^T + \gamma^{-2}\beta^2 C_S QPQC_S^T)^{-1}, \quad (3.14)$$

where

$$P_{a\infty} \triangleq B^T P + R_{12}^T + \gamma^{-2}R_{12\infty}^T QP + \gamma^{-2}R_{02\infty}^T D_1^T P. \quad (3.15)$$

4. Standard Output Feedback

To draw connections with the standard static output feedback literature, a series of specialisations of Theorem 3.1 are now given. First, as in Remark 3.4 we let $R_2 = \alpha^2\hat{R}_2$ and $R_{2\infty} = \beta^2\hat{R}_2$. Next, we assume that $K_S = 0, K_D = 0$ so that only nonnoisy measurements are fed back to weighted controls which corresponds to the standard output feedback setting.

Case 1: H_2/H_∞ Standard Static Output Feedback Problem. First, we state the standard static output feedback problem with $R_2 = \alpha^2\hat{R}_2$ and $R_{2\infty} = \beta^2\hat{R}_2$. In this case, (3.8) becomes

$$K_P = -\hat{R}_2^{-1}P_{a\infty}QC_S^T(\alpha^2 C_S Q C_S^T + \gamma^{-2}\beta^2 C_S QPQC_S^T)^{-1}, \quad (4.1)$$

with Q, P given by

$$0 = (\hat{A} + \gamma^{-2}D_1\tilde{R}_{01\infty})Q + Q(\hat{A} + \gamma^{-2}D_1\tilde{R}_{01\infty})^T + \gamma^{-2}Q\tilde{R}_\infty Q + V_{1\infty}, \quad (4.2)$$

$$0 = (\hat{A} + \gamma^{-2}D_1\tilde{R}_{01\infty} + \gamma^{-2}Q\tilde{R}_\infty)^T P + P(\hat{A} + \gamma^{-2}D_1\tilde{R}_{01\infty} + \gamma^{-2}Q\tilde{R}_\infty) + \tilde{R}. \quad (4.3)$$

Considerable simplification is further achieved by deleting the direct feedthrough term to the H_∞ performance, i.e., $E_\infty = 0$. In this case

$$K_P = -\hat{R}_2^{-1}P_{a\infty}S(\alpha^2 QC_S^T + \gamma^{-2}\beta^2 QPQC_S^T)(\alpha^2 C_S QC_S^T + \gamma^{-2}\beta^2 C_S QPQC_S^T)^{-1}, \quad (4.4)$$

$$0 = (A - B\hat{R}_2^{-1}P_{a\infty}S\nu_\infty - \gamma^{-2}\beta^2 QR_{12}\hat{R}_2^{-1}P_{a\infty}S\nu_\infty)Q + Q(A - B\hat{R}_2^{-1}P_{a\infty}S\nu_\infty - \gamma^{-2}\beta^2 QR_{12}\hat{R}_2^{-1}P_{a\infty}S\nu_\infty)^T + \gamma^{-2}QR_{1\infty}Q + \gamma^{-2}\beta^2 Q\nu_\infty^T S^T P_{a\infty}^T\hat{R}_2^{-1}P_{a\infty}S\nu_\infty Q + V_{1\infty}, \quad (4.5)$$

$$0 = (A + \gamma^{-2}Q[R_{1\infty} - \beta^2\nu_\infty^T S^T P_{a\infty}^T\hat{R}_2^{-1}R_{12}^T - \beta^2\nu_{\infty\perp}^T S^T P_{a\infty}^T\hat{R}_2^{-1}P_{a\infty}\nu_\infty])^T P + P(A + \gamma^{-2}Q[R_{1\infty} - \beta^2\nu_\infty^T S^T P_{a\infty}^T\hat{R}_2^{-1}R_{12}^T - \beta^2\nu_{\infty\perp}^T S^T P_{a\infty}^T\hat{R}_2^{-1}P_{a\infty}\nu_\infty]) + R_1 - S^T P_{a\infty}^T\hat{R}_2^{-1}P_{a\infty}S + \nu_{\infty\perp}^T S^T P_{a\infty}^T\hat{R}_2^{-1}P_a S\nu_{\infty\perp}, \quad (4.6)$$

where

$$S \triangleq (\alpha^2 I_n + \gamma^{-2}\beta^2 QP)^{-1}, \quad (4.7)$$

$$\nu_\infty \triangleq (\alpha^2 QC_S^T + \gamma^{-2}\beta^2 QPQC_S^T)(\alpha^2 C_S QC_S^T + \gamma^{-2}\beta^2 C_S QPQC_S^T)^{-1}C_S, \quad (4.8)$$

and

$$\nu_{\infty\perp} \triangleq I_n - \nu_\infty. \quad (4.9)$$

Note that $\nu_\infty^2 = \nu_\infty$ so that ν_∞ is a projection.

Case 2: H_2/H_∞ Full-State Feedback Problem. Next we specialise the results of Case 1 to the full state feedback problem. For simplicity we assume $R_{12}, R_{12\infty}$, and E_∞ are zero and $C_S = I_n$ so that $\nu_\infty = I_n$ and $\nu_{\infty\perp} = 0$. In this case

$$K_P = -\hat{R}_2^{-1}B^T PS, \quad (4.10)$$

with Q, P given by

$$0 = (A - \Sigma PS)Q + Q(A - \Sigma PS)^T + \gamma^{-2}QR_{1\infty}Q + \gamma^{-2}\beta^2 QS^T P\Sigma PSQ + V_{1\infty}, \quad (4.11)$$

$$0 = (A + \gamma^{-2}QR_{1\infty})^T P + P(A + \gamma^{-2}QR_{1\infty}) + R_1 - S^T P\Sigma PS, \quad (4.12)$$

where $\Sigma \triangleq B\hat{R}_2^{-1}B^T$.

Remark 4.1. It is interesting to note that even in the full-state feedback case the H_2/H_∞ problem involves *two* coupled equations, one modified Riccati equation and one modified Lyapunov equation.

Case 3: Petersen-Khargonekar-Zhou H_∞-Full State Feedback Theory [13,14]. In this case we completely eliminate the H_2 aspect in the design problem in order to establish connections between our approach and the results obtained in [13,14]. We completely eliminate the H_2 contribution by letting R_1, R_{12} and α (and thus R_2) approach zero. This asymptotic procedure serves to completely eliminate the H_2 contribution to the problem so that the resulting setting corresponds to a pure H_∞ design problem. Once again for simplicity we set $R_{12\infty}$ and E_∞ to zero. In this case by defining a new variable $Z \triangleq \gamma^2 Q^{-1}$, (4.4) becomes

$$K = -R_{2\infty}^{-1}B^T Z \quad (4.13)$$

and (4.5) and (4.6) collapse to

$$0 = A^T Z + ZA + R_{1\infty} + \gamma^{-2}ZV_1 Z - Z\Sigma Z, \quad (4.14)$$

which appears in [13,14].

Case 4: H_2 Static Output Feedback Theory [5]. In this case we specialise the results of Case 1 to the H_2 setting. Specifically, by sufficiently relaxing the H_∞ disturbance attenuation constraint, i.e., $\gamma \to \infty$, then (4.4)–(4.6) reduce to

$$K_P = -R_2^{-1}P_a QC_S^T(C_S QC_S^T)^{-1}, \quad (4.15)$$

$$0 = (A - BR_2^{-1}P_a\nu)Q + Q(A - BR_2^{-1}P_a\nu)^T + V_1, \quad (4.16)$$

$$0 = A^T P + PA + R_1 - P_a^T R_2^{-1}P_a + \nu_\perp^T P_a^T R_2^{-1}P_a\nu_\perp, \quad (4.17)$$

$$P_a \triangleq B^T P + R_{12}^T,\ \nu \triangleq QC_S^T(C_S QC_S^T)^{-1}C_S,\ \nu_\perp \triangleq I_n - \nu.$$

This case corresponds to the result of [5] with the added features of correlated plant/measurement noise (V_{12}) and cross weighting (R_{12}). Finally, by setting $C_S = I_n$, the LQR problem is recovered.

5. Fixed-Order Dynamic Compensation

We now consider the fixed-order problem (FoP) as a special case of the singular static output feedback problem considered in Section 3.

H_2/H_∞ Dynamic Compensation Problem. Given the n_cth-order stabilisable and detectable plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \quad (5.1)$$

$$y(t) = Cx(t) + Du(t) + D_2 w(t), \quad (5.2)$$

determine an n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (5.3)$$

$$u(t) = C_c x_c(t), \quad (5.4)$$

which satisfies the design criteria (i)–(iii). Using the results of Part 1 of this paper [1], the dynamic output feedback problem is recast as an output feedback problem of the form

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_c(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_c(t)\end{bmatrix} + \begin{bmatrix}0 & B\\ I & 0\end{bmatrix}\begin{bmatrix}u_S(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}D_1\\ 0\end{bmatrix}w(t), \quad (5.5)$$

$$\begin{bmatrix}y_S(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}0 & I\\ C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_c(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & D\end{bmatrix}\begin{bmatrix}u_S(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}0\\ D_2\end{bmatrix}w(t), \quad (5.6)$$

with a static output feedback control law

$$\begin{bmatrix}u_S(t)\\ u(t)\end{bmatrix} = \begin{bmatrix}K_S & K_D\\ K_P & 0\end{bmatrix}\begin{bmatrix}y_S(t)\\ y(t)\end{bmatrix}. \quad (5.7)$$

Note that it follows from (5.5)–(5.7) that $u_S(t) = \dot{x}_c(t)$ and $y_S(t) = x_c(t)$ so that

$$\dot{x}_c(t) = K_S x_c(t) + K_D y(t), \quad (5.8)$$

$$u(t) = K_P x_e(t). \tag{5.9}$$

Note that the singular gain K_S in the static problem corresponds to the compensator dynamics matrix A_e while K_D and K_P correspond to the Kalman gain B_e and regulator gain C_e. We rewrite (5.5) and (5.6) as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_S(t) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w(t), \tag{5.10}$$

$$y(t) = [C \quad 0] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + [0 \quad D] \begin{bmatrix} u_s(t) \\ u(t) \end{bmatrix} + D_2 w(t), \tag{5.11}$$

$$y_S(t) = [0 \quad I] \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \tag{5.12}$$

i.e., (2.1)-(2.3) with A, B replaced by $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}$, etc.

Next, as in Section 4 we give a series of specialisations of Theorem 3.1 to draw connections with the dynamic output feedback literature. Once again for simplicity in stating the main results we assume $R_2 = \alpha^2 \hat{R}_2$ and $R_{2\infty} = \beta^2 \hat{R}_2$ as in Remark 3.4. Thus using (5.7) with (5.10)-(5.12) along with Theorem 3.1 we obtain the following special cases.

Case 1: Bernstein-Haddad $\mathbf{H_2/H_\infty}$ Fixed-Order Dynamic Compensation Theory [18,19]. This case addresses the problem of fixed-order dynamic compensation, i.e., n_e may be less than the order of the plant n. As in [3] this constraint leads to an oblique projection which introduces coupling between the operations of regulation and observation. Specifically, partition $(n+n_c) \times (n+n_c)$ $\tilde{Q}, \tilde{P}$ as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \tag{5.13}$$

so that (3.11) with $Q = \tilde{Q}$ and $P = \tilde{P}$ yields

$$0 = P_{12}^T Q_{12} + P_2 Q_2 \tag{5.14}$$

or, equivalently,

$$-P_2^{-1} P_{12}^T Q_{12} Q_2^{-1} = I_{n_e}. \tag{5.15}$$

Next, defining $\Gamma \triangleq -P_2^{-1} P_{12}^T, G \triangleq Q_{12} Q_2^{-1}$ it follows that $\Gamma G^T = I_{n_e}$ so that $\tau \triangleq G^T \Gamma$ is idempotent, i.e., $\tau^2 = G^T \Gamma G^T \Gamma = G^T \Gamma = \tau$. This is precisely the condition that arises in fixed-order dynamic compensation theory [3]. Next using Theorem 3.1 we can specialise K_S, K_D, and K_P to obtain

$$\begin{aligned} A_c = \Gamma[&A - B\hat{R}_2^{-1} P_a S - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D \hat{R}_2^{-1} P_a S \\ &+ \gamma^{-2}(Q R_{1\infty} + D_1 R_{01\infty} - D_1 R_{02\infty} \hat{R}_2^{-1} P_a S \\ &- Q_a V_{2\infty}^{-1} D_2 R_{01\infty} + Q_a V_{2\infty}^{-1} D_2 R_{02\infty} \hat{R}_2^{-1} P_a S)] G^T, \end{aligned} \tag{5.16}$$

$$B_e = \Gamma Q_a V_{2\infty}^{-1}, \quad C_e = R_2^{-1} P_a S G^T. \tag{5.17),(5.18}$$

Furthermore, algebraic manipulation of (3.1) and (3.2) shows that Q, P and Γ are given by (see Theorem 6.1 of [19])

$$\begin{aligned} 0 = &(A + \gamma^{-2} D_1 R_{01\infty}) Q + Q(A + \gamma^{-2} D_1 R_{01\infty})^T \\ &+ \gamma^{-2} Q R_{1\infty} Q + V_{1\infty} - Q_a V_{2\infty}^{-1} Q_a^T + \tau_\perp Q_a V_{2\infty}^{-1} Q_a^T \tau_\perp^T, \end{aligned} \tag{5.19}$$

$$\begin{aligned} 0 = &(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2} \hat{Q} S^T P_a^T \hat{R}_2^{-1} R_{12\infty}^T)^T P \\ &+ P(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2} D_1 R_{01\infty} - \gamma^{-2} \hat{Q} S^T P_a^T \hat{R}_2^{-1} R_{12\infty}^T) \\ &+ R_1 - S^T P_a^T \hat{R}_2^{-1} P_a S + \tau_\perp^T S^T P_a^T \hat{R}_2^{-1} P_a S \tau_\perp, \end{aligned} \tag{5.20}$$

$$\begin{aligned} 0 = &(A - B\hat{R}_2^{-1} P_a S + \gamma^{-2} Q[R_{1\infty} - R_{12\infty} \hat{R}_2^{-1} P_a S] \\ &+ \gamma^{-2} D_1 [R_{01\infty} - R_{02\infty} \hat{R}_2^{-1} P_a S]) \hat{Q} \\ &+ \hat{Q}(A - B\hat{R}_2^{-1} P_a S + \gamma^{-2} Q[R_{1\infty} - R_{12\infty} \hat{R}_2^{-1} P_a S] \\ &+ \gamma^{-2} D_1 [R_{01\infty} - R_{02\infty} \hat{R}_2^{-1} P_a S])^T \\ &+ \gamma^{-2} \hat{Q}(R_{1\infty} - R_{12\infty} \hat{R}_2^{-1} P_a S - S^T P_a^T \hat{R}_2^{-1} R_{12\infty}^T \\ &+ \beta^2 S^T P_a \hat{R}_2^{-1} P_a S) \hat{Q} + Q_a V_{2\infty}^{-1} Q_a^T - \tau_\perp Q_a V_{2\infty}^{-1} Q_a^T \tau_\perp^T, \end{aligned} \tag{5.21}$$

$$\begin{aligned} 0 = &(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} + \gamma^{-2} Q R_{1\infty} \\ &- \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty})^T \hat{P} + \hat{P}(A - Q_a V_{2\infty}^{-1} C + \gamma^{-2} D_1 R_{01\infty} \\ &+ \gamma^{-2} Q_a V_{2\infty}^{-1} D_2 R_{01\infty}) \\ &+ S^T P_a^T \hat{R}_2^{-1} P_a S - \tau_\perp^T S^T P_a^T \hat{R}_2^{-1} P_a S \tau_\perp, \end{aligned} \tag{5.22}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_e, \tag{5.23}$$

where

$$S \triangleq (\alpha^2 I_n + \beta^2 \gamma^{-2} \hat{Q} \hat{P})^{-1} \tag{5.24}$$

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\#, \quad \tau_\perp = I_n - \tau. \tag{5.25}$$

$$\begin{aligned} Q_a \triangleq QC^T + V_{12\infty}, P_a \triangleq [&B^T + \gamma^{-2} R_{02\infty}^T D_1^T \\ &+ \gamma^{-2} R_{12\infty}^T (Q + \hat{Q})] P + R_{12}^T. \end{aligned} \tag{5.26}$$

Considerable simplification is achieved when $V_{12}, R_{12}, R_{12\infty}$, and E_∞ are zero. In this case (5.16)-(5.22) become

$$A_c = \Gamma(A - Q\Sigma - \Sigma P S + \gamma^{-2} Q R_{1\infty}) G^T, \tag{5.27}$$

$$B_c = \Gamma Q C^T V_2^{-1}, \; C_c = \hat{R}_2^{-1} B^T P S G^T, \tag{5.28),(5.29}$$

$$0 = AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q\Sigma Q + \tau_\perp Q \Sigma Q \tau_\perp^T, \tag{5.30}$$

$$\begin{aligned} 0 = &(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty})^T P + P(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty}) \\ &+ R_1 - S^T P \Sigma P S + \tau_\perp^T S^T P \Sigma P S \tau_\perp, \end{aligned} \tag{5.31}$$

$$\begin{aligned} 0 = &(A - \Sigma P S + \gamma^{-2} Q R_{1\infty}) \hat{Q} + \hat{Q}(A - \Sigma P S + \gamma^{-2} Q R_{1\infty})^T \\ &\gamma^{-2} \hat{Q}(R_{1\infty} + \beta^2 S^T P \Sigma P S) \hat{Q} + Q\Sigma Q - \tau_\perp Q \Sigma Q \tau_\perp^T, \end{aligned} \tag{5.32}$$

$$\begin{aligned} 0 = &(A - Q\Sigma + \gamma^{-2} Q R_{1\infty})^T \hat{P} + \hat{P}(A - Q\Sigma + \gamma^{-2} Q R_{1\infty}) \\ &+ S^T P \Sigma P S - \tau_\perp^T S^T P \Sigma P S \tau_\perp, \end{aligned} \tag{5.33}$$

where $\Sigma = B\hat{R}_2^{-1} B^T$ and $\Sigma = C^T V_2^{-1} C$. These are precisely the results of Theorem 6.1 of [18].

Remark 5.1. Equations (5.10)-(5.23) present the complete solution to the fixed-order $\mathbf{H_2/H_\infty}$ four block problem. If full-order controllers are of interest then set $n_c = n$, so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (5.19)-(5.22) can be deleted and (5.22) becomes superfluous so that Theorem 4.1 of [19] is recovered. Alternatively, if a "pure" $\mathbf{H_\infty}$ *reduced-order controller without an* $\mathbf{H_2}$ *design* aspect is sought, then as in Section 4, Case 3, let the $\mathbf{H_2}$ weights approach zero, i.e., R_1, R_{12}, and $\alpha \to 0$ to recover Theorem 7.1 of [19] which addresses the "pure" $\mathbf{H_\infty}$–optimal fixed-order problem.

Case 2. Glover-Doyle $\mathbf{H_\infty}$-Full-Order Dynamic Output Feedback Theory [15–17]. In this case we specialise the results of Theorem 3.1 or, equivalently, the results of Case 1 to [15–17]. First set $n_c = n$. Next, in order to eliminate the $\mathbf{H_2}$ contribution, let R_1, R_{12} and α approach zero. In this case by defining a new variable $Y_\infty \triangleq \gamma^{-1}(Q + \hat{Q})^{-1}$, (5.16)-(5.22) become

$$\begin{aligned} A_c = A &- BR_{2\infty}^{-1} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1} - Q_a V_{2\infty}^{-1} C \\ &+ Q_a V_{2\infty}^{-1} D R_{2\infty} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1} + \gamma^{-2}([Q R_{1\infty} \\ &+ D_1 R_{01\infty} - D_1 R_{02\infty} R_{2\infty}^{-1} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1} \\ &- Q R_{12\infty} R_{2\infty}^{-1} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1} - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} \\ &+ Q_a V_{2\infty}^{-1} D_2 R_{02\infty} R_{2\infty}^{-1} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1}] \end{aligned} \tag{5.34}$$

$$B_c = Q_a V_{2\infty}^{-1}, \; C_e = R_{2\infty}^{-1} Y_{\infty a}(I_n - \gamma^{-2} Q Y_\infty)^{-1}. \tag{5.35, 5.36}$$

$$\begin{aligned} 0 = &(A + \gamma^{-2} D_1 R_{01\infty}) Q + Q(A + \gamma^{-2} D_1 R_{01\infty})^T \\ &+ V_{1\infty} + \gamma^{-2} Q R_{1\infty} Q - Q_a V_{2\infty}^{-1} Q_a^T, \end{aligned} \tag{5.37}$$

$$\begin{aligned} 0 = &(A + \gamma^{-2} D_1 R_{01\infty})^T Y_\infty + Y_\infty(A + \gamma^{-2} D_1 R_{01\infty}) \\ &+ R_{1\infty} + \gamma^{-2} Y_\infty V_{1\infty} Y_\infty - Y_{\infty a} R_{2\infty}^{-1} Y_{\infty a}, \end{aligned} \tag{5.38}$$

$$\rho(Q Y_\infty) < \gamma^2, \tag{5.39}$$

$$Y_{\infty a} \triangleq B^T Y_\infty + \gamma^{-2} R_{02\infty}^T D_1^T Y_\infty + R_{12\infty}^T. \tag{5.40}$$

Further simplification can be achieved by setting $V_{12}, R_{12\infty}$, and E_∞ to zero so that (5.34)-(5.39) become

$$A_c = A - Q\Sigma - \Sigma Y_\infty(I_n - \gamma^{-2} Q Y_\infty)^{-1} + \gamma^{-2} Q R_{1\infty}, \tag{5.41}$$

$$B_e = QC^T V_2^{-1}, \tag{5.42}$$

$$C_e = -R_{2\infty}^{-1} B^T Y_\infty(I_n - \gamma^{-2} Q Y_\infty)^{-1}, \tag{5.43}$$

$$0 = AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q\Sigma Q, \tag{5.44}$$

$$0 = A^T Y_\infty + Y_\infty A + R_{1\infty} + \gamma^{-2} Y_\infty V_1 Y_\infty - Y_\infty \Sigma Y_\infty, \tag{5.45}$$

$$\rho(QY_\infty) < \gamma^2, \tag{5.46}$$

which yield the results of [16] and [18].

Remark 5.2. Note that an alternative approach to getting Case 2 from Case 1 appears in [18]. Specifically, it was shown that by equalising the $\mathbf{H}_2/\mathbf{H}_\infty$ weights the coupled equation form of Case 1 with $n_c = n$ could to transformed into the two decoupled Riccati equation form given by (5.44) and (5.45). However, this approach does not eliminate the $\mathbf{H}_2$ aspect from the design.

Case 3: Hyland-Bernstein $\mathbf{H}_2$/Optimal Projection Theory [3]. In this case we specialise the results of Case 1 to the pure $\mathbf{H}_2$ fixed-order dynamic compensation problem considered in [3]. Specifically, by sufficiently relaxing the $\mathbf{H}_\infty$ disturbance attenuation constraint, i.e., $\gamma \to \infty$, then (5.16)–(5.22) become

$$A_c = \Gamma(A - BR_2^{-1}P_a - Q_aV_2^{-1}C + Q_aV_2^{-1}DR_2^{-1}P_a)G^T, \tag{5.47}$$

$$B_c = \Gamma Q_a V_2^{-1},\ C_c = -R_2^{-1}P_aG^T, \tag{5.48,5.49}$$

$$0 = AQ + QA^T + V_1 - Q_aV_2^{-1}Q_a^T + \tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T, \tag{5.50}$$

$$0 = A^TP + PA + R_1 - P_a^TR_2^{-1}P_a + \tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{5.51}$$

$$0 = (A - BR_2^{-1}P_a)\hat{Q} + \hat{Q}(A - BP_2^{-1}P_a)^T + Q_aV_2^{-1}Q_a^T - \tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T, \tag{5.52}$$

$$0 = (A - Q_aV_2^{-1}C)^T\hat{P} + \hat{P}(A - Q_aV_2^{-1}C) + P_a^TR_2^{-1}P_a - \tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{5.53}$$

$$Q_a \triangleq QC^T + V_{12},\ P_a \triangleq B^TP + R_{12}^T. \tag{5.54}$$

This case corresponds to the results of [3] with the added features of correlated plant/measurement noise (V_{12}), cross weighting (R_{12}), and a direct transmission term (D) in the plant dynamics.

Finally, to recover the standard LQG theory which involves full-order controllers set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case (5.47)–(5.49) reduce to the standard LQG gains and the last term in (5.50)–(5.51) is zero yielding the standard observer and regulator Riccati equations. Furthermore, note that (5.53) and (5.54) are superfluous.

6. Numerical Algorithms

The results of [1] and the previous sections immediately suggest that a large class of constrained structure control laws can be designed by numerically solving the optimality condition for static output feedback. Considerable effort has been devoted to solving these optimality conditions, and a thorough survey can be found in [53]. There exist, however, serious unresolved problems concerning startup (i.e., the ability to find an initial stabilising controller), convergence, and global optimality (i.e., distinguishing the global minimum from local extremals). To this end, we consider an alternative approach based upon homotopy (i.e., continuation) algorithms. The theory and implementation of homotopy algorithms is well developed [54-59]. Because of the flexibility and global nature of such algorithms, they appear to be well-suited to addressing the issues of startup, convergence, and global optimality for the static output feedback problems. Results will appear in [60].

7. Further Extensions

1) decentralised and hierarchical feedback structures [52];

2) constrained structure discrete-time and sampled- data control laws [61,62] including both single and multirate architectures;

3) structured real-valued parameter uncertainty [63,66];

4) pole placement within prescribed regions [67];

5) nonlinear dynamic compensation via nonlinear static output feedback.

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H_2/H_∞ Controller Synthesis: Illustrative Numerical Results via Quasi-Newton Methods

Dorian R. Seinfeld
Department of Mechanical and Aerospace Engineering
Florida Institute of Technology
Melbourne, FL 32901

Wassim M. Haddad[1]
Department of Mechanical and Aerospace Engineering
Florida Institute of Technology
Melbourne, FL 32901

Dennis S. Bernstein[1]
Harris Corporation
Government Aerospace Systems Division, MS 22/4847
Melbourne, FL 32902

Carl N. Nett[2]
Department of Aerospace Engineering
Georgia Institute of Technology
Atlanta, GA 30332

Abstract

A quasi-Newton search technique is applied to the fixed-structure H_2/H_∞ control problem introduced in [1,2]. This quasi-Newton technique is based upon the BFGS variable metric algorithm developed in [3] for unconstrained nonlinear optimization. A barrier modification is used to enforce closed-loop stability, while a discrete homotopy is used to enforce an H_∞ constraint. Numerical results for three different illustrative problems are presented.

1. Introduction

In a recent paper [1] it was shown that a large class of fixed-structure control laws can be recast as static output feedback controllers for a suitably modified plant. Accordingly, a general theory was constructed in [2] for static output feedback control design. The results of [2] consider the usual H_2 performance measure and also an H_∞ criterion to permit simultaneous consideration of both H_2 and H_∞ design aspects. This formulation thus permits the treatment of loop shaping and unstructured uncertainty as well as nominal rms specifications. In this paper we present a numerical method for solving the systems of coupled Riccati equations that emanate from the theory developed in [1,2]. Results are presented for three benchmark problems.

2. Numerical Procedure

The numerical procedure utilizes software developed in [3] for unconstrained nonlinear optimization based on the BFGS variable metric algorithm combined with a standard LQG software package. The controller synthesis algorithm utilizes the BFGS quasi-Newton technique [4-6] to minimize the cost as a function of the controller gains. A number of modifications to the code have been implemented to address the controller synthesis problem developed in [2]. Specifically, a barrier modification is used to enforce stability, while a discrete homotopy procedure is used to enforce the H_∞ constraint. The starting point for the algorithm is usually chosen to be an LQG controller which is then transformed into the desired reduced-order/decentralized/H_∞-constrained solution.

As shown in [2], the cost functional depends on both the state covariance and its dual which are solutions of a modified Riccati equation and a modified Lyapunov equation. These coupled equations are solved using standard LQG software, as well as a Newton-Raphson method [7] developed for solving the H_∞ Riccati equation that enforces the H_∞ norm constraint. The BFGS algorithm is integrated with the LQG software to provide updated gains at each computation step (see Figure 1).

3. Design Examples

The first example is an H_2 optimal decentralized control problem involving a pair of simply supported Euler-Bernoulli flexible beams interconnected by a spring [8]. Each beam has one rate sensor and one force actuator. The 8th-order interconnected model includes two vibration modes in each beam. This problem was addressed in [8] and solved via homotopic continuation methods in [9]. As in [8,9], we obtain two decentralized 4th order compensators; one for each beam. The final H_2 cost is 11.9450 reflecting a slight improvement over the results of [8,9]. This result was obtained by optimizing both compensators simultaneously in a single iteration loop initialized with LQG gains. In contrast, the method of [8,9] requires a sequential design of each subcontroller.

The second design example is adopted from [10] and involves four coupled rotating disks with noncolocated sensors and actuators. The plant is of eighth order and has two neutrally stable poles and three right half plane zeros. The problem data are given in [10]. A series of mixed-norm H_2/H_∞ results was obtained for both full- and reduced-order controllers as well as pure H_2 reduced-order results. The mixed-norm full-order problem was solved via homotopic continuation methods in [10] for a net H_∞ performance improvement of 8.7 dB over the initial LQG controller. For this example, we were able to obtain an H_∞ disturbance attenuation constraint of 0.29 corresponding to an actual H_∞ performance of 0.28. This corresponds to an H_∞ performance improvement of 13.8 dB over the LQG controller (see Figure 2), or 5.1 dB over the full-order result obtained in [10]. We also obtained reduced-order compensators for the mixed-norm problem for the above example. For an H_∞ disturbance attenuation constraint of .44 we obtained an actual H_∞ performance of .42 with a 2nd-order compensator (see Figure 2). Controller characteristics for the reduced-order mixed-norm problem are given in Table 1 for several values of disturbance attenuation constraint γ.

For the same example, we also considered reduced-order H_2 designs as in [11] which compares the Optimal Projection methodology to five other methods to obtain stable reduced-order designs for varying values of the disturbance noise intensity q_2. Using stable initial designs we obtained stable designs for all of the cases considered in [11]. Table 2 shows the results for the second and fourth-order controller design along with their corresponding H_2 costs.

The final example considered is a model of a GE T700 turboshaft engine coupled to a helicopter rotor system [12]. This problem addresses the multi-input multi-output (MIMO) dynamic coordination of both fuel and engine compressor geometry in both an optimal H_2 and H_∞ sense. For this example we addressed the mixed sensitivity/complementary sensitivity minimization problem for the 95% N_g MIMO data presented in [12]. This data represents the simultaneous control of power turbine speed (in RPM) and inter-turbine gas temperature (in deg. R) for a linearized model operating at a 95% power level (% N_g). As reported in [12], the design specification on the loop gain transfer functions requires a crossover frequency no greater than 10 rad/sec with a low frequency performance barrier of 20 dB at frequencies at or below 1 rad/sec. In order to meet the design requirements we solved the mixed performance/robustness problem of the form (see [13] for further details)

$$\mathcal{F}(G,K) = \left\| \begin{matrix} W_1 S \\ W_2 T \end{matrix} \right\|_\infty$$ with weights (see Figure 3) $W_1 = w_1 I_2$, $W_2 = w_2 I_2$,

where $w_1(s) = (s^2 + 4s + 0.001)/(s^2 + 0.65)$, $w_2(s) = (60s + 500)/(s + 500)$. Using these weighting functions on the sensitivity transfer function matrix S and complementary-sensitivity transfer function matrix T our design satisfied the performance objectives as illustrated in Figures 4, 5 and 6.

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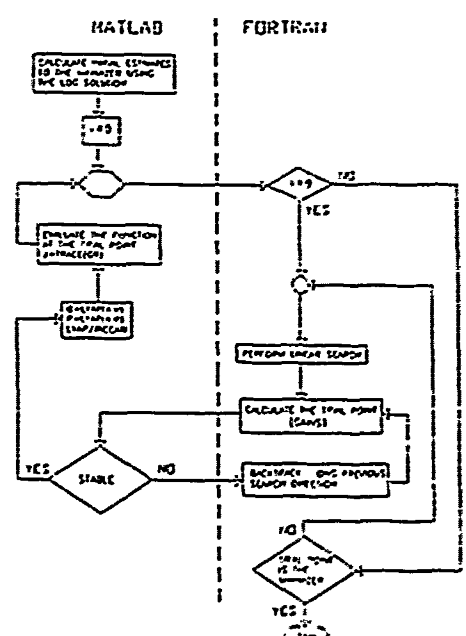

Figure 1

[1] Supported by the AFOSR. [2] Supported by NSF, GE CRD, and ONR.

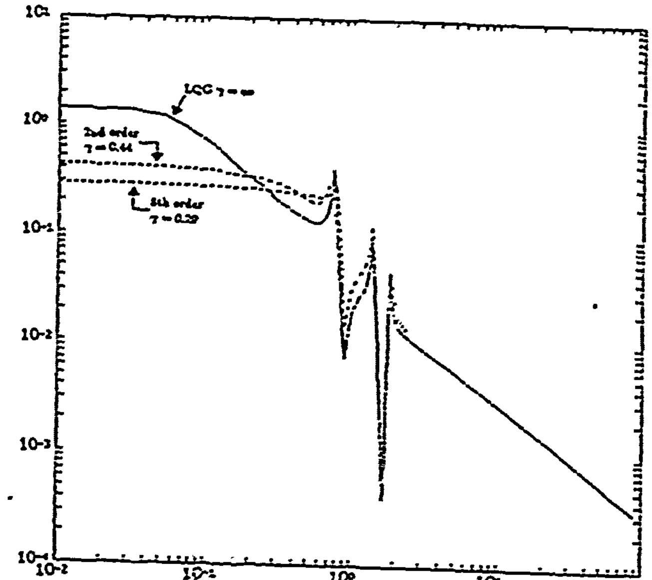

Figure 2.

Table 1.
2nd Order Controller Results
for 8th Order Coupled Rotating Disk Problem

H_∞ Attenuation Constraint γ	Actual H_∞ Attenuation $\|H(s)\|_\infty$	H_2 Performance Bound	Actual H_2 Performance
∞ (LQG)	1.380	—	.1433
2	1.203	.1655	.1499
1.5	1.078	.1803	.1550
1.0	.8538	.2237	.1747
.9	.7931	.2426	.1841
.8	.7237	.2697	.1979
.7	.6533	.3108	.2171
.440	.4257	.6829	.3579

Table 2. (q_2=2000)

Controller Order N_c	H_2 Cost (BFGS)	H_2 Cost from [11]
4	265.4	288.6
2	284.9	900.8

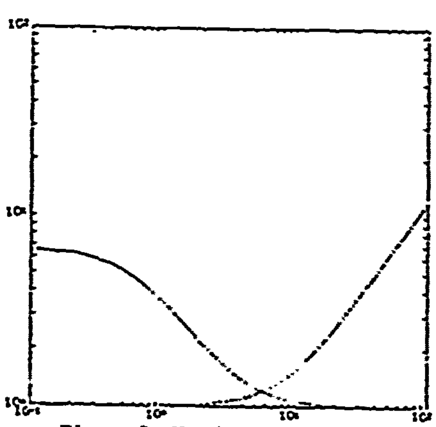

Figure 3 Weightings on S & T

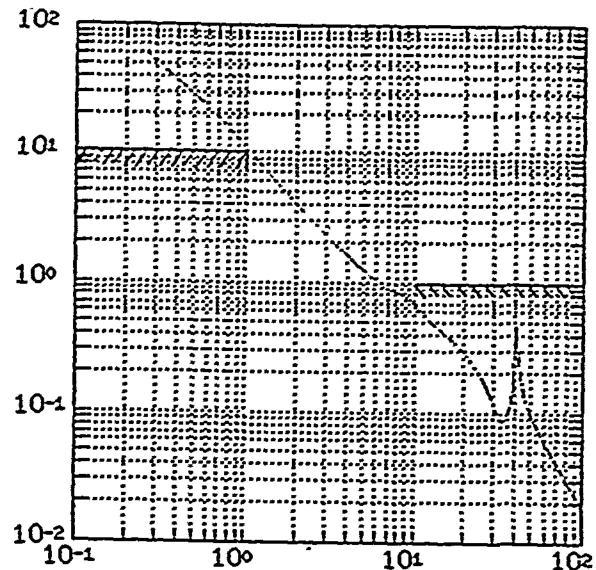

Figure 4 S.V. Plot Loop Gain (GK)

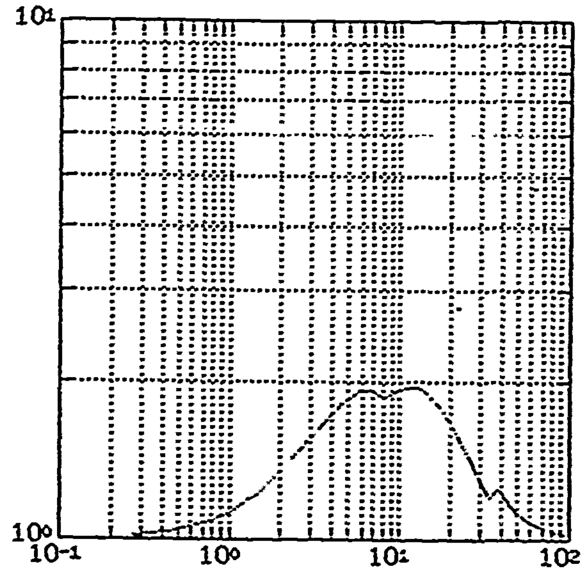

Figure 5 S.V. Plot Closed-Loop (T)

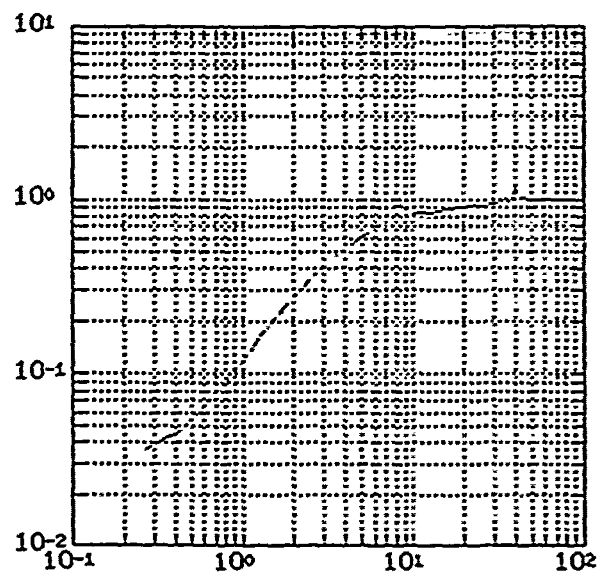

Figure 6 S.V. Plot Sensitivity (S)

Appendix G

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The Singular Linear-Quadratic Regulator Problem and the Goh-Riccati Equation

by

Dennis S. Bernstein
Harris Corporation
MS 22/4842
Melbourne, FL 32902

Vera Zeidan
Department of Mathematics
Michigan State University
East Lansing, MI 48824

Abstract

It is well known that Goh's transformation can be used to transform the singular linear-quadratic regulator problem into a nonsingular problem. The solution to this transformed problem is then given by a variant of the standard Riccati equation, called here the Goh-Riccati equation. The goal of this paper is to study the role of the Goh-Riccati equation in solving the singular linear-quadratic regulator problem. We approach this problem from several distinct points of view. These include perturbation methods and generalized Legendre-Clebsch conditions, as well as fixed-structure optimization.

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1. Introduction

Singular optimal control problems have been intensely studied for almost three decades now. Since the original analysis of Lawden's spiral [1–3], there has been significant interest in searching for necessary and sufficient conditions that establish the status of singular trajectories. References [1–83] are representative of research in this area, although this bibliography is far from exhaustive. The present paper is concerned with one particular, but fundamental, singular optimal control problem, namely, the time-invariant, infinite-horizon, singular linear-quadratic regulator problem without control constraints. This problem can be simply stated as follows.

Singular Linear-Quadratic Regulator Problem. Given the linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{1.1}$$

find a control $u(\cdot)$ that minimizes

$$J(u(\cdot), x_0) \triangleq \int_0^\infty x^{\mathrm{T}} R x \, \mathrm{d}t. \tag{1.2}$$

In general, solutions to this problem involve both impulsive controls and smooth feedback controls. For reasons discussed below, this paper is solely concerned with the latter.

There are both mathematical and engineering reasons for considering singular problems. Mathematically, attempts to solve singular problems have contributed to the development of a wide range of solution techniques of intrinsic value. These include transformation methods, perturbation techniques, higher-order optimality theory, and geometric methods.

From an engineering point of view, singular problems serve as idealized approximations to cheap control problems in which control effort is not treated as a limiting factor in feedback design. Solutions to such problems assist the control designer in assessing tradeoffs between available control effort and achievable system performance. For example, as shown in [35,36], the presence of right half plane plant zeros limits achievable performance in spite of unbounded controller effort. Analogous remarks apply to the dual problem of noiseless sensors.

The purpose of this paper is to study feedback control for the Singular Linear-Quadratic Regulator Problem by means of several quite differenct techniques, but with a unified focus. That focus is the Goh-Riccati equation, a variant of the standard Riccati equation which follows from applying Goh's transformation [18,19] to the singular problem to obtain a nonsingular problem. In

the present paper we derive the Goh-Riccati equation by means of four distinct methods, namely, Goh's transformation, perturbation methods, generalized Legendre-Clebsch conditions, and fixed-structure optimization.

The first derivation of the Goh-Riccati equation given in this paper is via Goh's transformation. Details of the transformation can be found in [44], pp. 81–85, while the Goh-Riccati differential equation appears as equation (4.5.27) of [44]. Generalizations of Goh's transformation were developed in [67] to obtain higher order extensions of the algebraic Goh-Riccati equation. As pointed out in [67], Goh's transformation was rediscovered in [31].

The second derivation of the Goh-Riccati equation is by means of a perturbation technique. For the algebraic equation case this approach was first carried out in [30]. This technique was subsequently generalized in [74] to obtain higher order extensions of the Goh-Riccati equation. Neither paper [30, 74], however, recognize connections with Goh's transformation.

The third derivation of the Goh-Riccati equation given herein is by means of higher order optimality conditions. The study of singular control problems by means of higher order optimality conditions has been quite extensive. Generalized Legendre-Clebsch conditions were developed in [12,21] and numerous other references. To our knowledge, however, connections between the generalized Legendre-Clebsch conditions and the Goh-Riccati equation have not been established. In the present paper we close this gap by deriving the Goh-Riccati equation be means of the Legendre-Clebsch conditions.

The fourth approach to deriving the Goh-Riccati equation considered in this paper is by means of fixed-structure optimization. By fixed-structure optimization we are referring to the method in which the feedback controller structure is fixed prior to the optimization [84–86]. The controller parameters, such as feedback gains, are then determined by the optimization procedure. There are two principal reasons for applying fixed-structure optimization to the Singular Linear-Quadratic Regulator Problem. Firstly, the fixed-structure approach allows us to focus directly on control signals of a specific class such as feedback controls. In a cheap control problem, feedback controls, as distinct from impulsive controls, are necessarily smooth and finite, and thus can be viewed as the singular portion of the optimal control. And secondly, the fixed-structure approach allows control designers to enforce control-structure constraints such as controller order, decentralization, etc. [85,86].

Recently, the fixed structure approach has been applied to control problems with singular control weighting and singular measurement noise [75,76,83]. These papers, however, did not establish connections with the Goh-Riccati equation. Thus our goal in the present paper is to derive the Goh-Riccati equation by means of the fixed-structure approach. Ultimately, we intend to merge the Goh-Riccati equation with the results of [75,76,83] as well as [84–86] to obtain a general theory of fixed-structure control for both singular and nonsingular problems.

The plan of the paper is as follows. In Section 2 we present a self-contained derivation of the nonsingular linear-quadratic regulator problem under weak assumptions. The fixed structure approach is used in this section to illustrate the technique that will subsequently be applied to the singular problem in Section 6. Goh's transformation is utilized in Section 3 to transform the singular problem into a nonsingular problem. The results of Section 2 are then applied to the transformed problem to obtain the Goh-Riccati equation. In Section 4 the singular problem is perturbed to obtain a nonsingular approximate problem. By assuming an asymptotic expansion for the solution to the standard Riccati equation, the Goh-Riccati equation is shown to characterize the zeroth-order term in the expansion. The generalized Legendre-Clebsch conditions are applied to the singular problem in Section 5. By introducing a transformation involving the solution to the Hamiltonian system, we obtain the Goh-Riccati equation. Finally, in Section 6 we return to the fixed-structure approach which is now applied to the singular problem. By introducing a series of transformations, we arrive at the Goh-Riccati equation.

2. The Nonsingular Linear-Quadratic Regulator Problem via Fixed-Structure Optimization

To begin, we study the nonsingular linear-quadratic regulator problem using fixed-structure techniques. Besides establishing notation, assumptions, and techniques that will be used in later sections, we shall focus on details that are not normally addressed in standard treatments of this problem.

Linear-Quadratic Regulator Problem. Given the linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{2.1}$$

with feedback control law

$$u = Kx, \tag{2.2}$$

determine $K \in \mathbb{R}^{m\times n}$ that minimizes

$$J(K,x_0) \triangleq \int_0^\infty [x^{\mathrm{T}}R_1x + 2x^{\mathrm{T}}R_{12}u + u^{\mathrm{T}}R_2u]\ \mathrm{d}t. \tag{2.3}$$

Several comments are in order. First, in (2.1) we mean $x = x(t) \in \mathbb{R}^n$ and $u = u(t) \in \mathbb{R}^m$, $t \geq 0$, with given initial condition $x(0) = x_0$. We assume throughout the paper that $m \leq n$. Furthermore, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $R_1 = R_1^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ (superscript "T" denotes transpose), $R_{12} \in \mathbb{R}^{n\times m}$, and $R_2 \in \mathbb{R}^{m\times m}$. The matrix $R_a \triangleq \begin{bmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$ is assumed to be nonnegative definite, and R_2 is assumed to be positive definite. Hence R_1 and $R_1' \triangleq R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}}$ are both nonnegative definite. Note that the admissible controls (2.2) are constrained a priori to be of the form of full-state feedback.

For arbitrary $K \in \mathbb{R}^{m\times n}$, the closed-loop system (2.1), (2.2) has the form

$$\dot{x} = \tilde{A}x, \quad x(0) = x_0, \tag{2.4}$$

where $\tilde{A} \triangleq A + BK$, while the cost (2.3) is given by

$$J(K,x_0) = \int_0^\infty x^{\mathrm{T}}\tilde{R}x\ \ \mathrm{d}t, \tag{2.5}$$

where

$$\tilde{R} \triangleq R_1 + R_{12}K + K^{\mathrm{T}}R_{12} + K^{\mathrm{T}}R_2K.$$

Since $\tilde{R} = \begin{bmatrix} I \\ K \end{bmatrix}^{\mathrm{T}} R_a \begin{bmatrix} I \\ K \end{bmatrix}$, where I or I_n denotes the $n \times n$ identity matrix, it follows that $\tilde{R}$ is nonnegative definite. Hence for arbitrary $K \in \mathbb{R}^{m\times n}$, $0 \leq J(K,x_0) \leq \infty$. Since the closed-loop system (2.4) has the solution

$$x(t) = e^{\tilde{A}t}x_0, \quad t \geq 0, \tag{2.6}$$

the cost (2.5) becomes

$$J(K,x_0) = \int_0^\infty x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}t}\tilde{R}e^{\tilde{A}t}x_0\ \mathrm{d}t. \tag{2.7}$$

Remark 2.1. A more general formulation of the regulator problem is obtained by averaging or summing $J(K,x_0)$ over values of x_0 [84]. It can be seen that this leads to (2.8) with $x_0x_0^{\mathrm{T}}$ replaced by an $n \times n$ nonnegative-definite matrix V. If V has rank 1, then the original problem is recovered. Allowing the rank of V to be greater than 1 provides a more general problem formulation that captures the possibility that x_0 belongs to a given subspace.

The expression (2.7) for the cost suggests the importance of the nonnegative-definite matrix

$$P \triangleq \int_0^\infty e^{\tilde{A}^{\mathrm{T}} t} \tilde{R} e^{\tilde{A} t} \, dt \tag{2.8}$$

when this integral exists, that is, is finite. If (2.8) exists, then (2.7) can be written as

$$J(K, x_0) = x_0^{\mathrm{T}} P x_0. \tag{2.9}$$

Of course, $J(K, x_0)$ may be finite even if (2.8) does not exist. It is well known that P given by (2.8) is intimately related to the Lyapunov equation

$$0 = \tilde{A}^{\mathrm{T}} P + P \tilde{A} + \tilde{R}. \tag{2.10}$$

The following result characterizes the class of nonnegative-definite solutions to (2.10).

Lemma 2.1. Let $K \in \mathbb{R}^{m \times n}$. Then the following statements are equivalent:

i) P given by (2.8) exists.

ii) There exists a nonnegative-definite solution to (2.10).

iii) The observable subspace of $(\tilde{R}, \tilde{A})$ is contained in the asymptotically stable subspace of $\tilde{A}$.

In this case the following properties are satisfied:

iv) P given by (2.8) satisfies (2.10).

v) P given by (2.8) is the only nonnegative-definite solution to (2.10) such that

$$\operatorname{rank} P = \operatorname{rank} \begin{bmatrix} \tilde{R} \\ \tilde{R}\tilde{A} \\ \vdots \\ \tilde{R}\tilde{A}^{n-1} \end{bmatrix}. \tag{2.11}$$

vi) Every nonnegative-definite solution to (2.10) is of the form $P + P_0$, where P is given by (2.8) and P_0 is an arbitrary nonnegative-definite matrix satisfying

$$0 = \tilde{A}^{\mathrm{T}} P_0 + P_0 \tilde{A}. \tag{2.12}$$

vii) If P_0 is a nonnegative-definite solution to (2.12), then $\operatorname{rank} P_0 = \beta + 2\gamma$, where $0 \le \beta \le \ell$, $0 \le \gamma \le \frac{q}{2}$, ℓ is the number of Jordan blocks of $\tilde{A}$ associated with the eigenvalue zero, and q is the number of Jordan blocks of $\tilde{A}$ associated with nonzero eigenvalues having zero real part.

viii) P given by (2.8) is the only nonnegative-definite solution to (2.10) if and only if every eigenvalue of $\tilde{A}$ has nonzero real part.

ix) $\tilde{A}$ is asymptotically stable if and only if $(\tilde{R}, \tilde{A})$ is detectable.

x) P given by (2.8) is positive definite if and only if $(\tilde{R}, \tilde{A})$ is observable.

Proof. The result follows from Theorems 3.1b, 3.2, 4.1b, and 4.2b of [88]. □

The previous result focused on the class of nonnegative-definite solutions to (2.10). In general, however, there may exist indefinite or nonsymmetric solutions. Such solutions are ruled out in the case $\tilde{A}$ is asymptotically stable. The following result is well known.

Lemma 2.2. Let $K \in \mathbb{R}^{m\times n}$ and suppose $\tilde{A}$ is asymptotically stable. Then P given by (2.9) exists and is the only solution of (2.10). Furthermore, (2.11) holds.

Remark 2.2. Lemma 2.1 shows that if P given by (2.8) exists, then P is the minimal nonnegative-definite solution to (2.10). Furthermore, *v)* implies that P given by (2.8) is the unique minimal-rank nonnegative-definite solution to (2.10) in the sense that all other solutions $P + P_0$, where P_0 is nonzero, nonnegative definite, and satisfies (2.10), must have rank greater than the rank of P.

Now define a Lagrangian

$$\mathcal{L}(K,P) \triangleq \lambda_0 \mathrm{tr}\; PV + \mathrm{tr}\; Q[\tilde{A}^{\mathrm{T}}P + P\tilde{A} + \tilde{R}],$$

where $\lambda_0 \geq 0$ and $Q \in \mathbb{R}^{n\times n}$ are Lagrange multipliers. Since the constraint equation (2.10) is symmetric, we require Q to also be symmetric. Then, viewing P as a free variable, we compute

$$\frac{\partial \mathcal{L}}{\partial K} = (B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + R_2 K)Q, \tag{2.13}$$

$$\frac{\partial \mathcal{L}}{\partial P} = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \lambda_0 x_0 x_0^{\mathrm{T}}. \tag{2.14}$$

Setting $\frac{\partial \mathcal{L}}{\partial K} = 0$ and $\frac{\partial \mathcal{L}}{\partial P} = 0$ yields

$$KQ = -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}})Q, \tag{2.15}$$

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \lambda_0 x_0 x_0^{\mathrm{T}}. \tag{2.16}$$

Next using (2.15) standard results imply that K is given by

$$K = -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}})QQ^{+} + M(I - QQ^{+}), \tag{2.17}$$

where Q^+ is the generalized inverse of Q and $M \in \mathbb{R}^{m\times n}$ is arbitrary. Conversely, for all $M \in \mathbb{R}^{n\times n}$, each matrix K given by (2.17) satisfies (2.15). Note that equation (2.16) for Q is dual to (2.11) so that Lemma 2.1 applies with obvious modifications.

Although the gain K given by (2.17) is not unique (since M is arbitrary), we can show that this nonuniqueness in irrelevant to the optimization problem. To show this we first present a result that characterizes the class of reachable states. Let "$\mathcal{R}$" denote range.

Lemma 2.3. Let $K \in \mathbb{R}^{m\times n}$, $x_0 \in \mathbb{R}^m$, and $t_1 > 0$. Then

$$\{e^{\tilde{A}t}x_0:\quad 0 \le t \le t_1\} \subset \mathcal{R}\big([x_0\ \tilde{A}x_0 \cdots \tilde{A}^{n-1}x_0]\big) = \mathcal{R}\Big(\int_0^t e^{\tilde{A}s}x_0x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}s}\,\mathrm{d}s\Big). \tag{2.18}$$

If, in addition, $\int_0^\infty e^{\tilde{A}s}x_0x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}s}\,\mathrm{d}s$ is finite, then

$$\mathcal{R}\Big(\int_0^{t_1} e^{\tilde{A}s}x_0x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}s}\,\mathrm{d}s\Big) = \mathcal{R}\Big(\int_0^{\infty} e^{\tilde{A}s}x_0x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}s}\,\mathrm{d}s\Big). \tag{2.19}$$

Proof. The inclusion in (2.18) follows from the Cayley-Hamilton theorem, while the equality in (2.18) is given by Theorem 3, p. 79, of [87]. Finally, (2.19) is easily shown. □

Next we show that the cost $J(K, x_0)$ is independent of the choice of M in (2.17). To do this first note that although Q satisfying (2.16) is symmetric, it does not necessarily follow that (2.16) has a nonnegative-definite solution. Hence we now make the assumption that the controllable subspace of $(\tilde{A}, x_0)$ is contained in the asymptotically stable subspace of $\tilde{A}$. Then by the dual of Lemma 2.1 it follows that Q given by

$$Q = \int_0^\infty e^{\tilde{A}t}x_0x_0^{\mathrm{T}}e^{\tilde{A}^{\mathrm{T}}t}\,\mathrm{d}t \tag{2.20}$$

exists and satisfies

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + x_0x_0^{\mathrm{T}} \tag{2.21}$$

that is, (2.16) with $\lambda_0 = 1$.

Next, it follows from Lemma 2.3 that the solution $x(t)$ of the closed-loop system (2.6) lies in the subspace $\mathcal{R}(Q)$. However, if $x \in \mathcal{R}(Q)$ then $x = Qz$ for some $z \in \mathbb{R}^n$ so that (2.17) implies

$$\begin{aligned} Kx &= -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}})QQ^+Qz + M(I - QQ^+)Qz \\ &= -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}})Qz \\ &= -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}})x. \end{aligned}$$

Hence the control $u(t) = Kx(t)$ is independent of the choice of M. Setting for convenience

$$M = -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}}) \tag{2.22}$$

yields

$$K = -R_2^{-1}(B^{\mathrm{T}}P + R_{12}^{\mathrm{T}}). \tag{2.23}$$

Note that the gain (2.23) agrees with (2.17) in the case that Q is positive definite. Hence the expression (2.23) is maximal in the sense that it characterizes optimal values of K regardless of Q.

Finally, substituting (2.23) into (2.10) yields the standard Riccati equation

$$0 = A^{\mathrm{T}}P + PA + R_1 - P_a^{\mathrm{T}}R_2^{-1}P_a, \tag{2.24}$$

where $P_a \triangleq B^{\mathrm{T}}P + R_{12}^{\mathrm{T}}$. Equation (2.24) can also be written as

$$0 = A'^{\mathrm{T}}P + PA' + R_1' - PBR_2^{-1}B^{\mathrm{T}}P, \tag{2.25}$$

where

$$A' \triangleq A - BR_2^{-1}R_{12}^{\mathrm{T}}, \quad R_1' \triangleq R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}}.$$

As noted previously, R_1' is nonnegative definite. If (A, B) is stabilizable (so that (A', B) is also stabilizable), it follows from [88] that (2.25) has a maximal nonnegative-definite solution P. Furthermore, if K is given by (2.23) with P the maximal soltuion of (2.25), then every eigenvalue of $\tilde{A} = A + BK$ lies in the closed left half plane. To illustrate this fact, let $A = 0$, $B = 1$, $R_1 = 0$, $R_{12} = 0$, and $R_2 = 1$. Then $P = 0$ is the only solution and hence the maximal solution to (2.25). Since $K = 0$, $\tilde{A} = A = 0$. If, however, (R_1', A') is detectable, then (2.25) has exactly one nonnegative-definite solution P, and the corresponding closed-loop system is asymptotically stable [89]. If (R_1', A') is not detectable, then (2.25) may have more than one nonnegative-definite solution. It then follows from the form of the cost (2.9) that $J(K, x_0)$ is minimized by choosing the minimal solution of (2.25). For related details see [91,92].[1]

[1] The key results in [91,92] can be summarized as follows: Assuming (A, B) stabilizable (so that (A', B) is also stabilizable), the Riccati equation (2.25) will have multiple nonnegative-definite solutions to the extent that (A', R_1') is undetectable. Detectable eigenvalues are stabilized by each such solution, while undetectable eigenvalues either remain invariant or are reflected into the left half plane by the various solutions. Moreover, every undetectable eigenvalue is reflected into the left half plane by the maximal solution. However, reflection does not affect undetectable eigenvalues lying on the imaginary axis which thus are not stabilized. If (A, B) is not stabilizable, then available results are less complete [93–96].

3. Goh's Transformation

In this section we review Goh's transformation [17,18,43] which transformd the Singular Linear-Quadratic Regulator Problem into a nonsingular problem. By applying the standard theory to the transformed problem we obtain the Goh-Riccati equation, a singular variant of the standard Riccati equation.

As in Section 2 we consider the linear equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{3.1}$$

with singular cost functional

$$J_{\mathrm{S}}(u(\cdot), x_0) = \int_0^\infty x^{\mathrm{T}} Rx \, \mathrm{d}t, \tag{3.2}$$

which is the same as (2.3) with $R_1 = R$, $R_{12} = 0$, and $R_2 = 0$. For Goh's transformation define

$$z(t) \triangleq x(t) - Bv(t), \quad v(t) \triangleq \int_0^t u(s) \, \mathrm{d}s + v_0, \tag{3.3}$$

so that

$$\dot{z}(t) = Ax(t), \quad \dot{v}(t) = u(t), \tag{3.4}$$

and

$$z(0) = x_0 - Bv_0, \quad v(0) = v_0. \tag{3.5}$$

Hence it follows that $z(t)$ satisfies

$$\dot{z}(t) = Az(t) + ABv(t), \quad z(0) = x_0 - Bv_0. \tag{3.6}$$

If closed-loop stability is desired, then it is necessary to determine whether (A, AB) is stabilizable.

Lemma 3.1. The following two statements are equivalent:

i) (A, B) is stabilizable and A is nonsingular,

ii) (A, AB) is stabilizable.

Proof. If (A, B) is stabilizable then so is (SAS^{-1}, SB) for all nonsingular $S \in \mathbb{R}^{n\times n}$. Choosing $S = A$, it follows that $(AAA^{-1}, AB) = (A, AB)$ is stabilizable. Conversely, suppose (A, AB) is stabilizable. Then by the PBH test it follows that

$$\operatorname{rank}_{\mathbb{C}}[\lambda I - A \quad AB] = n, \quad \lambda \in \mathbb{C}, \quad \operatorname{Re} \lambda \geq 0. \tag{3.7}$$

Setting $\lambda = 0$ this implies $\operatorname{rank}[-A \quad AB] = n$, which in turn implies, since $[-A \quad AB] = A[-I \quad B]$, that A is nonsingular. Now, since A is nonsingular, if (A, AB) is stabilizable, then so is $(A^{-1}AA, A^{-1}AB) = (A, B)$. □

For the transformed system (3.6) the cost (3.2) becomes

$$\begin{aligned} \hat{J}_S(v(\cdot), z(0)) &\triangleq J_S(u(\cdot), x_0) \\ &= J_S(\dot{v}(\cdot), z(0) + Bv_0) \\ &= \int_0^\infty (z + Bv)^T R (z + Bv)\, dt \\ &= \int_0^\infty [z^T R z + 2z^T R B v + v^T B^T R B v]\, dt. \end{aligned} \tag{3.8}$$

Note that (3.8) has the form of the cost function (2.2) with $R_{12} = RB$ and $R_2 = B^T RB$. Note that in this case R_a becomes

$$R_a = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} = \begin{bmatrix} R & RB \\ B^T R & B^T RB \end{bmatrix} = \begin{bmatrix} I_n \\ B^T \end{bmatrix} R \begin{bmatrix} I_n \\ B^T \end{bmatrix}^T,$$

which is nonnegative definite.

Now we assume that $B^T RB$ is positive definite so that $\hat{J}_S(v(\cdot), z(0))$ has the form of a nonsingular cost functional. Note that it follows from this assumption that $\operatorname{rank} B = m$ and $\operatorname{rank} R \geq m$. Assuming now a feedback solution of the form

$$v(t) = K' z(t), \tag{3.9}$$

it follows from (2.23) that

$$K' = -(B^T RB)^{-1}[(AB)^T P + (RB)^T]$$

or

$$K' = -(B^T RB)^{-1} B^T (A^T P + R), \tag{3.10}$$

where P satisfies

$$0 = A^T P + PA + R - [(AB)^T P + (RB)^T]^T (B^T RB)^{-1} [(AB)^T P + (RB)^T]$$

or

$$0 = A^T P + PA + R - (A^T P + R)^T B (B^T RB)^{-1} B^T (A^T P + R). \tag{3.11}$$

We call (3.11) the Goh-Riccati equation. An equivalent form of (3.11) is

$$\begin{aligned}0 = &[I_n - B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R]^{\mathrm{T}}A^{\mathrm{T}}P + PA[I_n - B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R] \\ &+ R - RB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R - PAB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}P.\end{aligned} \tag{3.12}$$

There are several consequences of the above relations that should be noted. First, it follows from $z(0) = x_0 - Bv_0$ and $v_0 = v(0) = K'x(0)$ that

$$(I_n + BK')z(0) = x_0 \tag{3.13}$$

and

$$(I_m + K'B)v_0 = K'x_0. \tag{3.14}$$

Next, forming $B^{\mathrm{T}}(3.11)B$ yields

$$\begin{aligned}0 =&B^{\mathrm{T}}A^{\mathrm{T}}PB + B^{\mathrm{T}}PAB + B^{\mathrm{T}}RB - (B^{\mathrm{T}}PA + B^{\mathrm{T}}R)B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}PB + RB) \\ =& - B^{\mathrm{T}}PAB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}PB,\end{aligned}$$

which implies that

$$B^{\mathrm{T}}PAB = 0, \quad B^{\mathrm{T}}A^{\mathrm{T}}PB = 0. \tag{3.15}$$

Next, form $(3.11)B$ and use (3.15) to obtain

$$0 = B^{\mathrm{T}}PA, \quad 0 = A^{\mathrm{T}}PB, \tag{3.16}$$

which is a stronger condition than (3.15). Now using (3.16) and (3.10) it follows that

$$K'B = -(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P + R)B = -I_m.$$

Hence

$$I_m + K'B = 0, \tag{3.17}$$

which, with (3.14), implies

$$K'x_0 = 0. \tag{3.18}$$

Now using (3.10) and (3.18) we obtain

$$B^{\mathrm{T}}(A^{\mathrm{T}}P + R)x_0 = 0, \tag{3.19}$$

while forming $(3.11)x_0$ yields

$$0 = (A^{\mathrm{T}}P + PA + R)x_0. \tag{3.20}$$

Next, it is useful to recognize the presence of two idempotent matrices. First, define the matrix $\rho \in \mathbb{R}^{n \times n}$ by

$$\rho \triangleq B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R,$$

which is idempotent since $\rho^2 = \rho$. Note that $\rho B = B$ and $R_\rho = \rho^{\mathrm{T}}R$. Since rank $B = m$ we have $m = \text{rank } B = \text{rank } \rho B \leq \text{rank } \rho \leq \text{rank } B = m$. Hence rank $\rho = m \leq n$. The complementary projection $\rho_\perp$ defined by

$$\rho_\perp \triangleq I_n - \rho = I_n - B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R$$

is also idempotent. Note that rank $\rho_\perp = n - m \geq 0$, rank $\rho_\perp < n$, and $\rho_\perp B = 0$. In terms of $\rho_\perp$, (3.12) can be written as

$$0 = \rho_\perp^{\mathrm{T}} A^{\mathrm{T}} P + PA\rho_\perp + R - RB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R - PAB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}P. \tag{3.21}$$

We can rewrite (3.21) as

$$0 = A_0^{\mathrm{T}}P + PA_0 + R_0 - PAB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}P, \tag{3.22}$$

where

$$A_0 \triangleq A\rho_\perp = A - AB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R,$$

$$R_0 \triangleq R\rho_\perp = \rho_\perp^{\mathrm{T}}R = \rho_\perp^{\mathrm{T}}R\rho_\perp = R - RB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}R.$$

Note that R_0 is nonnegative definite and that det $R_0 =$ (det R)det $\rho_\perp = 0$, so that R_0 is singular.

We can also define the matrix $\pi \in \mathbb{R}^{n \times n}$ by

$$\pi \triangleq -BK'$$

which is idemptotent because of (3.17). It follows from (3.10) that

$$\pi = \rho + \sigma, \tag{3.23}$$

where

$$\sigma \triangleq B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}P.$$

Using (3.16) we obtain $\sigma B = 0$ so that

$$\pi B = \rho B = B. \tag{3.24}$$

Hence we have $m = \text{rank } B = \text{rank } \pi B \leq \text{rank } \pi \leq \text{rank } B \leq m$ so that rank $\pi = m$. Defining

$$\pi_\perp \triangleq I_n - \pi = I_n + BK',$$

we see that rank $\pi_\perp = n - m$ and $\pi_\perp B = 0$. Also we see that

$$\sigma\rho = 0, \quad \rho\sigma = \sigma, \quad \sigma^2 = 0, \tag{3.25}$$

$$\pi x_0 = 0, \quad \pi_\perp x_0 = x_0, \quad \rho x_0 = -\sigma x_0. \tag{3.26}$$

Next note that the closed-loop system (3.6), (3.9) is given by

$$\dot{z} = \tilde{A}' z, \quad z(0) = x_0 - Bv_0, \tag{3.27}$$

where

$$\tilde{A}' \triangleq A + ABK'.$$

However, note that

$$\tilde{A}' = A(I_n + BK') = A\pi_\perp. \tag{3.28}$$

Since

$$\det \tilde{A}' = (\det A)(\det \pi_\perp) = 0, \tag{3.29}$$

it follows that the closed-loop dynamics matrix $\tilde{A}'$ is not stable even if A is nonsingular. Also note that

$$\tilde{A} x_0 = A x_0. \tag{3.30}$$

Hence we conclude that the feedback control $v = K'z$ given by (3.10) does not stabilize the transformed system even under the assumptions (A, AB) stabilizable and $(A, R^{\frac{1}{2}})$ detectable. To see why this is plausible in view of standard results [89,91,92], we first note that (A_0, AB) is stabilizable if and only if (A, AB) stabilizable. Next, note that by the PBH test with $\lambda = 0$ we have

$$\operatorname{rank} \begin{bmatrix} -A_0 \\ R_0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -A\rho_\perp \\ R\rho_\perp \end{bmatrix} \leq \operatorname{rank} \rho_\perp = n - m < n. \tag{3.31}$$

Hence (A_0, R_0) is not detectable. Thus, by [89], at best we can expect Re $\lambda(\tilde{A}') \leq 0$ corresponding to the maximal solution to the Goh-Riccati equation if A has eigenvalues on the imaginary axis. The lack of closed-loop stability, however, goes much deeper since (3.29) shows that det $\tilde{A}'$ is unstable even if A is stable, that is, if the open-loop system is stable.

Transforming back to the original system we have

$$u = \dot{v} = K'\dot{z} = K'Ax, \tag{3.32}$$

so that the control u is given by $u = Kx$ with

$$K = K'A = -(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P + R)A. \tag{3.33}$$

Hence the original closed-loop system consisting of (3.1) and (3.5) has the form

$$\dot{x} = \tilde{A}x, \tag{3.34}$$

where

$$\tilde{A} = A + BK = A + BK'A. \tag{3.35}$$

Note that

$$\tilde{A} = (I_n + BK')A = \pi_{\perp}A. \tag{3.36}$$

Comparing (3.36) to (3.28), we see that the original closed-loop system has the same closed-loop poles as the transformed system. Hence the original system is also not stabilized.

As a simple example of the preceding, consider the case $n = m = 1$, $A \neq 0$, $B \neq 0$, $R_1 > 0$. Then the Goh-Riccati equation is satisfied only by $P = 0$ and thus $K' = \frac{-1}{B}$ and $K = -\frac{A}{B}$. Thus $\tilde{A}' = A + ABK' = 0$ and $\tilde{A} = A + BK'A = 0$. Of course, for this problem a smooth minimizing control does not exist. Hence note that if A is nonsingular, then (3.16) implies

$$0 = B^{\mathrm{T}}P, \quad 0 = PB. \tag{3.37}$$

Using Sylvester's inequality we obtain

$$\operatorname{rank} B^{\mathrm{T}} + \operatorname{rank} P - n \leq \operatorname{rank} B^{\mathrm{T}}P = 0$$

which yields

$$\operatorname{rank} P \leq n - m. \tag{3.38}$$

If $B^{\mathrm{T}}RB = 0$, then Goh's transformation can be repeated. In this case $RB = 0$. Defining

$$z_1(t) \triangleq z(t) - Bv_1(t), \quad v_1(t) \triangleq \int_0^t v(s)\,\mathrm{d}s + v_{10}, \tag{3.39}$$

leads to

$$\dot{z}_1 = Az_1 + A^2Bv_1. \tag{3.40}$$

Assuming now that $B^{\mathrm{T}}A^{\mathrm{T}}RAB$ is positive definite, it follows that $v_1 = K''z_1$ is given by

$$K'' = -(B^{\mathrm{T}}A^{\mathrm{T}}RAB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}(A^{\mathrm{T}}P + R), \tag{3.41}$$

where P satisfies

$$0 = A^{\mathrm{T}}P + PA + R - (A^{\mathrm{T}}P + R)^{\mathrm{T}}A \quad B^{\mathrm{T}}A^{\mathrm{T}}RAB)^{-1}B^{\mathrm{T}}A^{\mathrm{T}}(A^{\mathrm{T}}P + R). \tag{3.42}$$

For the original system, $u = Kx$ is satisfied with

$$K = K''A^2. \tag{3.43}$$

4. Perturbation Approach

The purpose of this section is to provide an alternative and completely distinct derivation of the Goh-Riccati equation. The derivation is based upon the perturbation expansion technique pioneered in [30] and further developed in [74].

To begin, consider the nonsingular linear-quadratic regulator problem with $R_1 = R$, $R_{12} = 0$, and $R_2 = \epsilon^2 I_m$ so that $K = -R_2^{-1}B^{\mathrm{T}}P(\epsilon)$ with $P(\epsilon)$ given by

$$0 = A^{\mathrm{T}}P(\epsilon) + P(\epsilon)A + R - \frac{1}{\epsilon^2}P(\epsilon)BB^{\mathrm{T}}P(\epsilon). \tag{4.1}$$

Expanding

$$P(\epsilon) = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots \tag{4.2}$$

and substituting (4.2) into (4.1), we have

$$\begin{aligned} 0 = {} & A^{\mathrm{T}}(P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots) + (P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots)R \\ & - \frac{1}{\epsilon^2}(P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots)BB^{\mathrm{T}}(P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots). \end{aligned} \tag{4.3}$$

Terms of order ϵ^{-2} yield

$$0 = P_0 BB^{\mathrm{T}}P_0, \tag{4.4}$$

of order ϵ^{-1} yield

$$0 = P_0 BB^{\mathrm{T}}P_1 + P_1 BB^{\mathrm{T}}P_0, \tag{4.5}$$

and of order ϵ^0 yield

$$0 = A^{\mathrm{T}}P_0 + P_0 A + R - P_1 BB^{\mathrm{T}}P_1 - P_0 BB^{\mathrm{T}}P_2 - P_2 BB^{\mathrm{T}}P_0. \tag{4.6}$$

Note that (4.4) is equivalent to

$$B^{\mathrm{T}}P_0 = 0 \tag{4.7}$$

so that (4.5) holds for arbitrary P_1. Next using (4.7) in (4.6) implies

$$0 = A^{\mathrm{T}}P_0 + P_0A + R - P_1BB^{\mathrm{T}}P_1. \tag{4.8}$$

Next forming $B^{\mathrm{T}}(4.8)B$ and using (4.7) yields

$$0 = B^{\mathrm{T}}RB - B^{\mathrm{T}}P_1BB^{\mathrm{T}}P_1B,$$

that is,

$$B^{\mathrm{T}}RB = (B^{\mathrm{T}}P_1B)^2 \tag{4.9}$$

or

$$(B^{\mathrm{T}}RB)^{\frac{1}{2}} = B^{\mathrm{T}}P_1B. \tag{4.10}$$

Hence if $B^{\mathrm{T}}RB$ is positive definite, then so is $B^{\mathrm{T}}P_1B$. Next form $B^{\mathrm{T}}(4.8)$ to obtain

$$\begin{aligned} 0 &= B^{\mathrm{T}}A^{\mathrm{T}}P_0 + B^{\mathrm{T}}P_0A + B^{\mathrm{T}}R_1 - B^{\mathrm{T}}P_1BB^{\mathrm{T}}P_1 \\ &= B^{\mathrm{T}}(A^{\mathrm{T}}P_0 + R) - (B^{\mathrm{T}}RB)^{\frac{1}{2}}B^{\mathrm{T}}P_1 \end{aligned}$$

so that

$$B^{\mathrm{T}}P_1 = (B^{\mathrm{T}}RB)^{-\frac{1}{2}}B^{\mathrm{T}}(A^{\mathrm{T}}P_0 + R). \tag{4.11}$$

Substituting (4.11) into (4.8) now yields

$$0 = A^{\mathrm{T}}P_0 + P_0A + R_1 - (A^{\mathrm{T}}P_0 + R)^{\mathrm{T}}B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P_0 + R), \tag{4.12}$$

which is identical to (3.11), that is, the Goh-Riccati equation. Note that condition (4.7) is identical to (3.37) which was obtained under the assumption that A is nonsingular.

5. Generalized Legendre-Clebsch Conditions

The generalized Legendre-Clebsch necessary condition was first proved for scalar control in [8,12,13] and later for vector controls in [9,21] This condition states that if (x,u) is optimal, then

$$\frac{\partial}{\partial u}\frac{\mathrm{d}^p}{\mathrm{d}t^p}H_u(x(t),u(t),\lambda(t)) = 0, \quad t \geq 0, \tag{5.1}$$

where p is an odd positive integer, and

$$(-1)^q\frac{\partial}{\partial u}\frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}}H_u(x(t),u(t),\lambda(t)) \geq 0, \quad t \geq 0, \tag{5.2}$$

where q is a positive integer. In (5.1) and (5.2), H is the Hamiltonian associated with the problem and $\lambda(\cdot)$ is an adjoint variable corresponding to the optimal pair (x, u).

In this section we apply condition (5.2) with $q = 1$ to the problem consisting of

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{5.3}$$

with cost functional

$$J(u, x_0) = \int_0^\infty x^{\mathrm{T}} Rx \, \mathrm{d}t. \tag{5.4}$$

Note that this problem is normal since the final state endpoint is free. Hence the Lagrange multiplier associated with the cost can be set to unity. Applying (5.2) we obtain

$$\frac{\partial}{\partial u} \frac{\mathrm{d}^2}{\mathrm{d}t^2} H_u(x(t), u(t)), \lambda(t)) \leq 0, \quad t \geq 0, \tag{5.5}$$

where

$$H(x, u, \lambda) \triangleq \lambda^{\mathrm{T}}(Ax + Bu) + \frac{1}{2} x^{\mathrm{T}} Rx \tag{5.6}$$

and

$$-\dot{\lambda}(t) = H_x^{\mathrm{T}}(x(t), u(t), \lambda(t)) = Rx(t) + A\lambda(t). \tag{5.7}$$

Evaluating the left hand side of (5.5) we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} H_u\big(x(t), u(t), \lambda(t)\big) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} [\lambda^{\mathrm{T}}(t) B] \\ &= \frac{\mathrm{d}}{\mathrm{d}t} [-x^{\mathrm{T}}(t) RB - \lambda^{\mathrm{T}}(t) AB] \\ &= x^{\mathrm{T}}(t)(RA - A^{\mathrm{T}} R) B + \lambda^{\mathrm{T}}(t) A^2 B - u^{\mathrm{T}}(t) B^{\mathrm{T}} RB. \end{aligned} \tag{5.8}$$

Thus (5.5) implies that

$$B^{\mathrm{T}} RB \geq 0. \tag{5.9}$$

If (x, u) is optimal for the optimal control problem, then $(x(t), u(t), \lambda(t))$ satisfies, in addition to the adjoint equation, the minimality condition of the Pontryagin minimum principle, that is,

$$\min_{u \in \mathbb{R}^m} H(x(t), u, \lambda(t)) = H(x(t), u(t), \lambda(t)), \quad t \geq 0, \tag{5.10}$$

which implies

$$H_u(x(t), u(t), \lambda(t)) = \lambda^{\mathrm{T}}(t) B = 0, \quad t \geq 0. \tag{5.11}$$

Now (5.11) yields

$$\frac{d}{\mathrm{d}t} H_u\big(x(t), u(t), \lambda(t)\big) = 0, \quad t \geq 0, \tag{5.12}$$

which implies

$$x^{\mathrm{T}}(t)RB + \lambda^{\mathrm{T}}(t)AB = 0, \quad t \geq 0. \tag{5.13}$$

Furthermore, (5.12) implies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H_u(x(t), u(t), \lambda(t)) = 0, \quad t \geq 0, \tag{5.14}$$

which, by (5.8) is equivalent to

$$B^{\mathrm{T}}RBu(t) = B^{\mathrm{T}}(A^{\mathrm{T}}R - RA)x(t) + B^{\mathrm{T}}A^{2\mathrm{T}}\lambda(t). \tag{5.15}$$

Assuming now that the strengthened Legendre-Clebsch condition holds, that is, $B^{\mathrm{T}}RB > 0$, (5.15) implies

$$u(t) = (B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}[(A^{\mathrm{T}}R - RA)x(t) + A^{2\mathrm{T}}\lambda(t)], \tag{5.16}$$

which is equation (8.2.28), p. 251, of [45].

Next, substituting (5.16) into (5.1) and using (5.7) leads to the linear system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A + B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}R - RA) & B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{2\mathrm{T}} \\ -R & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}. \tag{5.17}$$

Consider the matrix system corresponding to (5.14), that is, where x and λ are replaced respectively by $n \times n$-matrix functions X and Λ. If X is invertible, define as in [97]

$$W \triangleq \Lambda X^{-1}, \tag{5.18}$$

which satisfies $Wx = \lambda$ and

$$\begin{aligned} \dot{W} &= \dot{\Lambda}X^{-1} - \Lambda X^{-1}\dot{X}X^{-1} \\ &= -R - A^{\mathrm{T}}W - W[A + B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}R - RA)] - WB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{2\mathrm{T}}W. \end{aligned} \tag{5.19}$$

Now consider the steady-state case $\dot{W} = 0$, that is, $W(t) = W_0 =$ constant. Then W_0 satisfies

$$0 = A^{\mathrm{T}}W_0 + W_0[A + B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}R - RA)] + W_0B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}A^{2\mathrm{T}}W_0. \tag{5.20}$$

Under the assumption that A is nonsingular, define

$$P = -(A^{\mathrm{T}}W_0 + R)A^{-1}, \tag{5.21}$$

so that

$$W_0 = -A^{-\mathrm{T}}(A^{\mathrm{T}}P + R)^{\mathrm{T}}. \tag{5.22}$$

By Lemma 2.1 and Lemma 2.3 it follows that $x_0 \in \mathcal{R}(Q)$ for each nonnegative-definite solution Q of (6.5). Hence (6.6) implies

$$B^{\mathrm{T}} P x_0 = 0. \tag{6.9}$$

Alternatively, (6.9) can be obtained by forming $B^{\mathrm{T}}P(6.8)PB$ and using (6.6). Next, forming $Q(6.7)Q$ yields

$$0 = Q(A^{\mathrm{T}}P + PA + R)Q \tag{6.10}$$

and hence

$$0 = x_0^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R)x_0. \tag{6.11}$$

Next, consider the minimal nonnegative-definite solution to (6.4) given by

$$P = \int_0^\infty e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t. \tag{6.12}$$

Letting $\epsilon > 0$, $\epsilon \approx 0$, (6.12) can be written as

$$\begin{aligned} P &= \int_0^\epsilon e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t + \int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t \\ &= \int_0^\epsilon [(I_n + O(t))] R [I_n + O(t)]\, \mathrm{d}t + \int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t \\ &= \epsilon R + O(\epsilon^2) + \int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t. \end{aligned} \tag{6.13}$$

Hence

$$B^{\mathrm{T}}PB = \epsilon B^{\mathrm{T}}RB + O(\epsilon^2) + B^{\mathrm{T}} \int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t} R e^{\tilde{A}t}\, \mathrm{d}t B. \tag{6.14}$$

Letting $\epsilon > 0$ be sufficiently small it follows from (6.14) that if $B^{\mathrm{T}}RB$ is positive definite, then $B^{\mathrm{T}}PB$ is also positive definite.

As in Section 3, we assume $B^{\mathrm{T}}RB$ is positive definite. Since $B^{\mathrm{T}}PB$ must be positive definite, it follows that

$$\operatorname{rank} PB = m. \tag{6.15}$$

Now (6.6) implies that

$$\operatorname{rank} Q \leq n - m. \tag{6.16}$$

Hence Q cannot be positive definite.

Now form $B^{\mathrm{T}}P(6.8)$ and use (6.6) and (6.9) to obtain

$$0 = B^{\mathrm{T}}P(A + BK)Q, \tag{6.17}$$

which implies

$$KQ = -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PAQ. \tag{6.18}$$

Furthermore, forming $B^{\mathrm{T}}(6.7)Q$ and using (6.6) and (6.18) yields

$$0 = B^{\mathrm{T}}(A^{\mathrm{T}}P + R)Q. \tag{6.19}$$

Arguing as in Section 2 we can take

$$K = -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA. \tag{6.20}$$

With K given by (6.20) the closed-loop dynamics matrix becomes

$$\tilde{A} = \nu_{\perp}A, \tag{6.21}$$

where $\nu_{\perp} \in \mathbb{R}^{n\times n}$ is defined by

$$\nu \triangleq B(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P, \quad \nu_{\perp} \triangleq I_n - \nu.$$

Note that ν satisfies

$$\nu^2 = \nu, \quad \operatorname{rank} \nu = m, \quad \operatorname{rank} \nu_{\perp} = n - m, \tag{6.22}$$

$$\nu B = B, \quad \nu_{\perp}B = 0, \quad P\nu = \nu^{\mathrm{T}}P, \tag{6.23}$$

$$\nu Q = 0, \quad Q = \nu_{\perp}Q\nu_{\perp}^{\mathrm{T}} = \nu_{\perp}Q = Q\nu_{\perp}^{\mathrm{T}}. \tag{6.24}$$

Also, define

$$P' \triangleq P\nu_{\perp} = \nu_{\perp}^{\mathrm{T}}P = \nu_{\perp}^{\mathrm{T}}P\nu_{\perp} = P - PB(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P. \tag{6.24}$$

Next, using (6.21) in (6.4) and (6.5) yields

$$0 = A^{\mathrm{T}}\nu_{\perp}^{\mathrm{T}}P + P\nu_{\perp}A + R, \tag{6.25}$$

$$0 = \nu_{\perp}AQ + QA^{\mathrm{T}}\nu_{\perp}^{\mathrm{T}} + x_0x_0^{\mathrm{T}}. \tag{6.26}$$

Note that (6.25) can be written as

$$0 = A^{\mathrm{T}}P' + P'A + R. \tag{6.27}$$

However, both (6.25) and (6.27) are wrong! Using (6.18) rather than (6.20) in (6.4)Q yields

$$0 = (A^{\mathrm{T}}P' + P'A + R)Q. \tag{6.28}$$

Next, note that since $\mathcal{R}(\tilde{A}Q) \subset \mathcal{R}(Q)$, it follows that $B^{\mathrm{T}}P\tilde{A}^{r}Q = 0$ for $r > 0$. Using this fact and forming $B^{\mathrm{T}}(6.4)\tilde{A}Q$ and solving for KQ yields

$$KQ = -(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}P\tilde{A} + RA)Q. \tag{6.29}$$

Now assuming A is nonsingular we see that

$$P' = A^{-\mathrm{T}}\tilde{A}^{\mathrm{T}}P\tilde{A}A^{-1} \tag{6.30}$$

and

$$KQ = -(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R)AQ. \tag{6.31}$$

Next define

$$\tilde{A}' \triangleq A\tilde{A}A^{-1}, \quad \tilde{R}' \triangleq A^{-\mathrm{T}}\tilde{A}^{\mathrm{T}}R\tilde{A}A^{-1},$$

and form $A^{-\mathrm{T}}\tilde{A}^{\mathrm{T}}(6.4)\tilde{A}A^{-1}$ to obtain

$$0 = \tilde{A}'^{\mathrm{T}}P' + P'\tilde{A}' + \tilde{R}'. \tag{6.32}$$

Since K operates on the reachable set of the closed-loop system, take

$$K = -(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R)A, \tag{6.33}$$

which is identical to (3.33) with P replaced by P'. From (6.33) we have

$$BKA^{-1} = -B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R). \tag{6.34}$$

Then $\tilde{A}'$ and $\tilde{R}'$ can be written as

$$\tilde{A}' = A + ABKA^{-1} = A - AB(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R), \tag{6.35}$$

$$\begin{aligned} \tilde{R}' &= (I + BKA^{-1})^{\mathrm{T}}R(I + BKA^{-1}) \\ &= [I - B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R)]^{\mathrm{T}}R[I - B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R)]. \end{aligned} \tag{6.36}$$

Now, with (6.35) and (6.36), equation (6.26) becomes

$$0 = A^{\mathrm{T}}P' + P'A + R - (A^{\mathrm{T}}P' + R)^{\mathrm{T}}B(B^{\mathrm{T}}RB)^{-1}B^{\mathrm{T}}(A^{\mathrm{T}}P' + R). \tag{6.37}$$

Note that (6.37) is the Goh-Riccati equation (3.11) with P replaced by P'.

Finally, comparing (6.21) to (3.26) we see that

$$\tilde{A} = \pi_\perp A = \nu_\perp A. \tag{6.38}$$

If A is nonsingular (6.38) implies

$$\pi = \nu. \tag{6.33}$$

We can show this directly. It is easy to see that

$$B = \nu B = \pi B = \pi\nu B = 0, \quad \nu Q = \pi Q = \pi\nu_\perp Q = 0,$$

which does not suffice, however.

Remark 6.1. In the case of a higher-order singularity, assume $B^{\mathrm{T}}RB = 0$. Then $B^{\mathrm{T}}R = 0$ and $RB = 0$. Hence

$$\begin{aligned}
B^{\mathrm{T}}PB &= B^{\mathrm{T}}\int_0^\epsilon [I + \tilde{A}t + O(t^2)]^{\mathrm{T}}R[I + \tilde{A}t + O(t^2)]\, \mathrm{d}tB + B^{\mathrm{T}}\int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t}Re^{\tilde{A}t}\, \mathrm{d}tB \\
&= B^{\mathrm{T}}\int_0^\epsilon [t^2\tilde{A}^{\mathrm{T}}R\tilde{A} + O(t^3)]\, \mathrm{d}tB + B^{\mathrm{T}}\int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t}Re^{\tilde{A}t}\, \mathrm{d}tB \\
&= \frac{\epsilon^3}{3}B^{\mathrm{T}}A^{\mathrm{T}}RAB + O(\epsilon^4) + B^{\mathrm{T}}\int_\epsilon^\infty e^{\tilde{A}^{\mathrm{T}}t}Re^{\tilde{A}t}\, \mathrm{d}tB,
\end{aligned}$$

which shows that if $B^{\mathrm{T}}A^{\mathrm{T}}RAB$ is positive definite, then so is $B^{\mathrm{T}}PB$.

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A RICCATI EQUATION APPROCH TO THE SINGULAR LQG PROBLEM

by

Yoram Halevi
Faculty of Mech. Engineering
Technion-Israel Institute of Technology
Haifa 32000,
Israel

Wassim M. Haddad
Department of Mechanical and
Aerospace Engineering
Florida Institute of Technology
Melbourne, FL 32902

Dennis S. Bernstein
Government Aerospace
System Division
Harris Corporation,
Melbourne, FL 32902

Abstract
The problem of optimal fixed-order dynamic compensators for the singular LQG problem is considered. Necessary conditions characterizing the optimal compensator for the most general case, i.e. both singular measurement and singular control weighting matrix, are given. The solution consists of a set of two algebraic Riccati equations and two Lyapunov equations couples by three projection matrices. One is the standard order reduction projection and the other two reflect the two types of singularity that exists in the system. The three projections are shown to satisfy disjointness conditions. In addition to order reduction, an advantage of the fixed-structure approach is that differentiation, which is often undesirable from a practical point of view and which may exist in the unconstrained optimal control, can be avoided. The fixed order compensator agrees wilth the unconstrained solution when the latter possesses the same number of differentiations as are included in the prespecified controller structure and when the order is selected appropriately.

1. INTRODUCTION

The singular LQG control problem has been of considerable interest for almost two decades ([1-14]). Such problems arise when some of the measurements are noise free or when some of the control signals are unweighted. This will be the case, for example, if the sensor noise is colored or if actuator dynamics are included. Augmentation of the plant dynamics by means of noise shaping filters or actuator dynamics thus leads directly to the singular problem formulation.

Most of the literature on the singular LQG problem is based upon either limiting procedures in which suitable weighting matrices and noise intensities approach zero, or differentiation of noise-free signals. These results demonstrate that the compensators which arise in the limiting solution may include differentiators. The dimension of the optimal compensators depends on the order of singularity of the problem, but in general the sum of differentiators and integrators is equal to the dimension of the system minus the number of noise-free measurements (unweighted control signals in the dual problem). In practical applications, however, it is often of interest to determine the optimal controller within a prespecified class of controllers. In particular, we consider the singular LQG problem in which the controller is *preconstrained* to possess a fixed dynamic feedback structure. The fixed structure includes the order of the compensator and the fact that it is proper. Hence an additional benefit of our approach is the ability to impose an upper bound on the number of differentiators to be included in the controller. That is, while certain measurement signals may be noise-free and hence differentiable, it may be undesirable in practice to implement more than one level of differentiation or, perhaps any differentiation at all.

Preliminary results for the singular LQG problem using the fixed structure approach were obtained in [15], The results there are incomplete, however, in that the gains associated with certain feedback paths were not given explicitly. For the corresponding singular estimation problem ([18]) this defect was remedied in [19] where all feedback gains were explicitly characterized. In addition, the solution obtained in [19] was shown to agree completely with results obtained using standard limiting methods when the (unconstrained) optimal singular estimator does not possess differentiators ([20]). The results of [19] thus provide for certain cases an alternative approach to the singular estimation problem considered in [20-23] and the numerous references therein. Preliminary and partial results of the present paper were reported in [16], where only the case of singular measurement was considered.

The contribution of the present paper is thus to complete the development of [15] by

incorporating the methods used in [19]. Accordingly, we derive a coupled system of modified Riccati and Lyapunov equations which explicitly characterize the feedback gains of the fixed-structure singular LQG controller. For generality we consider partial or total singularity in both the control weighting and measurement noise intensity matrices, and we allow the dynamic compensator to be of arbitrary dimension less than or equal to the number of plant states minus the number of noise-free measurements or minus the number of unweighted control signals, whichever maximal. In the special case in which only the measurement is singular, the order of the dynamic compensator is *equal* to the number of plant states minus the number of noise-free measurements (i.e., the quasi full-order case), and a certain matrix is nonsingular, then we show that the optimal solution decomposes (seaprates) into a reduced-order observer followed by state feedback. Furthermore, as in [19] we demonstrate connections with earlier results by showing that the fixed structure solution agrees with the standard limiting solution when the latter possesses the same number of differentiators as are included in the prespecified controller structure.

As in [17], the solution is given by a system of coupled algebraic Riccati and Lyapunov equations whose solutions (denoted by Q P, $\hat{Q}$, $\hat{P}$) are used to explicitly characterizethe optimal feedback gains. The coupling is due to three oblique projections (i.e., idempotent matrices) which arise as a direct consequence of the fixed structure constraint. The order-reduction projection τ defined by

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#},$$

where $(\ )^{\#}$ denotes group generalized inverse, appeared originally in [17], while the static projections ν_1 and ν_2, given by

$$\nu_1 \triangleq QC_2^T(C_2QC_2^T)^{-1}C_2, \qquad \nu_2 \triangleq B_2(B_2^TPB_2)^{-1}B_2^TP$$

are familiar from least square analysis. The three projections are shown to be disjoint, i.e. $\tau\nu_1 = \nu_2\tau = \nu_2\nu_1 = 0$.

The material is organized as follows: In section 2 the case of singular measurement and nonsingular performance index is considered. The results for the dual case of measurement and singular performance index are given in section 3. The general solution for the totally singular case, i.e. both singular mearuement and singular control weighting matrix is presented in section 4. Proofs of theorems are given in the appendices.

2. SINGULAR MEASUREMENT - NONSINGULAR PERFOMANCE INDEX.

We first consider the case where the measurement noise intensity is singular but the control weighting matrix is nonsingular. the system is given by

$$\dot{x} = Ax + Bu + w_1 \tag{2.1}$$

$$y_1 = C_1 x + w_2 \tag{2.2}$$

$$y_2 = C_2 x \tag{2.3}$$

$x \in R^n$, $u \in R^m$, $y_1 \in R^{r_1}$, $y_2 \in R^{r_2}$ and $w_1 \in R^n$, $w_2 \in R^{r_1}$ are uncorrelated white noise processes having intensities $V_1 \geq 0$ and $V_2 > 0$ respectively. This separation of the output noisy and noise-free components is not restrictive at all since any system having singular measurement can be brought to this structure by a static ouput transformation.
The cost function is

$$J = \lim_{t \to \infty} E \left\{ x^T(t) R_1 x(t) + u^T(t) R_2 u(t) \right\} \tag{2.4}$$

where $R_1 \geq 0$ and $R_2 > 0$. The optimization problem is to minimize J using the n_c - th order compensator.

$$\dot{x}_c = A_c x_c + B_c y \tag{2.5}$$

$$u = C_c x_c + D_c y \tag{2.6}$$

where $y = [y_1^T \ y_2^T]^T$. In the sequel we use the following partition

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}; \quad B_c = [B_{c1} \quad B_{c2}] ; \quad D_c = [D_{c1} \quad D_{c2}] \tag{2.7}$$

Note that as all modes of $u(t)$ are weighted, a direct transmission from the noisy measurement to the input will result in an infinite cost. Therefore only the noise-free output can have such a path and it follows that $D_{c1} = 0$.

The compensator (2.5) - (2.6) is proper, which is distinct from unconstrained singular control where derivatives of the noise-free measurement y_2 are often required. However a fixed

level of differentiation can be accommodated by carrying it out first and then redefining y_1 and y_2. The proper compensator acting on the modified output is equivalent to an improper comepnsator with the original output as its input. From implementation point of view this is advantageous because the level of allowed differentiation (which is often zero) is also prespecified.

Since J is independent of the internal realization of the compensator we restrict our attention to minimal realizations. Hence

Assumption 1: (A_c, B_c) is controllable, (A_C, C_C) is observable.

Combining the states of the plant and the compensator we obtain the augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} w_1 \\ B_{c1} w_2 \end{bmatrix} \tag{2.8}$$

which with obvious notation is written as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{w} \tag{2.9}$$

To guarantee the finiteness of J we make the following assumption:
Assumption 2: $\tilde{A}$ is a stability matrix, i.e. $\mathrm{Re}\,\lambda_i(\tilde{A}) < 0 \;\forall\, i$.

This includes implicitly the assumption that there exists an n_C-th order, proper compensator that stabilizes the plant. The stability of $\tilde{A}$ implies that the covariance matrix of $\tilde{x}$ reaches a steady state value $\tilde{Q}$. Patritioning $\tilde{Q}$ as

$$\lim_{t\to\infty} E\left\{\tilde{x}(t)\tilde{x}^T(t)\right\} = \tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix},\ Q_1 \in R^{n\times n},\ Q_2 \times R^{n_c\times n_c}$$

and defining $Q \stackrel{\Delta}{=} Q_1 - Q_{12}Q_2^{-1}Q_{12}^T$, we make an additional assumption.

Assumption 3: $Q_2 > 0$ and $C_2QC_2{}^T > 0$.

It can be shown that a sufficient (but not necessary) condition for $Q_2 > 0$ is that (A_C, B_{C1}) is controllable.

The following lemma is required for theorem 2.1, which states the main results of the section.

Lemma 2. 1 [16]: Suppose $n \times n$, $\hat{Q}$, $\hat{P}$ are nonnegative definite. Then $\hat{Q}\hat{P}$ is nonnegtive semisimple. If, in addition, rank $\hat{Q}\hat{P} = n_c$, then there exist $n_c \times n$, G, Γ and $n_c \times n_c$ invertible M such that

$$\hat{Q}\hat{P} = G^T M \Gamma; \qquad \Gamma G^T = I_{n_c}.$$

Since $\hat{Q}\hat{P}$ is semisimple it has a group inverse $(\hat{Q}\hat{P})^{\#} = G^T M^{-1} \Gamma$ and $\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T \Gamma$ is an oblique projection.

Theorem 2.1: Suppose (A_c, B_c, C_c, D_c) satisfy assumptions 1-3 and minimize J. Then they are given by

$$A_c = \Gamma(A - \Sigma_P P - Q\Sigma_Q)v_{1\perp}G^T \tag{2.10}$$

$$B_c = \Gamma\begin{bmatrix} QC_1^T & (A - \Sigma_P P - Q\Sigma_Q)QC_2^T \end{bmatrix}\bar{V}_2^{-1} \tag{2.11}$$

$$C_c = -R_2^{-1}B^T P v_{1\perp} G^T \tag{2.12}$$

$$D_c = -R_2^{-1}B^T P\begin{bmatrix} 0_{n \times r_1} & C_2^* \end{bmatrix} \tag{2.13}$$

where

$$\Sigma_P = BR_2^{-1}B^T \;;\quad \Sigma_Q = C_1^T V_2^{-1} C_1$$

$$\bar{V}_2 = \mathrm{diag}\{V_2, \quad C_2 Q C_2^T\}$$

$$C_2^* = QC_2^T(C_2 Q C_2^T)^{-1}$$

$$v_1 = C_2^* C_2 = QC_2^T(C_2QC_2^T)^{-1}C_2 \quad ; \quad v_{1\perp} = I_n - v_1$$

$$\tau = G^T\Gamma = \hat{Q}\hat{P}(\hat{Q}\hat{P})^\# \quad ; \quad \tau_\perp = I - \tau$$

Q, P, $\hat{Q}$, $\hat{P}$ are $n \times n$ nonnegative definite matrices satisfying.

$$[Av_{1\perp} + \tau_\perp(A - \Sigma_P P)v_1]Q + Q[Av_{1\perp} + \tau_\perp(A - \Sigma_P P)v_1]^T + V_1 - v_{1\perp}Q\Sigma_Q v_{1\perp}$$
$$+ \tau_\perp v_{1\perp}Q\Sigma_Q Q v_{1\perp}^T \tau_\perp^T = 0 \tag{2.14}$$

$$PA + A^TP + R_1 - P\Sigma_P P + \tau_\perp^T v_{1\perp}^T P\Sigma_P P v_{1\perp}^\tau = 0 \tag{2.15}$$

$$(A - \Sigma_P P)\hat{Q} + \hat{Q}(A - \Sigma_P P)^T + \tau(A - \Sigma_P P)v_1 Q + Qv_1^T(A - \Sigma_P P)^T\tau^T$$
$$+ v_{1\perp}Q\Sigma_Q Q v_{1\perp}^T - \tau_\perp v_{1\perp}Q\Sigma_Q Q v_{1\perp}^T \tau_\perp^T = 0 \tag{2.16}$$

$$\hat{P}(A - Q\Sigma_Q)v_{1\perp} + v_{1\perp}^T(A - Q\Sigma_Q)^T\hat{P} + v_{1\perp}^T P \quad \Sigma_P P v_{1\perp} - \tau_\perp^T v_\perp^T P\,\Sigma_P P\, v_{1\perp}\tau_\perp = 0 \tag{2.17}$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c \tag{2.18}$$

The proof is given in appendix A.

It is seen that the solution is given by a set of two algebraic Riccati equations and two Lyapunov equations coupled by two oblique projections τ and v_1. The dynamic projection τ arises as a direct consequence of the fixed-structure constraint, and appeared originally in [17]. The static projection v_1 is a consequence of the singularity of the problem. Similar matrices appear in least squares analysis. P, Q, τ and v_1 which are the solution of eqs. (2.14) - (2.17) are then used to obtain the various matrices of the compensator, given by eqs. (2.9) - (2.13).

Remark 2.1: A relation that is shown in the proof of theorem 2.1, and which is typical to singular observation order-reduction problems [15,18,19] is $\tau v_1 = 0$.

Remark 2.2: the nonsingular reduced order compensator [17] is recovered by setting $C_2 = 0$

and consequently $\nu_1 = 0, B_{C2} = 0, D_C = 0$.

Remark 2.3: The identity $\hat{P}\tau = \hat{P}$ was used in the derivation of eq. (2.17) to get the compact expression $\hat{P}(A - Q\Sigma_c)\nu_{1\perp}$. The disadvantage of this form is that the matrix that postmultiplies $\hat{P}$ is singular. Hence eq. (2.17) cannot be solved as a Lyapunov equation. This can be remedied by using the unsimplified $\hat{P}\left[A - Q\Sigma_Q - \tau(A - Q\Sigma_Q)\nu_1\right]$.

Remark 2.4: Note that

$$\begin{bmatrix} C_2 \\ \Gamma \end{bmatrix}\begin{bmatrix} QC_2^T(C_2QC_2^T)^{-1} & \nu_\perp G^T \end{bmatrix} = I_{n_c + r_2} \tag{2.19}$$

This is a degenerated form of a well known transformation used in optimal unconstrained singular estimation to generate a lower order nonsingular estimation problem. The difference between (2.19) and those cases is that here the matrices on the left-hand side are generally nonsquare.

Remark 2.4 leads to a discussion on the recovery of the quasi full order case from theorem 3.1. It is known from singular estimation theory [20 – 23] that if $C_2V_1C_2^T > 0$ then the optimal observer does not possess differentiators and its order is $n - r_2$. From the separation principle which holds in that case the optimal LQG compensator has the same order. Therefore the quasi full order case $n_c = n - r_2$ is the singular case equivalent of the full order case $n_c = n$ in the nonsingular case.

Now if $n_c = n - r_2$ then both matrices on the left-hand side of eq. (2.19) are square and therefore

$$\begin{bmatrix} QC_2^T(C_2QC_2^T)^{-1} & \nu_\perp G^T \end{bmatrix}\begin{bmatrix} C_2 \\ \Gamma \end{bmatrix} = \nu_1 + \nu_{1\perp}\tau = I_n \tag{2.20}$$

Or

$$\nu_{1\perp}\tau_\perp = 0 \tag{2.21}$$

Substituting this relation into eqs. (2.15) and (2.14) $\nu_{1\perp}^T$ we get

$$PA + A^T P + R_1 - P\Sigma_p P = 0 \tag{2.22}$$

$$A v_{1\perp} Q v_{1\perp}^T + Q v_{1\perp}^T A^T v_{1\perp}^T - v_{1\perp} Q \Sigma_Q Q v_{1\perp}^T + V_1 v_{1\perp}^T = 0 \tag{2.23}$$

Eqs. (2.22) - (2.23) lead to two immediate conclusions. First that Q and P are given by two uncoupled equations (v_1 depends solely on Q), i.e. separation holds in this case. Secondly, the control equation (2.22) is the standard Riccati equation. To further relate our result with optimal singular LQG we state the following lemma which summarizes results that were obtained in [19].

Lemma 2.2: Consider the system (2.1) – (2.3) and assume that $C_2 V_1 C_2^T > 0$. $\hat{x}$ is the optimal estimation of x and P_o is the corresponding error covariance matrix. Then if $n_c = n - r_2$ the following relations hold:

(i) $$v_{1\perp} Q = P_o \tag{2.24}$$

(ii) $$v = \left(V_1 + P_o A^T\right) C_2^T \left(C_2 V_1 C_2^T\right)^{-1} C_2 \tag{2.25}$$

(iii) $$\hat{x} = v_{1\perp} G^T \eta + C_2^* y_2 \tag{2.26}$$

$$\dot{\eta} = \Gamma\left(A - Q\Sigma_Q\right) v_{1\perp} G^T \eta + \Gamma\left[QC_1^T \ (A - Q\Sigma_Q) QC_2^T\right] \bar{V}_2^{-1} y + \Gamma B u \tag{2.27}$$

Note that by combining the optimal observer (2.26) - (2.27) and the optimal state feedback

$$u = -R_2^{-1} B^T P \hat{x} \tag{2.28}$$

We get the compensator matrices (2.10) - (2.13). Hence the quasi full order case is recovered from the results of theorem 2.1.

2. NONSINGULAR MEASUREMENT - SINGULAR PERFORMCE INDEX.

We consider now the case where there are linear combinations of the input which are not weighted in the cost function. The input is separated to a weighted component $u_1 \in R^{m_1}$ and an unweighted component $u_2 \in R^{m_2}$. The plant is now given by

$$\dot{x} = Ax + B_1u_1 + B_2u_2 + w_1 \tag{3.1}$$

$$y = Cx + w_2 \tag{3.2}$$

where the measurement is nonsingular, i.e. $V_2 > 0$. The performance index is

$$J = \lim_{t \to \infty} E\{x^T R_1 x + u_1^T R_2 u_1\} \qquad R_1 \geq 0,\ R_2 > 0. \tag{3.3}$$

We make the following assumption.

Assumption 4: $P_2 > 0$ and $B_2^T P B_2 > 0$ where P_2, P which are defined in the proof of theorem 2. 1. are analogous to Q_2 and Q respectively.
We have now the following result which is dual to theorem 2. 1.

Theorem 3. 1: Suppose (A_c, B, C_c, D_c) satisfy assumptions 1, 2 and 4 and minimize J. Then they are given by

$$A_c = \Gamma v_{2\perp}(A - Q\Sigma_Q - \Sigma_P P)G^T \tag{3.4}$$

$$B_c = \Gamma v_{2\perp} Q C^T V_2^{-1} \tag{3.5}$$

$$C_c = -\bar{R}_2^{-1} \begin{bmatrix} B_1^T P \\ B_2^T P(A - \Sigma_P P - \Sigma_Q Q) \end{bmatrix} G^T \tag{3.6}$$

$$D_c = -\begin{bmatrix} 0_{m_1 \times n} \\ B_2^* \end{bmatrix} Q C^T V_2^{-1} \tag{3.7}$$

where

$$\Sigma_P = B_1 R_2^{-1} B_1^T \ ; \ \Sigma_Q = C^T V_2^{-1} C$$

$$\bar{R}_2 = \text{diag}\left\{R_2, B_2^T P B_2\right\}$$

$$B_2^* = \left(B_2^T P B_2\right)^{-1} B_2^T P$$

$$\nu_2 = B_2 B_2^* = B_2 (B_2^T P B_2)^{-1} B_2^T P$$

τ, Γ and G^T are as in theorem 2.1 and Q P, $\hat{Q}$, $\hat{P}$ are nonnegative definite matrices satisfying

$$AQ + QA^T + V_1 - Q\Sigma_Q Q + \tau_\perp \nu_{2\perp} Q\Sigma_Q Q \nu_{2\perp}^T \tau_\perp^T = 0 \tag{3.8}$$

$$P[\nu_{2\perp} A + \nu_2 (A - \Sigma_Q Q)\tau_\perp] + [\nu_{2\perp} A + \nu_2 (A - \Sigma_Q Q)\tau_\perp]^T P + R_1$$
$$- \nu_{2\perp}^T P \Sigma_P P \nu_{2\perp} + \tau_\perp^T \nu_{2\perp}^T P \Sigma_P P \nu_{2\perp} \tau_\perp = 0 \tag{3.9}$$

$$\nu_{2\perp}(A - \Sigma_P P)\hat{Q} + \hat{Q}(A - \Sigma_P P)^T \nu_{2\perp}^T - \nu_{2\perp} Q \Sigma_C Q \nu_{2\perp}^T - \tau_\perp \nu_{2\perp} Q \Sigma_C Q \nu_{2\perp}^T \tau_\perp^T = 0 \tag{3.10}$$

$$\hat{P}(A - Q\Sigma_Q) + (A - Q\Sigma_Q)\hat{P} + P\nu_2 (A - Q\Sigma_Q)\tau + \tau^T (A - Q\Sigma_Q)^T \nu_2^T P$$
$$+ \nu_{2\perp}^T P \Sigma_P P \nu_{2\perp} - \tau_\perp^T \nu_{2\perp}^T P \Sigma_P P \nu_{2\perp} T_\perp = 0 \tag{3.11}$$

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q}\hat{P} = n_c \tag{3.20}$$

Remark 3.1: Though theoretically the problem is completely analogous to the singular measurement case there is a difference in the physical interpretation. Using a static gain for a noise free output is a sound engineering practice, allowing the variance of certain inputs to be infinite because they are not weighted is not. In case of included actuator dynamics this may be justified because u is only the command to the actuator. In other cases the results of this section should be viewed as theoretical bounds on the expected performance when near zero weights are assigned to the inputs which are unweighted here.

4. THE GENERAL CASE - SINGULAR MEASUREMENT AND PERFORMANCE INDEX.

The general case, which combines the two types of singularity, is now considered. The measurement contains noise-free components and some components of the input are not weighted. The system and the performance index are given by

$$\dot{x} = Ax + B_1u_1 + B_2u_2 + w \tag{4.1}$$

$$y_1 = C_1x + w_2 \tag{4.2}$$

$$y_2 = C_2x \tag{4.3}$$

$$J = \lim_{t \to \infty} \left\{x^TR_1x + u_1^TR_2u_1\right\} \qquad R_1 \geq 0,\ R_2 > 0 \tag{4.4}$$

The optimal compensator for this case is characterized by the following theorem.

Theorem 4.1: Consider the system (4.1) - (4.3) and the cost (4.4). Suppose (A_c, B_c, C_c, D_c) satisfy assumptions 1 - 4 and minimize J. then they are given by

$$A_c = \Gamma v_{2\perp}\left[\bar{A}v_{1\perp} - \left(QA^T + V_1\right)\bar{C}_2C_2\right]G^T \tag{4.5}$$

$$B_c = \Gamma v_{2\perp}\left[QC_1^T \quad \left(\bar{A}Q + QA^T + V_1\right)\right]\bar{V}_2^{-1} \tag{4.6}$$

$$C_c = -\bar{R}_2^{-1}\left[B_1^TP \quad B_2^T\left(P\bar{A} + A^TP + R_1\right)\right]v_{1\perp}G^T \tag{4.7}$$

$$D_c = -\bar{R}_2^{-1}\begin{bmatrix} 0_{m_1 \times r_1} & B_1^TPQC_2^T \\ B_2^TPQC_1^T & B_2^TP(\bar{A}Q + QA^T + V_1)C_2^T \end{bmatrix}\bar{V}_2^{-1} \tag{4.8}$$

where

$$\Sigma_Q = C_1^TV_2^{-1}C_1 \qquad \Sigma_P = B_1R_2^{-1}B_1^T$$

$$\bar{A} = A - Q\Sigma_Q - \Sigma_PP \qquad \bar{C}_2 = C_2^T\left(C_2QC_2^T\right)^{-1} \qquad \bar{B}_2 = \left(B_2^TPB_2\right)^{-1}B_2^T$$

and v_1, v_2, G^T, Γ, τ, $\bar{V}_2$ and $\bar{R}_2$ are as defined before. Q, P, $\hat{Q}$ and $\hat{P}$ are nonnegative matrices satisfying

$$[Av_{1\perp} + \tau_{\perp}v_{2\perp}(A-\Sigma_P P)v_1]Q + \dot{Q}[Av_{1\perp} + \tau_{\perp}v_{2\perp}(A-\Sigma_P P)v_1]^T + v_{1\perp}V_1v_{1\perp}^T$$

$$+ \tau_{\perp}v_{2\perp}V_1v_{2\perp}^T\tau_{\perp}^T - \tau_{\perp}v_{2\perp}v_{1\perp}V_1v_{1\perp}^Tv_{2\perp}^T\tau_{\perp}^T - Q\left(A^T\bar{C}_2C_2 + C_2^T\bar{C}_2^TA + v_{1\perp}^T\Sigma_Q v_{1\perp}\right)Q +$$

$$\tau_{\perp}v_{2\perp}Q\left(A^T\bar{C}_2C_2 + C_2^T\bar{C}_2^TA + v_{1\perp}^T\Sigma_Q v_{1\perp}\right)Qv_{2\perp}^T\tau_{\perp}^T = 0 \tag{4.9}$$

$$v_{2\perp}(A-\Sigma_P P)\hat{Q} + \hat{Q}(A-\Sigma_P P)^Tv_{2\perp}^T + v_{2\perp}v_{1\perp}Q\Sigma_Q Qv_{1\perp}^Tv_{2\perp}^T -$$

$$\tau_{\perp}v_{2\perp}v_{1\perp}Q\Sigma_Q Qv_{1\perp}^Tv_{2\perp}^T\tau_{\perp}^T + \tau v_{2\perp}[(A-\Sigma_P P)Q + QA^T + V_1]v_1^T$$

$$+ v_1[(A-\Sigma_P P)Q + QA^T + V_1]^Tv_{2\perp}^T\tau^T = 0 \tag{4.10}$$

$$P[v_{2\perp}A + v_2(A-Q\Sigma_Q)v_{1\perp}\tau_{\perp}] + [v_{2\perp}A + v_2(A-Q\Sigma_Q)v_{1\perp}\tau_{\perp}]^TP + v_{2\perp}^TR_1v_{2\perp}$$

$$+ \tau_{\perp}^Tv_{1\perp}^TR_1v_1\tau_{\perp} - \tau_{\perp}^Tv_{1\perp}^Tv_{2\perp}^TR_1v_{2\perp}v_{1\perp}\tau_{\perp} - P\left(A\bar{B}_2^TB_2^T + B_2\bar{B}_2A^T + v_{2\perp}\Sigma_P v_{2\perp}^T\right)P +$$

$$\tau_{\perp}^Tv_{1\perp}^TP\left(A\bar{B}_2^TB_2^T + B_2\bar{B}_2A^T + v_{2\perp}\Sigma_P v_{2\perp}^T\right)Pv_{1\perp}\tau_{\perp} = 0 \tag{4.11}$$

$$\hat{P}(A-Q\Sigma_Q)v_{1\perp} + v_{1\perp}^T(A-Q\Sigma_Q)^T\hat{P} + v_{1\perp}^Tv_{2\perp}^TP\Sigma_P Pv_{2\perp}v_{1\perp} -$$

$$\tau_{\perp}^Tv_{1\perp}^Tv_{2\perp}^TP\Sigma_P Pv_{2\perp}v_{1\perp}\tau_{\perp} + \tau^Tv_{1\perp}^T[(A-\Sigma_Q Q)P + PA + R_1]v_2 +$$

$$v_2^T[(A-\Sigma_Q Q)P + PA + R_1]^T\tau v_{1\perp} = 0 \tag{4.12}$$

Proof: The proof is given in Appendix B.

Remark 4.1: the three projections are disjoint as can be seen from the following relations which are derived in the proof of theorem 4.1

$$\tau\nu_1 = 0 \quad ; \quad \nu_2\tau = 0 \quad ; \quad \nu_2\nu_1 = 0 \tag{4.13a,b,c}$$

As a result, some combinations of the projection matrices, such as $\tau\nu_{2\perp}$ and $\nu_{1\perp}\tau$ are also projections. The results of theorem 4.1 may be interpreted using those combined projections. For example, $\Gamma\nu_2$ and G^T which premultiply and postmultiply respectively the bracketed term in the expression for A_c in eq. (4.5) are the factorization of the projection $\tau\nu_{2\perp}$. Eq. (4.13) gives also an order bound for the compensator. Eqs. (4.13a,b) state that

$$n_c \leq n - \max(r_2, m_2) \tag{4.14}$$

(4.13c) means that the sum of the singular variables, i.e. $r_2 + m_2$ should not exceed n.

Remark 4.2: To recover the results of sections 2 and 3, i.e. the one-side singular problems, the relations

$$\Gamma(QA^T + V_1)C_2^T = 0 \tag{4.15}$$

$$B_2^T(A^TP + R_1)G^T = 0 \tag{4.16}$$

which are proved in Appendix B should be used. In case either $\nu_1 = 0$ or $\nu_2 = 0$, eqs. (4.15) - (4.16) imply that all terms involving $QA^T + V_1$ and $A^TP + R_1$ vanish and the results of theorems 2.1 and 3.1 are recovered.

Remark 4.3: The expressions for the compensator matrices are not unique, which is in contrast to the results in the nonsingular and one-side singular cases. The matrices that correspond to the signals u_2 and y_2 have alternative expressions.

$$A_c = \Gamma\left[\nu_2\bar{A} + B_2\bar{B}_2(A^TP + R_1)\right]\nu_{1\perp}G^T \tag{4.17}$$

$$B_{c2} = \Gamma\left[\nu_2\bar{A} - B_2\bar{B}_2(A^TP + R_1)\right]C_2^* \tag{4.18}$$

$$C_{c2} = -B^*\left[\bar{A}\nu_{1\perp} - (QA^T + V_1)\bar{C}_2C_2\right]G^T \tag{4.19}$$

$$D_{c22} = -\left[B^*\bar{A} + \bar{B}(A^TP + R_1)\right]C^* \tag{4.20}$$

This is a consequence of the relation in eqs. (B.27)

$$B_2^T(QA^T + V_1)C_2^T = B_2^T(T_c + A^TP)QC_2^T$$

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Appendix A : Proof of Theorem 2.1

From eq. (2.9) and by assumption 2 it follows that

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0 \tag{A.1}$$

where

$$\tilde{V} = \begin{bmatrix} V_1 & 0 \\ 0 & B_{C_1} V_2 B_{C_1}^T \end{bmatrix}$$

Defining

$$\tilde{R} = \begin{bmatrix} R_1 + C_2^T D_{C2}^T R_2 D_{C2} C_2 & C_2^T D_{C2}^T R_2 C_C \\ C_C^T R_2 D_{C2} C_2 & C_C^T R_2 C_C \end{bmatrix}$$

The cost J may be written as $J = \mathrm{tr}\tilde{Q}\tilde{R}$

Adding the constraint (A.1) we define the Lagrangian

$$H = \mathrm{tr}[\tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}]. \tag{A.2}$$

$\partial H / \partial \tilde{P} = 0$ leads to (A . 1), the other partial derivative yield

$$\partial H / \partial \tilde{Q} = \tilde{P}\tilde{A} + \tilde{A}^T \tilde{P} + \tilde{R} = 0 \tag{A.3}$$

$$\partial H / \partial A_C = Q_2 P_2 + Q_{12}^T P_{12} = 0 \tag{A.4}$$

$$\partial H / \partial B_{C_1} = C_1 Q_1 P_{12} + C_1 Q_{12} P_2 + V_2 B_{C_1}^T P_2 = 0 \tag{A.5}$$

$$\partial H / \partial B_{C_2} = C_2 Q_1 P_{12} + C_2 Q_{12} P_2 = 0 \tag{A.6}$$

$$\partial H / \partial C_C = Q_{12}^T P_1 B + Q_2 P_{12}^T B + Q_2 C_C^T R_2 + Q_{12}^T C_2^T D_C^T R_2 = 0 \tag{A.7}$$

$$\partial H / \partial D_C = C_2 Q_1 P_1 B + C_2 Q_{12} P_{12}^T B + C_2 Q_1 C_2^T D_C^T R_2 + C_2 Q_{12} C_C^T R_2 = 0 \tag{A.8}$$

Subblocks of (A.1) and (A.3) give

$$A_d Q_1 + Q_1 A_d^T + BC_C Q_{12}^T - Q_{12} C_C^T B^T + V_1 = 0 \tag{A.9}$$

$$A_d Q_{12} + BC_C Q_2 + Q_1 C^T B_C^T + Q_{12} A_C^T = 0 \tag{A.10}$$

$$B_C C Q_{12} + A_C Q_2 + Q_2 A_C^T + Q_{12}^T C^T B_C^T + B_{C_1} V_2 B_{C_1}^T = 0 \tag{A.11}$$

$$P_1 A_d + A_d^T P_1 + P_{12} B_C C + C^T B_C^T P_{12}^T + R_1 + C_2^T D_C^T R_2 D_C C_2 = 0 \tag{A.12}$$

$$P_1 BC_C + P_{12} A_C + A_d^T P_{12} + C^T B_C^T P_2 + C_2^T D_C^T R_2 C_C = 0 \tag{A.13}$$

$$P_{12}^T BC_C + C_C^T B^T P_{12} + P_2 A_C + A_C^T P_2 + C_C^T R_2 C_C = 0 \tag{A.14}$$

where $A_d \triangleq A + BD_{C2} C_2$

Writing (A.14) as

$$P_2(A_C + P_2^+ P_{12}^T BC_C) + (A_C + P_2^+ P_{12} BC_C)^T P_2 + C_C^T R_2 C_C = 0 \tag{A.15}$$

where P_2^+ is the Moore-Penrose generalized inverse of P_2, it follows from the observability of (C_C, A_C) that P_2 is positive definite, hence invertible, see [17].

Define now

$$G^T = Q_{12} Q_2^{-1}; \Gamma = -P_2^{-1} P_{12}^T; \ \tau = G^T \Gamma \ ; \ \hat{Q} = Q_{12} Q_2^{-1} Q_{12}^T \ ; \qquad \hat{P} = P_{12} P_2^{-1} P_{12}^T$$

$$Q = Q_1 - \hat{Q} \quad ; \quad P = P_1 - P$$

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H_2/H_∞ Controller Synthesis with Singular Control Weighting and Singular Measurement Noise

by

Frederick C. Parker and Wassim M. Haddad
Department of Mechanical
and Aerospace Engineering
Florida Institute of Technology
Melbourne, Florida 32901

Dennis S. Bernstein
Harris Corporation
MS 22/4842
Melbourne, Florida 32902

Abstract

This paper considers the problem of fixed-order dynamic controller synthesis with both H_2 and H_∞ performance criteria. Sufficient conditions characterizing the optimal compensator for both singular control weighting and singular measurement noise are given. The solution consists of a set of two algebraic Riccati equations and two algebraic Lyapunov equations coupled by three projection matrices. One is the standard order reduction projection, while the other two reflect the two types of singularity present in the system. The three projections are shown to satisfy disjointness conditions. In addition to order reduction, an advantage of the fixed-structure approach is that differentiation, which is often undesirable from a practical point of view and which arises due to the singularities, can be avoided. The fixed-order compensator agrees with the unconstrained solution when the latter possesses the same number of differentiations as are included in the prespecified controller structure and when the order is selected appropriately.

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Appendix B: Proof of theorem 4.1

Following the development in Appendix A we get the Lagrangian (A.2) with

$$\tilde{V} = \begin{bmatrix} V_1 + B_2 D_{C21} V_2 D_{C21}^T B_2^T & B_2 D_{C21} V_2 B_{C1}^T \\ B_{C1} V_2 D_{C21}^T B_2^T & B_{C1} V_2 B_{C1}^T \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} R_1 + C_2^T D_{C12}^T R_2 D_{C12} C_2 & C_2^T D_{C12}^T R_2 C_{C_1} \\ C_{C1}^T R_2 D_{C12} C_2 & C_{C1}^T R_2 C_{C_1} \end{bmatrix} .$$

Now

$$\partial H / \partial A_C = Q_{12}^T P_{12} + Q_2 P_2 = 0 \qquad \text{(B. 1)}$$

$$\partial H / \partial B_{C1} = C_1 Q_1 P_{12} + C_1 Q_{12} P_2 + V_2 D_{C21}^T B_2^T P_{12} + V_2 B_{C1}^T P_2 = 0 \qquad \text{(B.2)}$$

$$\partial H / \partial B_{C2} = C_2 Q_1 P_{12} + C_2 Q_{12} P_2 = 0 \qquad \text{(B.3)}$$

$$\partial H / \partial C_{C1} = Q_{12}^T P_1 B_1 + Q_2 P_{12}^T B_2 + Q_2 C_{C1}^T R_2 + Q_{12}^T C_2^T D_{c12}^T R_2 = 0 \qquad \text{(B.4)}$$

$$\partial H / \partial C_{C2} = Q_{12}^T P_1 B_2 + Q_2 P_{12}^T B_2 = 0 \qquad \text{(B 5)}$$

$$\partial H / \partial D_{C12} = C_2 Q_1 P_1 B_1 + C_2 Q_{12} P_{12}^T B_1 + C_2 Q_{12} C_{C1}^T R_2 + C_2 Q_1 C_2^T D_{c12}^T R_1 = 0 \qquad \text{(B.6)}$$

$$\partial H / \partial D_{C21} = C_1 Q_1 P_1 B_2 + C_1 Q_{12} P_{12}^T B_2 + V_2 D_{C21}^T B_2^T P_1 B_2 + V_2 B_{C1}^T P_{12}^T B_2 = 0 \qquad \text{(B7)}$$

$$\partial H / \partial D_{C22} = C_2 Q_1 P_1 B_2 + C_2 Q_{12} P_{12}^T B_2 = 0 \qquad \text{(B.8)}$$

$$\partial H / \partial \tilde{P} = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0$$

$$\partial H / \partial \tilde{Q} = \tilde{P}\tilde{A} + \tilde{A}^T \tilde{P} + \tilde{R} = 0$$

Submatrices of these equations give

$$(A+BD_cC)Q_1+Q_1(A+BD_CC)^T+BC_CQ_{12}^T+Q_{12}C_C^TB^T+V_1+B_2D_{221}V_2D_{C21}^TB_2^T=0 \quad (B.9)$$

$$(A+BD_CC)Q_{12}+BC_CQ_2+Q_1C^TB_C^T+Q_{12}A_C^T+B_2D_{C21}V_2B_{C1}^T=0 \quad (B.10)$$

$$A_cQ_2+Q_2A_C^T+B_CCQ_{12}+Q_{12}^TC^TB_C^T+B_{C1}V_2B_{C1}^T=0 \quad (B.11)$$

$$P_1(A+BD_CC)+(A+BD_CC)^TP_1+P_{12}B_CC+C^TB_C^TP_{12}^T+R_1+C_2^TD_{C12}^TR_2D_{C12}C_2=0 \quad (B.12)$$

$$P_1BC_C+P_{12}A_C+(A+BD_CC)^TP_{12}+C^TB_C^TP_2+C_2^TD_{C12}^TR_2C_{C1}=0 \quad (B.13)$$

$$P_2A_C+A_C^TP_2+P_{12}^TBC_C+C_C^TB^TP_{12}+C_{C1}^TR_2C_{C1}=0 \quad (B.14)$$

Eq. (B. 1) gives the projection $\tau=G^T\Gamma$. From $(B.3)B_2^{-1}$, using $Q_{12}=\hat{Q}\Gamma^T$ we obtain $\Gamma QC_2^T=0$, or $\tau\nu_1=0$. Similarly $Q_2^{-1}(B.5)$ leads to $B_2^TPG^T=0$ or $\nu_2\tau=0$. Using these results together with $Q_{12}P_{12}^T=\hat{Q}\hat{P}$ we get out of eq. (B. 8), $C_2QPB_2^T=0$ or $\nu_2\nu_1=0$. So the partial derivatives with respect to the " singular" coefficients lead to the disjointness conditions. $(B.2)\Gamma B_2+(B.7)$, noting that $\hat{Q}PB_2=0$, gives

$$D_{C21}=-B_2^*QC_1^TV_2^{-1} \quad (B.15)$$

The dual $(B.6)-C_2G^T(B.4)$ leads to the dual expression

$$D_{C12}=-R_2^{-1}B_1^TPC_2^* \quad (B.16)$$

Substituting D_{C21} and D_{C12} into eqs, (B. 2) and (B. 4) respectively we have

$$B_{C1}=\Gamma\nu_{2\perp}QC_1^TV_2^{-1} \quad (B.17)$$

$$C_{C1}=-R_2^{-1}B_1^TP\nu_{1\perp}G^T \quad (B.18)$$

$(B.9) - (B.10) - G^T(B.10)^T + G^T(B.11)G$ gives

$$(A + BD_CC)Q + Q(A + BD_CC)^T - G^TB_CCQ - QC^TB_C^TG + V_1 + B_2D_{C21}V_2D_{C21}^TB_2^T$$
$$- B_2D_{C21}V_2B_{C1}^TG - G^TB_{C1}V_2D_{C21}^TB_2^T + G^TB_{C1}V_2B_{C1}^TG = 0 \tag{B.19}$$

$\Gamma(B.15)C_2^T + P_2^{-1}(B.13)QC_2^T$, using $\Gamma QC_2^T = 0$, yields

$$\Gamma\left[Q\left(A + BD_CC - G^TB_CC\right)^T + V_1 + B_2D_{C21}V_2\left(D_{C21}^TB_2^T - B_{C1}^TG\right)\right]C_2^T$$
$$- B_{C1}V_2\left(D_{C21}^TB_2^T - B_{C1}^TG^T\right)C_2^T + P_2^{-1}C_C^TB^TPQC_2^T + P_2^{-1}C_{C1}^TR_2D_{C12}C_2QC_2^T = 0 \tag{B.20}$$

Substituting $D_{C12}D_{C21}$, B_{C1}, C_{C1} and using $\Gamma QC_2^T = 0$, $B_2^TPQC_2^T = 0$, we obtain

$$\Gamma\left(QA^T + V_1\right)C_2^T = 0 \tag{B. 21}$$

Similar operations on eqs. (B.12) - (B.14) and (B.10) give the dual relation

$$B_2^T\left(A^TP + R_1\right)G^T = 0 \tag{B. 22}$$

Substituting C_{C12}, D_{C21}, B_{C1} and C_{C1} into (B.19) and rearranging we have

$$\left(A - \Sigma_PPv_1 + B_2D_{C22}C_2 - G^TB_{C2}C_2\right)Q + Q\left(A - \Sigma_PPv_1 + B_2D_{C22}C_2 - G^TB_{C2}C_2\right)^T$$
$$+ V_1 - Q\Sigma_QQ + \tau_\perp v_{2\perp}Q\Sigma_QQv_{2\perp}^T\tau_\perp^T = 0 \tag{B.23}$$

$B_2^TP(B.23)C_2^T$ gives, using $B_2^TPG^T = 0$, $B_2^TPQC_2^T = 0$

$$B_2^TP(A - \Sigma_PPv_1 + B_2D_{c22}C_2)QC_2^T + B_2^TP\left(QA^T + V_1\right)C_2^T - B_2^TPQ\Sigma_QC_2^T = 0 \tag{B. 24}$$

Premultiplying by $\left(B_2^TPB_2\right)^{-1}$ and postmultiplying by $\left(C_2QC_2^T\right)^{-1}$ we get

$$D_{C22} = -B_2^*(A - \Sigma_PP - Q\Sigma_Q)C_2^* - B_2^*\left(QA^T + V_1\right)\bar{C}_2 \tag{B. 25}$$

Constructing the dual of (B.18), similar operations lead to

$$D_{C22} = -B_2^*(A - \Sigma_P P - Q\Sigma_Q)C_2^* - \bar{B}_2(A^T P + R_1)C_2^* \tag{B. 26}$$

Therefore

$$B_2^T P(QA^T + V_1)C_2^T = B_2^T(A^T P + R_1)QC_2^T \tag{B.27}$$

This realtion is the source of the alternative expressions in remark 4.3.
Γ(B. 18)$\bar{C}_2$ gives, after substitution of D_{C22}

$$B_{C2} = \Gamma(A - \Sigma_P P - Q\Sigma_Q)C_2^* + \Gamma B_2 D_{C22} = \Gamma v_{2\perp}(A - \Sigma_P P - Q\Sigma_Q)C_2^* - \Gamma v_2(A^T P + R_1)\bar{C}_2 \tag{B.28}$$

(B.21) implies that $-\Gamma v_2$ may be replaced by $\Gamma v_{2\perp}$, hence (4.6). The expression for C_{C2} is obtaiend by operating in a similar fashion on the dual of (B.23).

Substituting D_{C22} and B_{C2} into (B.23) we have

$$\begin{aligned}
&\left[A - \Sigma_P P v_1 - \tau_\perp v_2(A - \Sigma_P P - Q\Sigma_Q)v_1 - \tau_\perp v_2(QA^T + V_1)\bar{C}_2 C_2 - \tau(A - \Sigma_P P - Q\Sigma_Q)v_1\right]Q \\
&+ Q\left[A - \Sigma_P P v_1 - \tau_\perp v_2(A - \Sigma_P P - Q\Sigma_Q)v_1 - \tau_\perp v_2(QA^T + V_1)\bar{C}_2 C_2 - \tau(A - \Sigma_P P - Q\Sigma_Q)v_1\right]^T \\
&+ V_1 - Q\Sigma_Q Q + \tau_\perp v_{2\perp} Q\Sigma_Q Q v_{2\perp}^T \tau_\perp^T = 0
\end{aligned} \tag{B. 29}$$

(B. 29) $- G^T$(B. 21)$C_2^{*T} - C_2$(B. 21)$^T G$ gives, using $\tau_\perp v_2 + \tau = I - \tau_\perp v_{2\perp}$

$$\begin{aligned}
&[Av_\perp + \tau_\perp v_{2\perp}(A - \Sigma_P P)]Q + Q[Av_{1\perp} + \tau_\perp v_{2\perp}(A - \Sigma_P P)]^T + (I - \tau_\perp v_{2\perp})Q(\Sigma_Q v_1 \\
&- A^T \bar{C}_2 C_2)Q - (I - \tau_\perp v_{2\perp})V_1 v_1 + Q(v_1^T \Sigma_Q - C_2^T \bar{C}_2^T A)Q(I - v_{2\perp}^T \tau_\perp^T) \\
&+ V_1 - Q\Sigma_Q Q + \tau_\perp v_{2\perp} Q\Sigma_Q Q v_{2\perp}^T \tau_\perp^T = 0
\end{aligned} \tag{B. 30}$$

The following relations are used to get (4.9) from (B.30):

$$\tau_{\perp}v_{2\perp}QC_2^T = QC_2^T,\ \ \tau_{\perp}v_{2\perp} - v_1 = \tau_{\perp}v_{2\perp}v_{1\perp},\ \ I - \tau_{\perp}v_{2\perp} = v_{1\perp} - \tau_{\perp}v_{2\perp}v_{1\perp}$$

(B. 9) – (B. 19) gives, using $Q_{12}^T = \Gamma\hat{Q}$

$$(A + BD_CC)\hat{Q} + \hat{Q}(A + BD_CC)^T + GC_C\Gamma\hat{Q} + \hat{Q}\Gamma^TC_C^TB + G^TB_CCQ + QC^TB_C^TG$$

$$- B_2D_{C21}V_2B_{C1}^TG - G^TB_{C1}V_2D_{C21}^TB_2^T + G^TB_{C1}V_2B_{C1}^TG = 0 \qquad \text{(B. 31)}$$

Substitution of D_C, C_C and B_C, using $\tau\hat{Q} = \hat{Q}$ leads to (4. 10). Eqs. (4. 11) and (4. 12) are dual to (4. 9) – (4. 10) and are achieved by the same procedure. Finally to find A_C we take $[(B.11) - \Gamma v_{2\perp}(B.10)]\,Q_2^{-1}$ and have

$$A_C = -B_CCG^T + \Gamma v_{2\perp}QC_1^TB_{C1}^T - B_{C1}V_2B_{C1}^T + \Gamma v_{2\perp}(A + BD_CC)\,Q_{12} + \Gamma v_{2\perp}BC_CQ_2 \qquad \text{(B.32)}$$

Substituting B_C, C_C, D_C into (B.32) we get (4.5).

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H_2/H_∞ Controller Synthesis with Singular Control Weighting and Singular Measurement Noise

by

Frederick C. Parker and Wassim M. Haddad
Department of Mechanical
and Aerospace Engineering
Florida Institute of Technology
Melbourne, Florida 32901

Dennis S. Bernstein
Harris Corporation
MS 22/4842
Melbourne, Florida 32902

Abstract

This paper considers the problem of fixed-order dynamic controller synthesis with both H_2 and H_∞ performance criteria. Sufficient conditions characterizing the optimal compensator for both singular control weighting and singular measurement noise are given. The solution consists of a set of two algebraic Riccati equations and two algebraic Lyapunov equations coupled by three projection matrices. One is the standard order reduction projection, while the other two reflect the two types of singularity present in the system. The three projections are shown to satisfy disjointness conditions. In addition to order reduction, an advantage of the fixed-structure approach is that differentiation, which is often undesirable from a practical point of view and which arises due to the singularities, can be avoided. The fixed-order compensator agrees with the unconstrained solution when the latter possesses the same number of differentiations as are included in the prespecified controller structure and when the order is selected appropriately.

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1. Introduction

The Riccati equation approach to H_∞ controller synthesis is now well established [1–7]. The purpose of the present paper is to go beyond earlier work by addressing the problems of singular measurement noise and singular control weighting. The approach we take is based upon fixed-structure controller synthesis with both H_2 and H_∞ performance criteria as in [5,7]. It is well known that singularities in the problem formulation may lead to improper compensators, that is, compensators with differentiators [8]. The fixed-structure approach, however, allows us to specify the compensator structure a priori, thus eliminating differentiation of measurement signals if so desired. If differentiation is desired, then the differentiated signals can be included as additional measurement signals and then treated within the fixed structure approach as "original" signals.

The treatment of singular problems in the present paper is a direct extension of results obtained for singular H_2 problems in [9–13]. The reader is referred to these papers for discussions of singular control issues as well as extensive references to the singular control and estimation literature.

Notation

Note: All matrices have real entries.

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{E}$	real numbers, $r\times s$ real matrices, expected value
$I_r, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$r\times r$ identity matrix, transpose $r\times s$ zero matrix, $0_{r\times r}$
tr	trace
$\mathbb{N}^r, \mathbb{P}^r$	$r\times r$ nonnegative-definite, positive-definite matrices
$n, m, \ell, \ell_1, \ell_2, n_c, d, d_\infty, q, q_\infty; \tilde{n}$	positive integers; $n+n_c(n_c\leq n)$
$x, u, y_1, y_2, x_c, \tilde{x}$	$n, m, \ell_1, \ell_2, n_c, \tilde{n}$-dimensional vectors
$y, \tilde{x}$	$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\begin{bmatrix} x \\ x_c \end{bmatrix}$
A, B, C_1, C_2	$n\times n, n\times m, \ell_1\times n, \ell_2\times n$ matrices
C	$[C_1 \quad C_2]$
$A_c, B_{c1}, B_{c2}, C_c, D_{c11}, D_{c12}$	$n_c\times n_c, n_c\times \ell_1, n_c\times\ell_2, m\times n_c, m\times\ell_1, m\times\ell_2$ matrices
B_c, D_c	$[B_{c1} \quad B_{c2}]$, $[D_{c11} \quad D_{c12}]$
$\tilde{A}$	$\begin{bmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$
$w(\cdot)$	d-dimensional standard white noise
D_1, D_2	$n\times d, \ell_1\times d$ matrices; $D_1D_2^{\mathrm{T}}=0$
$D_{1\infty}, D_{2\infty}$	$n\times d_\infty, \ell\times d_\infty$ matrices; $D_{1\infty}D_{2\infty}^{\mathrm{T}}=0$

V_1, V_2	$D_1 D_1^{\mathrm{T}}, D_2 D_2^{\mathrm{T}};\ V_2 \in \mathbb{P}^{\ell_1}$
$V_{1\infty}, V_{2\infty}$	$D_{1\infty} D_{1\infty}^{\mathrm{T}},\ D_{2\infty} D_{2\infty}^{\mathrm{T}}$
$\tilde{D}$	$\begin{bmatrix} D_1 \\ B_{c1} D_2 \end{bmatrix}$
$\tilde{V}$	$\begin{bmatrix} V_1 & 0_{n \times n_c} \\ 0_{n_c \times n} & B_{c1} V_2 B_{c1}^{\mathrm{T}} \end{bmatrix}$
E_1, E_2	$q \times n, q \times m$ matrices; $E_1^{\mathrm{T}} E_2 = 0$
$\tilde{E}$	$[E_1 + E_2 D_c C \quad E_2 C_c]$
R_1, R_2	$E_1^{\mathrm{T}} E_1,\ E_2^{\mathrm{T}} E_2;\ \ R_2 \in \mathbb{P}^m$
$\tilde{R}$	$\begin{bmatrix} R_1 + C^{\mathrm{T}} D_c^{\mathrm{T}} R_2 D_c C & C^{\mathrm{T}} D_c^{\mathrm{T}} R_2 C_c \\ C_c^{\mathrm{T}} R_2 D_c C & C_c^{\mathrm{T}} R_2 C_c \end{bmatrix}$
$E_{1\infty}, E_{2\infty}$	$q_\infty \times n,\ q_\infty \times m$ matrices; $E_{1\infty}^{\mathrm{T}} E_{2\infty} = 0$
$\tilde{R}_\infty$	$\begin{bmatrix} R_{1\infty} + C^{\mathrm{T}} D_c^{\mathrm{T}} R_{2\infty} D_c C & C^{\mathrm{T}} D_c^{\mathrm{T}} R_{2\infty} C_c \\ C_c^{\mathrm{T}} R_{2\infty} D_c C & C_c^{\mathrm{T}} R_{2\infty} C_c \end{bmatrix}$
$\Sigma, \bar{\Sigma}$	$B R_2^{-1} B^{\mathrm{T}},\ C_1^{\mathrm{T}} V_2^{-1} C_1$
β, γ	nonnegative constant, positive constant

2. Problem Statement

In this section we consider the H_2 dynamic-compensation control problem with constrained H_∞ disturbance attenuation for the case in which the measurement noise intensity is singular but the control weighting matrix is nonsingular. Given the nth order stabilizable and detectable plant (see Fig. 1)

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{2.1}$$

$$y_1(t) = C_1 x(t) + D_2 w(t), \tag{2.2a}$$

$$y_2(t) = C_2 x(t), \tag{2.2b}$$

determine an n_cth order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t) + D_c y(t), \tag{2.4}$$

that satisfies the following design criteria:

i) the closed loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable,

ii) the $q_\infty \times p$ closed-loop transfer function

$$G(s) \triangleq \tilde{E}_\infty(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{2.5}$$

from $w(t)$ to $z_\infty(t) \triangleq E_{1\infty}x(t) + E_{2\infty}u(t)$ satisfies the constraint

$$\|G(s)\|_\infty \leq \gamma, \tag{2.6}$$

where $\gamma > 0$ is a given constant; and

iii) the performance functional

$$J(A_c, B_c, C_c, D_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\Big\{\int_0^t [x^{\mathrm{T}}(t)R_1x(t) + u^{\mathrm{T}}(t)R_2u(t)]\mathrm{d}t\Big\}. \tag{2.7}$$

is minimized.

The closed-loop system (2.1)–(2.4) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \tag{2.8}$$

while (2.7) becomes

$$J(A_c, B_c, C_c, D_c) \triangleq \lim_{t\to\infty} \mathbb{E}\Big\{(\tilde{E}\tilde{x}(t))^{\mathrm{T}}(\tilde{E}\tilde{x}(t))\Big\}. \tag{2.9}$$

Since all of the components of $u(t)$ are weighted, a direct transmission term from noisy measurements to weighted inputs will result in an infinite H_2 cost. Therefore, we set $D_{c1} = 0$. Note also that the problem statement involves both H_2 and H_∞ performance weights. For convenience we define $R_1 \triangleq E_1^{\mathrm{T}}E_1$ and $R_2 \triangleq E_2^{\mathrm{T}}E_2$. Although an H_2 cross weighting term of the form $2x^{\mathrm{T}}(t)R_{12}u(t)$ can be included, we shall not do so here to facilitate the presentation.

For the H_∞ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$ and $E_{2\infty}$ for the state and control variables. The matrices $R_{1\infty} \triangleq E_{1\infty}^{\mathrm{T}}E_{1\infty}$ and $R_{2\infty} \triangleq E_{2\infty}^{\mathrm{T}}E_{2\infty}$ are thus the H_∞ counterparts of the H_2 weights R_1 and R_2. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to R_1 and R_2 we shall require that $R_{2\infty} = \beta^2 R_2$ where the nonnegative scalar β is a design variable. Finally, the condition $E_{1\infty}^{\mathrm{T}}E_{2\infty} = 0$ precludes an H_∞ cross-weighting term which again facilitates the presentation. Similar remarks apply to the disturbance and sensor noise intensities $V_1 \triangleq D_1D_1^{\mathrm{T}}$, $V_2 \triangleq D_2D_2^{\mathrm{T}}$, $V_{1\infty} \triangleq D_{1\infty}D_{1\infty}^{\mathrm{T}}$, and $V_{2\infty} \triangleq D_{2\infty}D_{2\infty}^{\mathrm{T}}$ for the H_2 and H_∞

designs respectively. Finally for convenience, we assume $D_1 D_2^{\mathrm{T}} = 0$ which effectively implies that the plant disturbance and sensor noise are uncorrelated.

If $\tilde{A}$ is asymptotically stable for a given compensator (A_c, B_c, C_c, D_c), then the performance (2.9) is given by

$$J(A_c, B_c, C_c, D_c) = \operatorname{tr} \tilde{Q}\tilde{R},$$

where the steady-state closed-loop covariance defined by

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}\{\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t)\} \tag{2.10}$$

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{2.11}$$

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.11) by an algebraic Riccati equation which overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.

Lemma 2.1. Let (A_c, B_c, C_c, D_c) be given and assume there exists an $\tilde{n} \times \tilde{n}$ nonnegative-definite matrix $\mathcal{Q}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + \tilde{V}. \tag{2.12}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.13}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \tag{2.14}$$

In this case,

$$\|G(s)\|_\infty \le \gamma \tag{2.15}$$

and

$$\tilde{Q} \le \mathcal{Q}. \tag{2.16}$$

Consequently,

$$J(A_c, B_c, C_c, D_c) \le J(A_c, B_c, C_c, D_c, \mathcal{Q}), \tag{2.17}$$

where

$$J(A_c, B_c, C_c, D_c, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{2.18}$$

Proof. See [5]. □

3. The Auxiliary Minimization Problem

As shown in the previous section, replacing (2.11) by (2.12) enforces the H_∞ disturbance attenuation constraint and yields an upper bound for the H_2 performance criterion. That is, given a compensator (A_c, B_c, C_c, D_c) for which there exists a nonnegative-definite solution to (2.12), the *actual* H_2 performance $J(A_c, B_c, C_c, D_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(A_c, B_c, C_c, D_c, \mathcal{Q})$. Hence $\mathcal{J}(A_c, B_c, C_c, D_c, \mathcal{Q})$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, D_c, \mathcal{Q})$ that minimizes $\mathcal{J}(A_c, B_c, C_c, D_c, \mathcal{Q})$ subject to (2.12).

It follows from Lemma 2.1 that the satisfaction of (2.12) along with the generic condition (2.13) leads to : (1) closed-loop stability; (2) prespecified H_∞ disturbance attenuation; and (3) an upper bound for the H_2 performance criterion. Hence, it remains to determine $(A_c, B_c, C_c, D_c, \mathcal{Q})$ that minimizes $\mathcal{J}(A_c, B_c, C_c, D_c, \mathcal{Q})$, and thus provides an optimal bound for the actual H_2 performance $J(A_c, B_c, C_c, D_c)$.

4. Sufficient Conditions for H_∞ Disturbance Attenuation with Singular Measurement Noise

In this section we state sufficient conditions for characterizing fixed-order, (i.e., full- and reduced-order) controllers guaranteeing closed-loop stability, constrained H_∞ disturbance attenuation, and an optimized H_2 performance bound. For arbitrary matrices $Q, P, \hat{Q} \in \mathbb{R}^{n\times n}$ define the notation

$$S_1 \triangleq (I_n + \beta^2\gamma^{-2}\hat{Q}P)^{-1},$$

$$\nu_1 \triangleq QC_2^{\mathrm{T}}(C_2QC_2^{\mathrm{T}})^{-1}C_2, \quad \nu_{1\perp} \triangleq I_n - \nu_1,$$

$$\nu_{1\infty} \triangleq QC_2^{\mathrm{T}}(C_2QC_2^{\mathrm{T}} + \gamma^{-2}C_2QPS_1QC_2^{\mathrm{T}})^{-1}C_2(I_n + \gamma^{-2}QPS_1), \quad \nu_{1\infty\perp} \triangleq I_n - \nu_{1\infty},$$

$$\Sigma \triangleq BR_2^{-1}B^{\mathrm{T}}, \quad \bar{\Sigma} \triangleq C_1^{\mathrm{T}}V_2^{-1}C_1,$$

$$\hat{C}_2 \triangleq QC_2^{\mathrm{T}}(C_2QC_2^{\mathrm{T}} + \gamma^{-2}C_2QPS_1QC_2^{\mathrm{T}})^{-1},$$

when the indicated inverses exist.

Allowing the compensator to be of fixed dimension n_c, which may be less than the plant order n, leads to an oblique projection that introduces additional coupling in the design equations. The following lemma is required.

Lemma 4.1. Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ G, Γ and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma, \tag{4.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_c}. \tag{4.2}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}} \Gamma, \tag{4.3}$$

$$\tau_\perp \triangleq I_n - \tau, \tag{4.4}$$

are idempotent and have rank n_c and $n - n_c$ respectively.

Proof. See [14]. □

Theorem 4.1. Suppose there exist matrices $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying

$$\begin{aligned}
0 = {} & [A\nu_{1\perp} + \tau_\perp(A - \Sigma P S_1 \hat{C}_2 C_2 + \gamma^{-2} Q R_{1\infty})\nu_1]Q \\
& + Q[A\nu_{1\perp} + \tau_\perp(A - \Sigma P S_1 \hat{C}_2 C_2 + \gamma^{-2} Q R_{1\infty})\nu_1]^{\mathrm{T}} + V_1 \\
& - \gamma^{-2}\nu_1 Q R_{1\infty} Q \nu_1^{\mathrm{T}} + \nu_{1\perp} Q R_{1\infty} Q \nu_{1\perp}^{\mathrm{T}} \\
& + \gamma^{-2}\beta^2 \nu_1 Q (\hat{C}_2 C_2)^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 (\hat{C}_2 C_2) Q \nu_1^{\mathrm{T}} \\
& - \nu_{1\perp} Q \bar{\Sigma} Q \nu_{1\perp}^{\mathrm{T}} + \tau_\perp \nu_{1\perp} Q \bar{\Sigma} Q \nu_{1\perp}^{\mathrm{T}} \tau_\perp^{\mathrm{T}},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
0 = {} & [A + \gamma^{-2}(Q + \hat{Q}) R_{1\infty}]^{\mathrm{T}} P + P[A + \gamma^{-2}(Q + \hat{Q}) R_{1\infty}] \\
& + R_1 - (\nu_{1\infty\perp} + \hat{C}_2 C_2)^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 (\nu_{1\infty\perp} + \hat{C}_2 C_2) \\
& + \tau_\perp^{\mathrm{T}} \nu_{1\infty\perp}^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 \nu_{1\infty\perp} \tau_\perp,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
0 = {} & \{A - \Sigma P S_1(\nu_{1\infty\perp} + \hat{C}_2 C_2) + \gamma^{-2} Q [R_{1\infty} + \beta^2 (\hat{C}_2 C_2)^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 (\hat{C}_2 C_2)]\}\hat{Q} \\
& + \hat{Q}\{A - \Sigma P S(\nu_{1\infty\perp} + \hat{C}_2 C_2) + \gamma^{-2} Q [R_{1\infty} \beta^2 (\hat{C}_2 C_2)^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 (\hat{C}_2 C_2)]\}^{\mathrm{T}} \\
& + \tau(A - \Sigma P S_1 \hat{C}_2 C_2 + \gamma^{-2} Q R_{1\infty}) Q \nu_1^{\mathrm{T}} \\
& + \nu_1 Q (A - \Sigma P S_1 \hat{C}_2 C_2 + \gamma^{-2} Q R_{1\infty})^{\mathrm{T}} \tau^{\mathrm{T}} \\
& + \gamma^{-2} Q [R_{1\infty} + \beta^2 (\nu_{1\infty\perp} + \hat{C}_2 C_2)^{\mathrm{T}} S_1^{\mathrm{T}} P \Sigma P S_1 (\nu_{1\infty\perp} + \hat{C}_2 C_2)] \hat{Q}
\end{aligned}$$

$$+\nu_{1\perp}Q\bar{\Sigma}Q\nu_{1\perp}^{\mathrm{T}}-\tau_{\perp}\nu_{1\perp}Q\bar{\Sigma}Q\nu_{1\perp}^{\mathrm{T}}\tau_{\perp}^{\mathrm{T}}, \tag{4.7}$$

$$\begin{aligned}0=&[(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})\nu_{1\perp}-\gamma^{-2}\beta^{2}Q(\hat{C}_2C_2)^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1(\hat{C}_2C_2)^{\mathrm{T}}\hat{P}\\&+\hat{P}[(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})\nu_{1\perp}-\gamma^{-2}\beta^{2}Q(\hat{C}_2C_2)^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1(\hat{C}_2C_2)]\\&+\nu_{1\infty\perp}^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1\nu_{1\infty\perp}-\tau_{\perp}^{\mathrm{T}}\nu_{1\infty\perp}^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1\nu_{1\infty\perp}\tau_{\perp},\end{aligned} \tag{4.8}$$

and let $(A_c, B_c, C_c, D_c, \mathcal{Q})$ be given by

$$A_c=\Gamma[(A-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})\nu_{1\perp}-\Sigma PS_1\nu_{1\infty\perp}]G^{\mathrm{T}}, \tag{4.9}$$

$$B_c=\Gamma[QC_1^{\mathrm{T}}V_2^{-1}(A-\Sigma PS_1\hat{C}_2C_2-Q\bar{\Sigma}+\gamma^{-2}QR_{1\infty})QC_2^{\mathrm{T}}(C_2QC_2^{\mathrm{T}})^{-1}], \tag{4.10}$$

$$C_c=-R_2^{-1}B^{\mathrm{T}}PS_1\nu_{1\infty\perp}G^{\mathrm{T}}, \tag{4.11}$$

$$D_c=[0_{m\times \ell_1}-R_2^{-1}B^{\mathrm{T}}PS_1\hat{C}_2], \tag{4.12}$$

$$\mathcal{Q}=\begin{bmatrix}Q+\hat{Q} & \hat{Q}\Gamma^{\mathrm{T}}\\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}}\end{bmatrix}. \tag{4.13}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $G(s)$ satisfies the H_∞ disturbance attenuation constraint (2.15) and the H_2 performance criterion (2.7) satisfies the bound

$$\begin{aligned}J(A_c,B_c,C_c,D_c)\le \operatorname{tr}[&(Q+\hat{Q})R_1+Q(\hat{C}_2C_2)^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1(\hat{C}_2C_2)\\&+\hat{Q}(\nu_{1\infty\perp}+\hat{C}_2C_2)^{\mathrm{T}}S_1^{\mathrm{T}}P\Sigma PS_1(\nu_{1\infty\perp}+\hat{C}_2C_2)].\end{aligned} \tag{4.14}$$

Proof. The proof follows as in [5] with the additional terms arising due to the singular structure of the problem. For further details see [12]. □

Remark 4.1. By setting $n_c = n$, it follows that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case, the last term in each of (4.5)–(4.8) can be deleted yielding the full-order H_2/H_∞ singular measurement noise control problem. Alternatively, by retaining the reduced-order constraint and letting $\gamma \to \infty$ we recover Theorem 2.1 of [12].

Remark 4.2. The nonsingular reduced order compensator [14] is recovered by setting $C_2 = 0$ and, consequently, $\nu_1 = 0$, $B_{c2} = 0$, $D_c = 0$.

5. Sufficient Conditions for H_∞ Disturbance Attenuation with Singular Control Weighting

In this section we consider the dual problem involving unweighted controls. The input is separated into a weighted component $u_1 \in \mathbb{R}^{m_1}$ and an unweighted component $u_2 \in \mathbb{R}^{m_2}$. The plant now is given by (see Fig. 2)

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + D_1w(t), \tag{5.1}$$

$$y(t) = Cx(t) + D_2w(t), \tag{5.2}$$

with the compensator structure as in (2.3)–(2.4), where

$$B \triangleq [B_1 \quad B_2], \quad C_c \triangleq \begin{bmatrix} C_{c1} \\ C_{c2} \end{bmatrix}, \quad D_c \triangleq \begin{bmatrix} D_{c11} \\ D_{c21} \end{bmatrix}. \tag{5.3}$$

We thus seek to minimize

$$J(A_c, B_c, C_c, D_c) \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\{\int_0^t [x^{\mathrm{T}}(t)R_1x(t) + u_1^{\mathrm{T}}R_2u_1(t)]\mathrm{d}t\}, \tag{5.4}$$

while constraining

$$\|\hat{G}(s)\|_\infty \le \gamma, \tag{5.5}$$

where

$$\hat{G}(s) \triangleq \hat{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D}_\infty, \tag{5.6}$$

and

$$\hat{E} \triangleq [E_1 \quad E_2C_{c1}], \quad \tilde{D}_\infty \triangleq \begin{bmatrix} D_{1\infty} + B_2D_{c21}D_{2\infty} \\ B_cD_{2\infty} \end{bmatrix}.$$

For the statement of the next result we need the following definitions. For arbitrary matrices $Q, P, \hat{P} \in \mathbb{R}^{n\times n}$ define the notation

$$S_2 \triangleq (I_n + \beta^2\gamma^{-2}Q\hat{P})^{-1},$$

$$\nu_2 \triangleq B_2(B_2^{\mathrm{T}}PB_2)^{-1}B_2^{\mathrm{T}}P, \quad \nu_{2\perp} \triangleq I_n - \nu_2,$$

$$\nu_{2\infty} \triangleq (I_n + S_2QP)B_2(B_2^{\mathrm{T}}PB_2 + \gamma^{-2}PS_2QPB_2)^{-1}B_2^{\mathrm{T}}P, \quad \nu_{2\infty\perp} \triangleq I_n - \nu_{2\infty},$$

$$\Sigma \triangleq B_1R_2^{-1}B_1^{\mathrm{T}}, \quad \bar{\Sigma} \triangleq C^{\mathrm{T}}V_2^{-1}C,$$

$$\hat{B}_2 \triangleq (B_2^{\mathrm{T}}PB_2)^{-1}B_2^{\mathrm{T}}P,$$

when the indicated inverses exist.

Theorem 5.1. Suppose there exist matrices $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying

$$0 = [A + \gamma^{-2}(P + \hat{P})V_{1\infty}]Q + Q[A + \gamma^{-2}(P + \hat{P})V_{1\infty}] + V_1 - (\nu_{2\infty\perp} + B_2\hat{B}_2)S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}(\nu_{2\infty\perp} + B_2\hat{B}_2)^{\mathrm{T}} + \tau_\perp\nu_{2\infty\perp}S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\nu_{2\infty\perp}^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{5.7}$$

$$0 = [\nu_{2\perp}A + \nu_2(A - B_2\hat{B}_2S_2Q\bar{\Sigma} + \gamma^{-2}\nu_{1\infty}P)\tau_\perp]^{\mathrm{T}}P + P[\nu_{2\perp}A + \nu_2(A - B_2\hat{B}_2S_2Q\bar{\Sigma} + \gamma^{-2}V_{1\infty}P)\tau_\perp] + R_1 - \gamma^{-2}\nu_2^{\mathrm{T}}PV_{1\infty}P\nu_2 + \nu_{2\perp}^{\mathrm{T}}PV_{1\infty}P\nu_{2\perp} + \gamma^{-2}\beta^2\nu_2^{\mathrm{T}}PB_2\hat{B}_2S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\hat{B}_2^{\mathrm{T}}B_2^{\mathrm{T}}P\nu_2^{\mathrm{T}} - \nu_{2\perp}^{\mathrm{T}}P\Sigma P\nu_{2\perp} + \tau_\perp^{\mathrm{T}}\nu_{2\perp}^{\mathrm{T}}P\Sigma P\nu_{2\perp}\tau_\perp, \tag{5.8}$$

$$0 = [\nu_{2\perp}(A - \Sigma P + \gamma^{-2}V_{1\infty}P) - \gamma^{-2}\beta^2B_2\hat{B}_2S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\hat{B}_2^{\mathrm{T}}B_2P]\hat{Q} + \hat{Q}[\nu_{2\perp}(A - \Sigma P + \gamma^{-2}V_{1\infty}P) - \gamma^{-2}\beta^2B_2\hat{B}_2S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\hat{B}_2^{\mathrm{T}}B_2P]^{\mathrm{T}} + \nu_{2\infty\perp}S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\nu_{2\infty\perp}^{\mathrm{T}} - \tau_\perp\nu_{2\infty\perp}S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\nu_{2\infty\perp}^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \tag{5.9}$$

$$0 = [A - (B_2\hat{B}_2 + \nu_{2\infty\perp})S_2Q\bar{\Sigma} + \gamma^{-2}(\nu_{1\infty} + \beta^2B_2\hat{B}_2S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\hat{B}_2^{\mathrm{T}}B_2)]^{\mathrm{T}}\hat{P} + \hat{P}[A - (B_2\hat{B}_2 + \nu_{2\infty\perp})S_2Q\bar{\Sigma} + \gamma^{-2}(\nu_{1\infty} + \gamma^{-2}B_2\hat{B}_2S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}\hat{B}_2^{\mathrm{T}}B_2)] + \nu_2^{\mathrm{T}}P(A - B_2\hat{B}_2S_2Q\bar{\Sigma} + \gamma^{-2}V_{1\infty}P)\tau + \tau^{\mathrm{T}}(A - B_2\hat{B}_2S_2Q\bar{\Sigma} + \gamma^{-2}V_{1\infty}P)^{\mathrm{T}}P\nu_2 + \gamma^{-2}P[V_{1\infty} + \beta^2(\nu_{2\infty\perp} + B_2\hat{B}_2)S_2Q\bar{\Sigma}QS_2^{\mathrm{T}}(\nu_{2\infty\perp} + B_2\hat{B}_2)^{\mathrm{T}}]P + \nu_{2\perp}^{\mathrm{T}}P\Sigma P\nu_{2\perp} - \tau_\perp^{\mathrm{T}}\nu_{2\perp}^{\mathrm{T}}P\Sigma P\nu_{2\perp}\tau_\perp, \tag{5.10}$$

and let $(A_c, B_c, C_c, D_c, \mathcal{Q})$ be given by

$$A_c = \Gamma[\nu_{2\perp}(A - \Sigma P + \gamma^{-2}V_{1\infty}P) - \nu_{2\infty\perp}S_2Q\bar{\Sigma}]G^{\mathrm{T}}, \tag{5.11}$$

$$B_c = \Gamma\nu_{2\infty\perp}S_2QC^{\mathrm{T}}V_2^{-1}, \tag{5.12}$$

$$C_c = \begin{bmatrix} -R_2^{-1}B_1^{\mathrm{T}}P \\ -(B_2^{\mathrm{T}}PB_2)^{-1}B_2^{\mathrm{T}}P(A - B_2\hat{B}_2S_2Q\bar{\Sigma} - \Sigma P + \gamma^{-2}V_{1\infty}P) \end{bmatrix} G^{\mathrm{T}}, \tag{5.13}$$

$$D_c = \begin{bmatrix} 0_{m_1\times \ell} \\ -\hat{B}_2S_2QC_1^{\mathrm{T}}V_2^{-1} \end{bmatrix}, \tag{5.14}$$

$$\mathcal{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix}. \tag{5.15}$$

Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $\hat{G}(s)$ satisfies the H_∞ disturbance attenuation constraint (5.5) and the H_2 performance criterion (2.7) satisfies the bound

$$J(A_c, B_c, C_c, D_c) \leq \mathrm{tr}[(Q + \hat{Q})R_1 + \hat{Q}P\Sigma P]. \tag{5.16}$$

Proof. The result is dual to Theorem 4.1 and thus is proved in an analogous manner. □

Remark 5.1. Although theoretically the problem is completely analogous to the singular measurement case there is a difference in the physical interpretation. Using a static gain for a noise-free output is sound engineering practice, whereas allowing the variance of certain inputs to be infinite because they are not weighted is not. In the case of actuator dynamics, however, this may be justified because u represents the command to the actuator.

6. Directions for Further Research

Several generalizations remain to be explored. These include

1) Cross weighting and cross correlation of the plant and measurement disturbance,

2) direct transmission terms in the plant dynamics and feedforward terms from disturbances to *both* H_2 and H_∞ performance variables,

3) the general H_2/H_∞ problem with both types of singularities included as shown in Figure 3,

4) and finally, the connection of the present results to the mixed performance/robustness weighted sensitivity/complimentary sensitivity problem.

The extension mentioned in (3) has been studied in [12] for the H_2 case.

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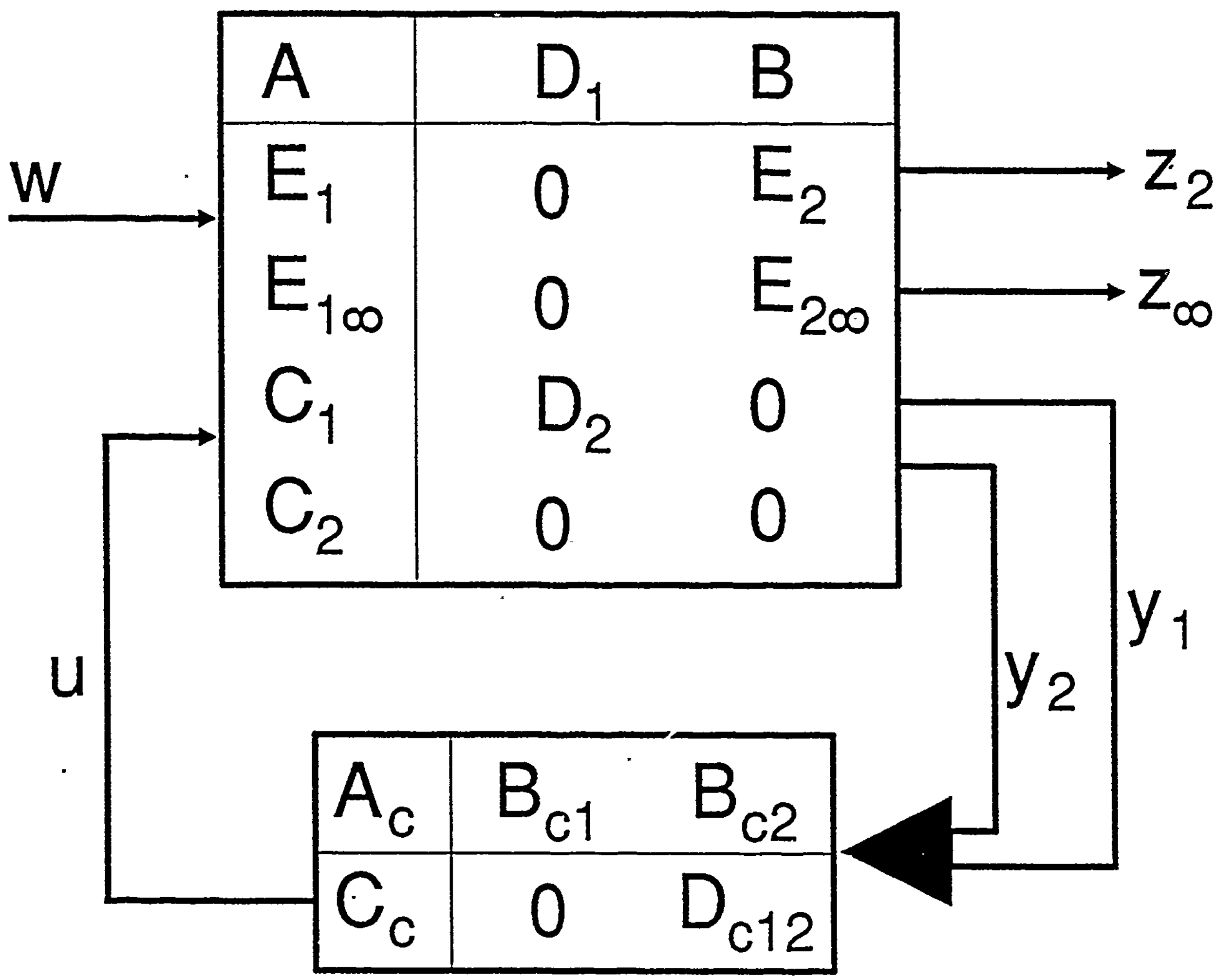


Figure 1.

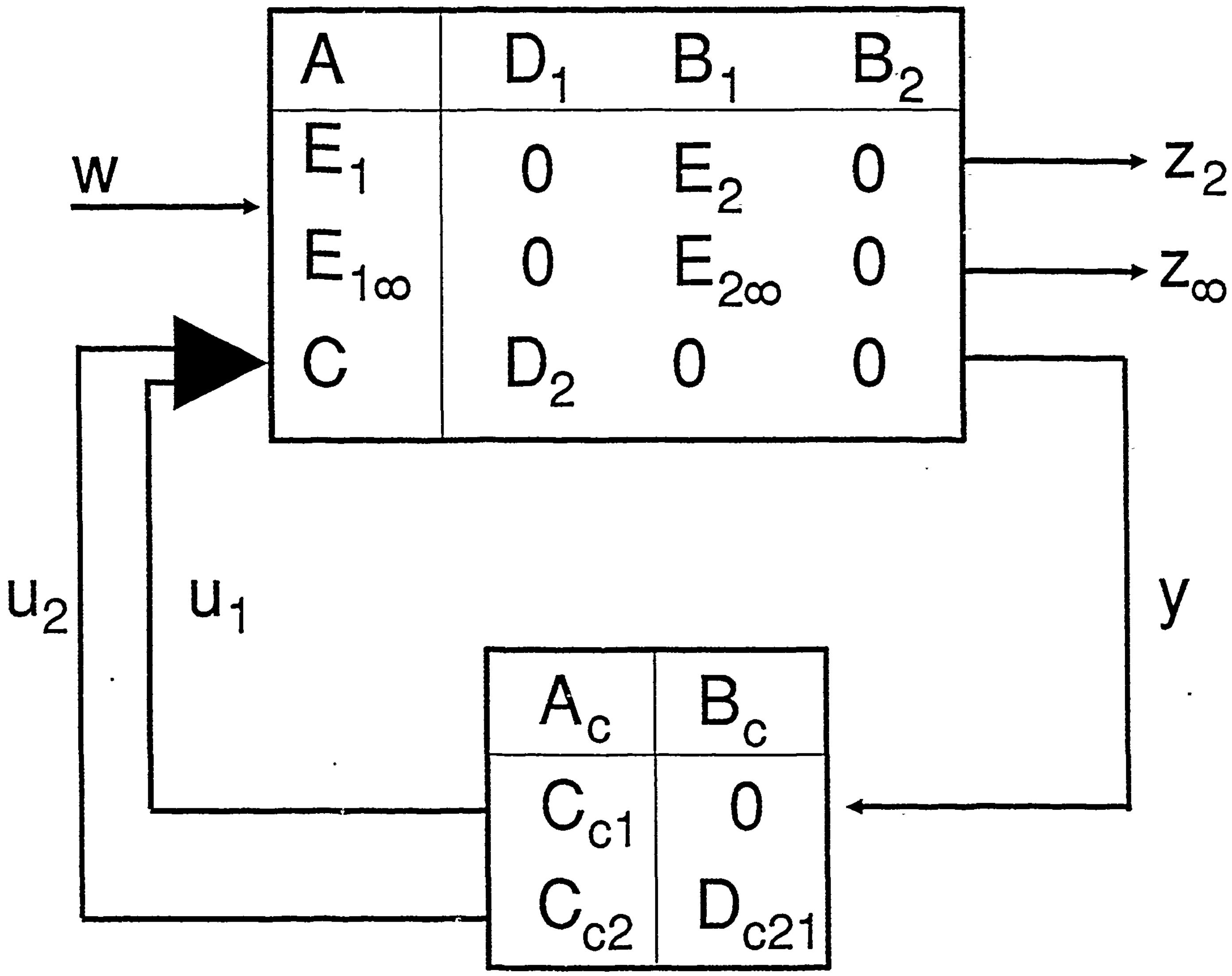
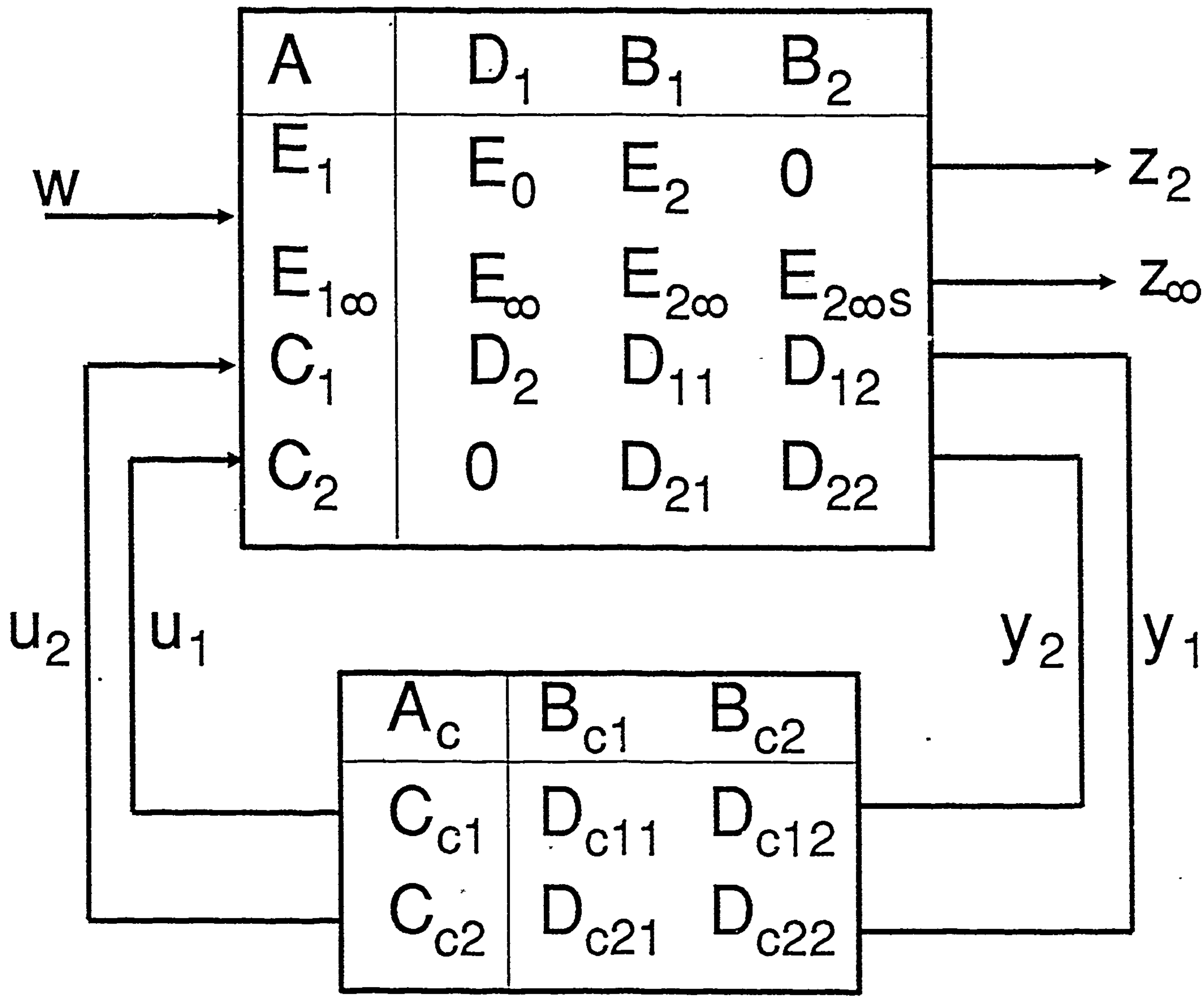


Figure 2.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z∞
u2
u1
y2
y1
Ac Bc1 Bc2
Cc1 Dc11 Dc12
Cc2 Dc21 Dc22

Figure 3.